ON $\mathcal{M}$-HARMONIC SPACE $B_p^n$

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Dedicated to the memory of Professor Slobodan Aljančić

Abstract. We give several characterizations of the Besov space $B_p^n$ of $\mathcal{M}$-harmonic functions in the open unit ball in $\mathbb{C}^n$.

1. Introduction and results

In [4] Hahn and Youssi considered the boundary behavior in the Besov spaces $B_p^n$ of $\mathcal{M}$-harmonic functions in the open unit ball $B$ in $\mathbb{C}^n$. In this paper we deal with several characterizations of the spaces $B_p^n$. As a consequence we have:

1) If $s > n$, then $B_p^n = A_p^n$, where $A_p^n$ is the weighted Bergman space.

2) If $s = n$, the spaces $B_p^n$ are closely related to the Hardy spaces $\mathcal{H}^p$ of $\mathcal{M}$-harmonic functions in $B$.

3) For $0 < s < n$, $B_p^n$ are Besov spaces ($B_p^0$ is the diagonal Besov space).

4) For $-p < s < 0$ the functions in the space $B_p^n$ have Lipschitz continuity of order $-s/p$ and thus extend continuously to the closed unit ball (see also Theorem 1.4 of [4]).

5) If $s \leq -p$ then $B_p^n = \{\text{constants}\}$.

Let $B = B_n$ be the open unit ball in $\mathbb{C}^n$ and $S = \partial B$ the unit sphere in $\mathbb{C}^n$. We denote by $\nu$ the normalized Lebesgue measure on $B$ and by $\sigma$ the rotation invariant probability measure on $S$.

Let $\tilde{\Delta}$ be the invariant Laplacian on $B$. That is, $\tilde{\Delta}f(z) = \Delta(f \circ \varphi_z)(0)$, $f \in C^2(B)$, where $\Delta$ is the ordinary Laplacian and $\varphi_z$ the standard automorphism of $B$, $\varphi_z \in \text{Aut}(B)$, taking $0$ to $z$ (see [9]).

The $C^2$-functions $f$ that are annihilated by $\tilde{\Delta}$ are called $\mathcal{M}$-harmonic ($f \in \mathcal{M}$).

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Definition 1.1. For $0 < p < \infty$, and $s \in \mathbb{R}$, the weighted Bergman space $\mathcal{A}_p^s$ is defined as the space of $\mathcal{M}$-harmonic functions $f$ on $B$ for which

$$
\|f\|_{\mathcal{A}_p^s} = \left[ \int_B (1 - |z|^2)^s |f(z)|^p d\lambda(z) \right]^{1/p} < \infty,
$$

where $d\lambda(z) = (1 - |z|^2)^{-n-1} d\sigma(z)$ is the measure on $B$ that is invariant under the group $\text{Aut}(B)$.

For $f \in C^1(B)$, $Df = (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n})$, denotes the complex gradient of $f$,

$$
\nabla f = (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}),
$$

$z_k = x_{2k-1} + i x_{2k}$, $k = 1, 2, \ldots, n$, denotes the real gradient of $f$.

For $f \in C^1(B)$ let $\bar{D}f(z) = D(f \circ \varphi_z)(0)$, $z \in B$, and $\bar{\nabla}f(z) = \nabla(f \circ \varphi_z)(0)$, $z \in B$, be the invariant complex gradient of $f$ and the invariant real gradient of $f$ respectively.

Definition 1.2. For $0 < p < \infty$, and $s \in \mathbb{R}$, the $\mathcal{M}$-harmonic Dirichlet space $\mathcal{D}_p^s$ is defined as the space of $\mathcal{M}$-harmonic functions $f$ on $B$ for which

$$
\int_B |\nabla f(z)|^p (1 - |z|^2)^s d\lambda(z) < \infty.
$$

The (differential) Bergman metric $b : B \times \mathbb{C}^n \rightarrow \mathbb{R}$ is defined by

$$
b(z, \xi) = \left( \frac{(1 - |z|^2)|\xi|^2 + |(z, \xi)|^2}{(1 - |z|^2)^2} \right)^{1/2}.
$$

For $f \in C^1(B)$, define the functional quantity

$$
Qf(z) = \sup_{|\xi| = 1} \frac{\nabla f(z) \cdot \xi}{b(z, \xi)} = \sup_{|\xi| = 1} \frac{\langle Df(z) \cdot \xi \rangle}{b(z, \xi)},
$$

$\xi \in \mathbb{C}^n$. This quantity is invariant under $\text{Aut}(B)$, that is $Q(f \circ \varphi) = Q(f) \circ \varphi$, for all $C^1$-functions $f$ in $B$ and $\varphi \in \text{Aut}(B)$ (see [5, 6]).

