AN ESTIMATE FOR COEFFICIENTS OF POLYNOMIALS IN $L^2$ NORM. II

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Dedicated to the memory of Professor S. Aljančić

Abstract. Let $\mathcal{P}_n$ be the class of algebraic polynomials $P(x) = \sum_{k=0}^{n} a_k x^k$ of degree at most $n$ and $\|P\|_{d\nu} = \left(\int_{\mathbb{R}} |P(x)|^2 d\nu(x)\right)^{1/2}$, where $d\nu(x)$ is a nonnegative measure on $\mathbb{R}$. We determine the best constant in the inequality $|a_k| \leq C_{n,k} \|P\|_{d\nu}$, for $k = 0, 1, \ldots, n$, when $P \in \mathcal{P}_n$ and such that $P(\xi_k) = 0$, $k = 1, \ldots, m$. The cases $C_{n,n}(d\nu)$ and $C_{n,n-1}(d\nu)$ were studied by Milovanović and Guessab [6]. In particular, we consider the case when the measure $d\nu(x)$ corresponds to generalized Laguerre orthogonal polynomials on the real line.

1. Introduction

Let $\mathcal{P}_n$ be the class of algebraic polynomials $P(x) = \sum_{k=0}^{n} a_k x^k$ of degree at most $n$. The first inequality of the form $|a_k| \leq C_{n,k} \|P\|$ was given by Markov [3]. Namely, if $\|P\| = \|P\|_\infty = \max_{x \in [-1,1]} |P(x)|$ and $T_n(x) = \sum_{\nu=0}^{n} t_{n,\nu} x^\nu$ denotes the $n$-th Chebyshev polynomial of the first kind, then Markov proved that

\[
|a_k| \leq \begin{cases} |t_{n,k}| \cdot \|P\|_\infty & \text{if } n-k \text{ is even, } \\ |t_{n-1,k}| \cdot \|P\|_\infty & \text{if } n-k \text{ is odd.} \end{cases}
\]  

(1.1)

For $k = n$ (1.1) reduces to the well-known Chebyshev inequality

\[
|a_n| \leq 2^{n-1} \|P\|_\infty.
\]  

(1.2)

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Using a restriction on the polynomial class like \( P(1) = 0 \) or \( P(-1) = 0 \), Schur [8] found the following improvement of (1.2)

\[
|a_n| \leq 2^{n-1} \left( \cos \frac{\pi}{4n} \right)^{2n} \|P\|_{\infty}.
\]

This result was extended by Rahman and Schmeisser [7] for polynomials with real coefficients, which have at most \( n - 1 \) distinct zeros in \((-1, 1)\).

Similarly in \( L^2 \) norm,

\[
\|P\| = \|P\|_2 = \left( \int_{-1}^{1} |P(x)|^2 \, dx \right)^{1/2},
\]

Tariq [10] improved the following result of Lavelle [2]

\[
|a_k| \leq \frac{1 \cdot 3 \cdot 5 \cdots (2k - 1)}{k!} \left( k + \frac{1}{2} \right)^{1/2} \left( \frac{(n - k/2) + k + 1/2}{(n - k/2)} \right) \|P\|_2. \tag{1.3}
\]

for \( P \in P_n \) and \( 0 \leq k \leq n \), where the symbol \([x]\) denotes as usual the integral part of \( x \). Equality in this case is attained only for the constant multiplies of the polynomial

\[
\sum_{\nu=0}^{[n-k/2]} (-1)^\nu (4\nu + 2k + 1) \left( k + \nu - 1/2 \right) P_{k+2\nu}(x),
\]

where \( P_m(x) \) denotes the Legendre polynomial of degree \( m \).

Under restriction \( P(1) = 0 \), Tariq [10] proved that

\[
|a_n| \leq \frac{n}{n+1} \cdot \frac{(2n)!}{2^n(n!)^2} \left( \frac{2n + 1}{2} \right)^{1/2} \|P\|_2, \tag{1.4}
\]

with equality case \( P(x) = P_n(x) - \frac{1}{n^2} \sum_{\nu=0}^{n-1} (2\nu + 1) P_{\nu}(x) \). Also, he obtained that

\[
|a_{n-1}| \leq \frac{(n^2 + 2)!}{n+1} \cdot \frac{(2n - 2)!}{2^n(n-1)!^2} \left( \frac{2n - 1}{2} \right)^{1/2} \|P\|_2, \tag{1.5}
\]

with equality case

\[
P(x) = \frac{2n + 1}{n^2 + 2} P_n(x) - P_{n-1}(x) + \frac{1}{n^2 + 2} \sum_{\nu=0}^{n-2} (2\nu + 1) P_{\nu}(x).
\]
In the absence of the hypothesis $P(1) = 0$ the factor $(n^2 + 2)^{1/2} / (n + 1)$ appearing on the right-hand side of (1.5) is to be dropped.

This result was extended by Milovanović and Guessab [4] for polynomials with real coefficients, which have $m$ zeros on real line.

In this paper we consider more general problem including $L^2$ norm of polynomials with respect to a nonnegative measure on the real line $\mathbb{R}$. The generalized Laguerre measure is also included.