Definition 1.3. For $0 < p < \infty$, $s \in \mathbb{R}$, let $\mathcal{B}_p^s$ be the space of $\mathcal{M}$-harmonic functions $f$ on $B$ such that

$$
\|f\|_{\mathcal{B}_p^s} = \left( \int_B (Qf)^p(z) (1 - |z|^2)^s d\lambda(z) \right)^{1/p} < \infty.
$$

Theorem 1.4. Let $0 < p < \infty$, $s > n - p/2$ and $f \in \mathcal{M}$. Then the following statements are equivalent:

(i) $f \in \mathcal{D}_p^s$

(ii) $f \in \mathcal{B}_p^s$

(iii) $\int_B |\nabla f(z)|^p (1 - |z|^2)^s d\lambda(z) < \infty$,

(iv) $\int_B (1 - |z|^2)^{s+p} (|Rf(z)| + |\bar{R}f(z)|)^p d\lambda(z) < \infty$. 


As usual, \( Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j} \), is the radial derivative of \( f \) and \( \overline{Rf}(z) = \sum_{j=1}^{n} \overline{z}_j \frac{\partial f}{\partial \overline{z}_j} \).

### 2. Proof of Theorem

If \( 0 < r < 1 \), we set \( E_r(z) = \{ w \in B : |\varphi_z(w)| < r \} = \varphi_z(rB) \). It is easy to see that \( E_r(z) \) is an ellipsoid and its volume is given by \( \nu(E_r(z)) = \frac{r^{2n} |z|^n}{(1 - r|z|)^{n+1}} \) (see [9, p. 30]). We set \( |E_r(z)| = \nu(E_r(z)) \).

For the proof of Theorem 1.4 the following lemmas will be needed.

**Lemma 2.1.** [7] Let \( 0 < r < 1 \). There is a constant \( C > 0 \) such that if \( f \in \mathcal{M} \) then

\[
\begin{align*}
(i) \quad & |T_{ij} Rf(w)| \leq C(1 - |w|^2)^{-1/2} \int_{E_r(w)} |Rf(z)| d\lambda(z), \quad w \in B, \\
(ii) \quad & |T_{ij} \overline{Rf}(w)| \leq C(1 - |w|^2)^{-1/2} \int_{E_r(w)} |\overline{Rf}(z)| d\lambda(z), \quad w \in B, \\
(iii) \quad & |T_{ij} f(w)| \leq C(1 - |w|^2)^{-1/2} \int_{E_r(w)} |f(z)| d\lambda(z), \quad w \in B.
\end{align*}
\]

Here, as usual, \( T_{ij} = \frac{\partial}{\partial z_j} - \frac{\partial}{\partial \overline{z}_i} \), \( T_{ij} = z_i \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial \overline{z}_i} \) are tangential derivatives.

Here and elsewhere constants are denoted by \( C \) which may indicate a different constant from one occurrence to the next.

**Lemma 2.2.** If \( s > 1 \), then

\[
\int_{0}^{1} \frac{dt}{|1 - t(z,w)|^s} \leq \frac{C}{|1 - \langle z, w \rangle|^{s-1}}, \quad z, w \in B.
\]

**Lemma 2.3.** [9, p. 17] If \( \alpha > 0 \), then

\[
\int_{S} \frac{d\alpha(\xi)}{|1 - \langle \xi, z \rangle|^{n+\alpha}} = O \left( \frac{1}{(1 - |z|)^{\alpha}} \right), \quad z \in B.
\]

It is easy to see that \( |\nabla f(z)| = Qf(z) \). Hence, \( \mathcal{D}_p^\alpha = \mathcal{B}_p^\alpha \), for all \( 0 < p < \infty \) and \( \alpha \in \mathbb{R} \).

From the inequality \( Qf(z) \geq (1 - |z|^2)|\nabla f(z)| \) (see [4, p. 221]) it follows that

(iii) \( \implies \) (iv) It is easy to see that if (iii) holds then

\[
\begin{align*}
\int_{B} (1 - |z|^2)^{s+1} \left| \frac{\partial f}{\partial z_j}(z) \right|^p d\lambda(z) & < \infty, \quad 1 \leq j \leq n, \\
\int_{B} (1 - |z|^2)^{s+1} \left| \frac{\partial f}{\partial \overline{z}_j}(z) \right|^p d\lambda(z) & < \infty, \quad 1 \leq j \leq n,
\end{align*}
\]
which in turn implies that
\[
\int_B (1 - |z|^2)^{s+p}|Rf(z)|^p d\lambda(z) < \infty
\]
\[
\int_B (1 - |z|^2)^{s+p} |\overline{Rf(z)}|^p d\lambda(z) < \infty.
\]
Thus, (iii) \implies (iv).