2. Main results

Let $d\sigma(x)$ be a given nonnegative measure on the real line $\mathbb{R}$, with compact or infinite support, for which all moments $\mu_k = \int_{\mathbb{R}} x^k \, d\sigma(x)$, $k = 0, 1, \ldots$, exist and are finite, and $\mu_0 > 0$. In that case, there exist a unique set of orthonormal polynomials $\pi_n(\cdot) = \pi_n(\cdot; d\sigma)$, $n = 0, 1, \ldots$, defined by

$$
\pi_n(x) = b_n^{[n]}(d\sigma)x^n + b_{n-1}^{[n]}(d\sigma)x^{n-1} + \cdots + b_0^{[n]}(d\sigma), \quad b_n^{[n]}(d\sigma) > 0,
$$

where

$$
(f, g) = \int_{\mathbb{R}} f(x)g(x) \, d\sigma(x) \quad (f, g \in L^2(\mathbb{R})). \tag{2.1}
$$

For $P \in P_n$, we define

$$
\|P\|_{d\sigma} = \sqrt{(P, P)} = \left( \int_{\mathbb{R}} |P(x)|^2 \, d\sigma(x) \right)^{1/2}. \tag{2.2}
$$

Also, for $\xi_k \in \mathbb{C}$, $k = 1, \ldots, m$, we define a restricted polynomial class

$$
P_n(\xi_1, \ldots, \xi_m) = \{ P \in P_n \mid P(\xi_k) = 0, \; k = 1, \ldots, m \} \quad (0 \leq m \leq n).
$$

In the case $m = 0$ this class of polynomials reduces to $P_n$. The case $m = n$ is trivial. If $\xi_1 = \cdots = \xi_k = \xi$ ($1 \leq k \leq m$) then the restriction on polynomials at the point $x = \xi$ becomes $P(\xi) = P'(\xi) = \cdots = P^{[k-1]}(\xi) = 0$.

Let

$$
\prod_{i=1}^m (x - \xi_i) = x^m - s_1 x^{m-1} + \cdots + (-1)^{m-1} s_{m-1} x + (-1)^m s_m
$$

where $s_k$ denotes elementary symmetric functions of $\xi_1, \ldots, \xi_m$, i.e.,

$$
s_k = \sum_{\ell=k}^m \xi_{\ell-k} \cdots \xi_k \quad \text{for} \quad k = 1, \ldots, m. \tag{2.3}
$$

For $k = 0$ we have $s_0 = 1$, and $s_k = 0$ for $k > m$ or $k < 0$. 
Theorem 2.1. Let $P \in \mathcal{P}_n(\xi_1, \ldots, \xi_m)$ and $s_1, \ldots, s_m$ be given by (2.3). If the measure $d\sigma(x)$ is given by

$$d\sigma(x) = \prod_{k=1}^{m} |x - \xi_k|^2 d\sigma(x)$$

(2.4)

and $||P||_{d\sigma}$ is defined by (2.2), then

$$|a_{n-k}| \leq \left( \sum_{j=0}^{k} \left( \sum_{i=j}^{k} (-1)^{k-i} s_k \hat{b}_{n-m-i}^{(n-m-j)} \right)^2 \right)^{1/2} ||P||_{d\sigma},$$

(2.5)

for $k = 0, 1, \ldots, n$, where $\hat{b}_{i}^{(\nu)} = b_{\nu}^{(\nu)}(d\sigma)$, $\nu = 0, 1, \ldots, \mu$, are the coefficients in the orthonormal polynomial $\pi_{\nu}(\cdot) = \pi_{\nu}(\cdot; d\sigma)$.

Inequality (2.5) is sharp and becomes an equality if and only if $P(x)$ is a constant multiple of the polynomial

$$\left( \sum_{j=0}^{k} \hat{\pi}_{n-m-j}(x) \sum_{i=j}^{k} (-1)^{k-i} s_k \hat{b}_{n-m-i}^{(n-m-j)} \right) \prod_{k=1}^{m} (x - \xi_k).$$

Proof. At first we consider the inner product (2.1). Then the polynomial $P(x) = \sum_{\nu=0}^{n} a_{\nu} x^{\nu} \in \mathcal{P}_n$ can be represented in the form $P(x) = \sum_{\nu=0}^{n} \alpha_{\nu} \pi_{\nu}(x; d\sigma)$, where $\alpha_{\nu} = (P, \pi_{\nu})$, $\nu = 0, 1, \ldots, n$. Then we have

$$a_{n-k} = \sum_{i=0}^{k} \alpha_{n-i} \hat{b}_{n-k}^{(n-i)}(d\sigma) = \left( P, \sum_{i=0}^{k} \hat{b}_{n-k}^{(n-i)}(d\sigma) \pi_{n-i} \right), \quad k = 0, 1, \ldots, n,$$

(2.6)

where $\pi_{\nu}(\cdot) = \pi_{\nu}(\cdot; d\sigma)$.