(iv) \implies (i) Assume now that
\[
\int_B (1 - |z|^2)^{s+p} (|Rf(z)| + |\overline{Rf(z)}|)^p d\lambda(z) < \infty.
\]
It is easy to check that \(|z|^2 |Df(z)|^2 = |Rf(z)|^2 + \sum_{i<j} |T_{ij}f(z)|^2\). Using this and the equality
\[
|\nabla f(z)|^2 = 2(|D f(z)|^2 + |D\overline{f}(z)|^2)
\]
\[
= 2(1 - |z|^2) (|D f(z)|^2 - |Rf(z)|^2 + |D\overline{f}(z)|^2 - |\overline{Rf}(z)|^2)
\]
(see [8]) we find that
\[
|z|^2 |\nabla f(z)|^2 = 2(1 - |z|^2) \left[ (1 - |z|^2) (|Rf(z)|^2 + |\overline{Rf}(z)|^2) + \sum_{i<j} |T_{ij}f(z)|^2 + \sum_{i<j} |T_{ij}\overline{f}(z)|^2 \right].
\]
Hence, to show that \(f \in \mathcal{D}^s_p\) it is sufficient to show that
\[
\int_B (1 - |z|^2)^{s+p/2} (|T_{ij}f(z)|^p + |T_{ij}\overline{f}(z)|^p) d\lambda(z) < \infty, \quad 1 \leq i < j \leq n.
\]
Integration by parts shows that
\[
f(z) = \int_0^1 [Rf(tz) + \overline{Rf(tz)} + f(tz)] dt.
\]
From this we conclude that it is sufficient to prove that
\[
\int_B (1 - |z|^2)^{s+p/2} \left( \int_0^1 |T_{ij}u(tz)| dt \right)^p d\lambda(z) < \infty, \quad 1 \leq i < j \leq n,
\]
where \(u\) is \(Rf\) or \(\overline{Rf}\) or \(\overline{Rf}\) or \(\overline{Rf}\) or \(f\).

We will show that, for fixed \(1 \leq i < j \leq n,\)
\[
I = \int_B (1 - |z|^2)^{s+p/2} \left( \int_0^1 |T_{ij}Rf(tz)| dt \right)^p d\lambda(z) < \infty.
\]
The remaining cases may be treated analogously.
Using Lemma 2.1, Fubini’s theorem and Lemma 2.2 we find that for any \( a > 0 \)
\[
\int_0^1 |T_{ij}rf(tz)| dt \leq C \int_0^1 \left( \int_{E_r(tz)} \frac{|Rf(w)|(1 - |w|^2)^a}{|1 - t \langle z, w \rangle |^{n+a+3/2}} dv(w) \right) dt \\
\leq C \int_0^1 \left( \int_B \frac{|Rf(w)|(1 - |w|^2)^a dv(w)}{|1 - t \langle z, w \rangle |^{n+a+3/2}} \right) dt \\
= C \int_B \left| Rf(w) \right|(1 - |w|^2)^a \left( \int_0^1 \frac{dt}{|1 - t \langle z, w \rangle |^{n+a+3/2}} \right) dv(w) \\
\leq C \int_B \left| Rf(w) \right|(1 - |w|^2)^a dv(w).
\]

Assume now \( 1 < p < \infty \). Applying the continuous form of Minkowski’s inequality we obtain
\[
I \leq C \int_0^1 (1 - r)^{s+p/2-n-1} \left( \int_S \left( \int_S |Rf(\rho \xi)| (1 - \rho)^a d\sigma(\xi) \right)^p d\rho \right)^{1/p} dr.
\]

By Hölder’s inequality
\[
\int_S |Rf(\rho \xi)| d\sigma(\xi) \\
\leq \left( \int_S |Rf(\rho \xi)|^p d\sigma(\xi) \right)^{1/p} \left( \int_S \frac{d\sigma(\xi)}{|1 - \langle r \rho \xi \rangle |^{n+a+1/2}} \right)^{1/p'} \\
\leq \frac{C}{(1 - r \rho)(a+1/2)/p'} \left( \int_S \frac{|Rf(\rho \xi)|^p d\sigma(\xi)}{|1 - \langle r \rho \xi \rangle |^{n+a+1/2}} \right)^{1/p}, \quad \text{by Lemma 2.3.}
\]
(Here \( 1/p + 1/p' = 1 \)).