Suppose now that $P \in \mathcal{P}_n(\xi_1, \ldots, \xi_m)$. Then we can write

$$P(x) = Q(x) \prod_{k=1}^{m} (x - \xi_k),$$

(2.7)

where $Q(x) = a_{n-m} x^{n-m} + a_{n-m-1} x^{n-m-1} + \ldots + a_0 \in \mathcal{P}_{n-m}$. Also, we have

$$\prod_{k=1}^{m} (x - \xi_k) = x^m - s_1 x^{m-1} + \ldots + (-1)^{m-1} s_{m-1} x + (-1)^m s_m$$

where $s_k$, $k = 0, 1, \ldots, m$, denotes elementary symmetric functions (2.3). Now, putting this in (2.7), we obtain

$$P(x) = \sum_{i=0}^{n-m} \sum_{\nu=0}^{m} a_{i}^{(n)} b_{\nu}^{(n-m)} x^{n-k},$$
where
\[ a_{n-k} = \sum_{i=0}^{k} a^i_{n-m-i} (-1)^{k-i} s_{k-i}, \quad k = 0, 1, \ldots, n, \tag{2.8} \]
and \( a^i_k = 0 \) for \( k < 0 \) and \( k > n - m \).

Now, the corresponding equalities (2.6) for polynomial \( Q \) in the measure \( d\tilde{\sigma}(x) \), given by (2.4), become
\[ a^i_{n-m-i} = \left( Q, \sum_{j=0}^{i} \hat{b}^{(n-m-j)}_{n-m-i} \hat{\pi}_{n-m-j} \right), \quad i = 0, 1, \ldots, n - m, \tag{2.9} \]
where \( \hat{\pi}_i(.) = \pi_i(\cdot ; d\tilde{\sigma}) \).

According to (2.7), we have
\[ a_{n-k} = \sum_{i=0}^{k} (-1)^{k-i} s_{k-i} \left( Q, \sum_{j=0}^{i} \hat{b}^{(n-m-j)}_{n-m-i} \hat{\pi}_{n-m-j} \right) = (Q, W_{n-m}) \tag{2.10} \]
where
\[ W_{n-m}(x) = \sum_{i=0}^{k} (-1)^{k-i} s_{k-i} \sum_{j=0}^{i} \hat{b}^{(n-m-j)}_{n-m-i} \hat{\pi}_{n-m-j}(x) \]
\[ = \sum_{j=0}^{k} \hat{\pi}_{n-m-j}(x) \sum_{i=j}^{k} (-1)^{k-i} s_{k-i} \hat{b}^{(n-m-j)}_{n-m-i} \]
and \( b^{(u)}_{\nu} = 0 \) for \( \nu < 0 \). Now, using Cauchy inequality we get
\[ |a_{n-k}| \leq C_{n,n-k}||Q||_{d\tilde{\sigma}} \]
where \( C_{n,n-k} = ||W_{n-m}||_{d\tilde{\sigma}} = \left( \sum_{j=0}^{k} \left( \sum_{i=j}^{k} (-1)^{k-i} s_{k-i} \hat{b}^{(n-m-j)}_{n-m-i} \right)^2 \right)^{1/2} \).

Since
\[ ||Q||^2_{d\tilde{\sigma}} = \int_{\mathbb{R}} |Q(x)|^2 d\tilde{\sigma}(x) = \int_{\mathbb{R}} |P(x)|^2 d\sigma(x) = ||P||^2_{d\sigma} \]
we obtain inequality (2.5).

The extremal polynomial is \( x \mapsto W_{n-m}(x) \prod_{k=1}^{m} (x - \xi_k) \). \( \square \)

**Remark 2.1.** For \( k = 0 \) and \( k = 1 \) Theorem 2.1 gives the results obtained by Mlovanović and Gusevskii [4] (see also [6, pp. 432–439]).

Consider now the generalized Laguerre measure \( d\sigma(x) = x^\alpha e^{-x} dx, \alpha > -1 \), on \((0, +\infty)\). With \( \tilde{L}_{n}^{(\alpha)}(x) \) we denote the generalized orthonormal Laguerre polynomial. The coefficient \( b_k^{(n)} \) of \( x^k \) in \( \tilde{L}_{n}^{(\alpha)}(x) \) is given by
\[ b_k^{(n)} = (-1)^{n-k} \binom{n}{k} \frac{(\alpha + k + 1)_{n-k}}{\sqrt{n!} \Gamma(n + \alpha + 1)}. \]

As a direct corollary of Theorem 2.1, we have:
Corollary 2.2. Under restriction $P^{(i)}(0) = 0$, $i = 0, 1, \ldots, m - 1$, we have that

$$|a_{n-k}| \leq \sqrt{A_{n,k}} ||P||_1,$$

where

$$A_{n,k} = \frac{1}{(n - m - k)!T(n + m - k + \alpha + 1)} \sum_{j=0}^{k} \binom{n + m - j + \alpha}{k-j} \binom{n - m - j}{k-j}$$

for $n - k \geq m$, and $A_{n,k} = 0$ for $n - k < m$. The equality is attained if and only if $P(x)$ is a constant multiple of the polynomial

$$x^m \sum_{j=0}^{k} \hat{b}_{n-m-k}^{(n-m-j)} \tilde{f}_{n-m-j}(x).$$

 References


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