Now we substitute (2.2) into (2.1) and use Fubini’s theorem and Lemma 2.3 to get
\[
I \leq C \int_0^1 (1 - r)^{s+p/2-n-1} \left( \int_0^1 \frac{(1 - \rho)^a}{(1 - r \rho)(a+1/2)/p'} \right) \\
\cdot \left( \int_S \left( \int_S |Rf(\rho \xi)|^p d\sigma(\xi) \right)^{1/p} d\rho \right)^p dr \\
= C \int_0^1 (1 - r)^{s+p/2-n-1} \left( \int_0^1 \frac{(1 - \rho)^a}{(1 - r \rho)(a+1/2)/p'} \right) \\
\cdot \left( \int_S |Rf(\rho \xi)|^p d\sigma(\xi) \int_S \frac{d\sigma(\xi)}{|1 - \langle r \rho \xi \rangle |^{n+a+1/2}} \right)^{1/p} d\rho \right)^p dr \\
\leq C \int_0^1 (1 - r)^{s+p/2-n-1} \left( \int_0^1 \frac{(1 - \rho)^a}{(1 - r \rho)(a+1/2)} \right) \\
\cdot \left( \int_S |Rf(\rho \xi)|^p d\sigma(\xi) \right)^{1/p} d\rho \right)^p dr.
\]
A simple observation shows that it is possible to select positive parameters \(a, t_1, t_2, t_3, t_4\) such that

(i) \(a = t_1 + t_2 = t_3 + t_4\),
(ii) \(\frac{1}{p'} < t_3 - t_1 < \frac{s}{p} + \frac{3}{2} - \frac{n + 1}{p}\),
(iii) \(t_2 > 1 + \frac{s}{p} - \frac{n + 1}{p}\).

Note that here again we used the assumption that \(s > n - p/2\).

Applying Hölder’s inequality on (2.3) and Lemma 2.3 we obtain

\[
I \leq C \int_0^1 (1-r)^{s+3p/2-n-2-(t_1-t_3)p} \left( \int_0^1 \frac{(1-r)^{t_1'} \rho}{(1-r^t)^{(t_1+t_2)/p}} \left( \int_S |Rf(\rho \xi)|^p d\sigma(\xi) \right) d\rho \right) dr
\]

\[
\leq C \int_0^1 (1-r)^{s+3p/2-n-2-(t_1-t_3)p} \left( \int_0^1 \frac{(1-r)^{t_2} \rho}{(1-r^t)^{(t_1+t_2)/p}} \left( \int_S |Rf(\rho \xi)|^p d\sigma(\xi) \right) d\rho \right) dr
\]

\[
= C \int_0^1 \left( \int_0^1 (1-r)^{s+3p/2-n-2-(t_1-t_3)p} \right) d\rho
\]

\[
\leq C \int_B (1-|z|^2)^{s+p}|Rf(z)|^p d\nu(z) < \infty.
\]

If \(p = 1\), then

\[
I \leq C \int_B (1-|z|^2)^{s+1/2} \left( \int_B \frac{|Rf(w)| (1-|w|^2)^a d\nu(w)}{1-\langle z, w \rangle |w|^{n+1/2}} \right) d\lambda(z)
\]

\[
\leq C \int_0^1 (1-r)^{s+1/2-n-1} \left( \int_0^1 \frac{(1-r)^a \rho}{1-\langle r \xi, \rho \xi \rangle |\rho|^2} \left( \int_S |Rf(\rho \xi)|^p d\sigma(\xi) \right) d\rho \right) dr
\]

\[
\leq C \int_0^1 (1-r)^{s-n-1/2} \left( \int_0^1 \frac{(1-r)^a \rho}{1-\langle r \xi, \rho \xi \rangle |\rho|^2} \left( \int_S |Rf(\rho \xi)|^p d\sigma(\xi) \right) d\rho \right) dr
\]

\[
= C \int_0^1 (1-r)^{s-n-1/2} \left( \int_0^1 \frac{(1-r)^a \rho}{1-\langle r \xi, \rho \xi \rangle |\rho|^2} \left( \int_S |Rf(\rho \xi)|^p d\sigma(\xi) \right) d\rho \right) dr
\]

\[
\leq C \int_B (1-|w|^2)^{s-n}|Rf(w)| d\nu(w) < \infty.
\]

(We may assume that \(a > \max \{s-n, 0\}\).)

For the case \(0 < p < 1\) the following lemma will be needed.

**Lemma 2.4.** Let \(0 < r < 1\) and \(0 < p < \infty\). There is a constant \(C\) such that if \(f \in \mathcal{M}\), then

(i) \[
\frac{|Rf(w)|^p}{1-\langle z, w \rangle} \leq C \int_{E_r(w)} \frac{|Rf(\xi)|^p}{1-\langle z, \xi \rangle} d\lambda(\xi), \quad z, w \in B
\]
(ii) \[
\left( \frac{|Rf(w)|}{|1 - \langle z, w \rangle|} \right)^p \leq C \int_{E_r(w)} \left( \frac{|Rf(\xi)|}{|1 - \langle z, \xi \rangle|} \right)^p d\lambda(\xi), \quad z, w \in B
\]

Note that the constant \( C \) is independent of \( z \) and \( w \).
We will prove (i). The proof of (ii) is similar. By the formula (1.3) in [1]
\[
Rf(w) = \int_B \frac{Rf(\varphi_w(p\xi)) d\varphi(\xi)}{1 - \langle p\xi, w \rangle}, \quad w \in B, \quad 0 < p < 1.
\]
Multiplying this equality by \( 2n p^{2n-1} (1 - p^2)^{-n-1} h(\rho) d\rho \), where \( h \) is a radial function which belongs to \( C^\infty(B) \) with compact support in \( B \) such that \( \int_B h(z) d\lambda(z) = 1 \), then integrating from 0 to 1 and using the invariance of the measure \( \lambda \), we get
\[
Rf(w) = \int_B h(\varphi_w(z)) \frac{Rf(z)}{1 - \langle \varphi_w(z), w \rangle} d\lambda(z) = \int_B h(\varphi_w(z)) \frac{1 - \langle z, w \rangle}{1 - |w|^2} Rf(z) d\lambda(z)
\]
by Theorem 2.2.5 [9, p. 28]. By a suitable choice of a function \( h \) we obtain
\[
|Rf(w)| \leq C \int_{E_r(w)} |Rf(\xi)| d\lambda(\xi), \quad w \in B, \quad \text{for some } 0 < r < 1.
\]
Since \( |1 - \langle z, w \rangle| \geq |1 - \langle z, \xi \rangle| \), if \( \xi \in E_r(w) \), we have
\[
\frac{|Rf(w)|}{|1 - \langle z, w \rangle|} \leq C \int_{E_r(w)} \frac{|Rf(\xi)|}{|1 - \langle z, \xi \rangle|} d\lambda(\xi),
\]
and consequently,
\[
\left( \frac{|Rf(w)|}{|1 - \langle z, w \rangle|} \right)^p \leq C \int_{E_r(w)} \left( \frac{|Rf(\xi)|}{|1 - \langle z, \xi \rangle|} \right)^p d\lambda(\xi), \quad z, w \in B
\]
(see [8]).

To finish the proof of Theorem 1.4 assume that \( 0 < p < 1 \). Applying Theorem 3.2 (iii) [3] to the function
\[
F(w) = \left( \frac{|Rf(w)|}{|1 - \langle z, w \rangle|} \right)^{n/2}, \quad w \in B \quad (z \in B - \text{fixed})
\]
and replacing \( p, r, k, q \) by \( 2, 2/p, 2/p, p(a + n + 1) - n \) respectively and using Lemma 2.4 we find that
\[
\left( \int_B \frac{|Rf(w)|}{|1 - \langle z, w \rangle|^{n+1/2}} d\nu(w) \right)^p \leq C \int_B \frac{|Rf(w)|^p (1 - |w|^2)^{p(a+n+1)-n+1}}{|1 - \langle z, w \rangle|^{p(a+n+1)/2}} d\nu(w).
\]
Thus, assuming that \( a > s/p - n \),
\[
I \leq C \int_B (1 - |z|^2)^{s+n/p/2} \left( \int_B \frac{|Rf(w)|^p (1 - |w|^2)^{p(a+n+1)-n-1}}{|1 - \langle z, w \rangle|^{p(a+n+1)/2}} d\nu(w) \right) d\lambda(z)
\]
\[
= C \int_B |Rf(w)|^p (1 - |w|^2)^{p(a+n+1)-n} \left( \int_B \frac{(1 - |z|^2)^{s+n/2-n-1}}{|1 - \langle z, w \rangle|^{p(a+n+1)/2}} d\nu(w) \right)
\]
\[
\leq C \int_B (1 - |w|^2)^{p+n-1} |Rf(w)|^p d\lambda(w) < \infty.
\]
This finishes the proof of Theorem 1.4.

REFERENCES

3. P. Beatrous, J. Burbea, Holo

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