A REMARK ON THE PARTIAL SUMS IN HARDY SPACES

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Abstract. We prove that a function $f$, analytic in the unit disc, belongs to the Hardy space $H^1$ if and only if

$$\sum_{j=0}^{n} \frac{1}{j+1} \|s_jf\| = O(\log n) \quad (n \to \infty),$$

where $s_jf$ are the partial sums of the Taylor series of $f$. As a corollary we have that, for $f \in H^1$,

$$\sum_{j=0}^{n} \frac{1}{j+1} \|f - s_jf\| = o(\log n),$$

The analogous facts for $L^1$ do not hold.

For a function $f$ analytic in the unit disc $D$ let

$$P_n f = \frac{1}{A_n} \sum_{j=0}^{n} \frac{1}{j+1} s_jf \quad (n = 0, 1, 2, \ldots),$$

where

$$A_n = \sum_{j=0}^{n} \frac{1}{j+1}$$

and $s_jf$ are the partial sums of the Taylor series of $f$,

$$s_jf(z) = \sum_{k=0}^{j} \hat{f}(k) z^k.$$

As usual, we denote by $H^1$ the space of those functions $f$, analytic in $D$, such that

$$\|f\| = \sup_{r<1} I(f, r) < \infty,$$

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where
\[ I(f, r) = \frac{2\pi}{\int_0^r |f(re^{it})|dt / 2\pi}. \]

For the properties of \( H^1 \) see [1] and [2].

It is well known that \( \| s_n f \| \leq \text{const.} A_n \| f \| \) and that \( A_n \) is “best possible”. (Note that \( A_n \) behaves like \( \log n \) as \( n \to \infty \).) A direct consequence is that, for \( n \geq 2 \),
\[ \frac{1}{A_n} \sum_{j=0}^n \frac{1}{j+1} \| s_j f \| \leq C \| f \| \quad (f \in H^1, n \geq 0). \tag{1} \]

where \( C \) is an absolute constant. In this note we prove, by using an inequality of Hardy and Littlewood, that (1) can be improved to get that
\[ \frac{1}{A_n} \sum_{j=0}^n \frac{1}{j+1} \| s_j f \| \leq C \| f \| \quad (f \in H^1, n \geq 0). \tag{2} \]

Moreover, we prove the following characterization of the space \( H^1 \).

**Theorem 1.** For a function \( f \) analytic in \( D \) the following assertions are equivalent.

(i) \( f \) belongs to \( H^1 \);
(ii) \( \sup_n \frac{1}{A_n} \sum_{j=0}^n \frac{1}{j+1} \| s_j f \| < \infty \);
(iii) \( \sup_n \| P_n f \| < \infty \).

Remark. It follows from the proof that the quantities occurring in (ii) and (iii) are “proportional” to the original norm in \( H^1 \); in particular there holds (2).

Before proving the theorem we give some immediate consequences and also consider the analogous facts in the Lebesgue space \( L^1 = L^1 (\partial D) \).

**Theorem 2.** If \( f \in H^1 \), then
\[ \lim_n \frac{1}{A_n} \sum_{j=0}^n \frac{1}{j+1} \| f - s_j f \| = 0 \tag{3} \]
and, consequently,
\[ \lim_n \frac{1}{A_n} \sum_{j=0}^n \frac{1}{j+1} \| s_j f \| = \| f \|. \tag{4} \]

**Proof.** It is easy to verify that (3) holds when \( f \) is a polynomial. Then, the result is deduced in a standard way from (2) and the fact that the polynomials are dense in \( H^1 \) (cf. [1]). \( \square \)

**Corollary 1.** If \( f \in H^1 \), then
\[ \lim_{n \to \infty} \inf \| f - s_n f \| = 0. \tag{5} \]
In fact, one can prove somewhat more: for each $\varepsilon > 1$ there is a sequence 
$\{k_n\}_{n=0}^{\infty}$ of integers such that 
$\lim_n \|f - s_{k_n} f\| = 0$ and $n^\varepsilon \leq k_n \leq (n + 1)^{\varepsilon}$ for sufficiently large $n$. We omit the easy proof.

The case of $L^1$. The space $H^1$ can be realized, via the Poisson integral, as 
the subspace of $L^1 = L^1(\partial D)$ consisting of those $f \in L^1$ for which $f(j) = 0$ for $j < 0$, 
where $\hat{f}$ is the Fourier transformation of $f$. However, not one of the relations (2), 
(3), (4), (5) is valid in $L^1$, and this follows from the fact that there is a function 
$f \in L^1$ such that $\lim_n \|f - s_n f\| = \infty$; such an example is given by 
$$f(w) = \sum_j (\log j)^{-1/2} e^{jw} \quad (w = e^{it}).$$
Since the sequence $\{(\log j)^{-1/2}\}$ is convex, the function belongs to $L^1$ ([2], Theorem 4.1). Furthermore, using the standard technique, one shows that $\|f - s_n f\| \geq \text{const.} (\log n)^{1/2}$. We omit the details.

It should be noted that inequality (1) is the best possible in $L^1$ in the sense that 
$log n$ cannot be replaced by any $\psi(n)$ (independent of $f$) such that $\psi(n) = o(\log n)$ ($n \to \infty$). To see this we take $f$ to be the Poisson kernel, 
$$f(w) = \frac{l - r^2}{|w - r|^2} \quad (|w| = 1, \ 0 < r < 1),$$
then let $r$ tend to $l$ and use the norm estimate for the Dirichlet kernel.

Let $h^1$ denote the class of harmonic functions satisfying the condition $\|f\| = \sup_{r < 1} I(f, r) < \infty$. The Poisson integral provides an isometric isomorphism of $L^1$ 
into $h^1$ (cf. [1]). Using Fejér’s theorem one shows, by summation by parts, that 
if $f \in h^1$, then $\sup_n \|P_n f\| < \infty$, where $P_n$ is extended to harmonic function in 
the obvious way. Conversely, it follows from the proof of Theorem 1 that if $f$ is 
harmonic in $D$ and $\sup_n \|P_n f\| < \infty$, then $f \in h^1$.

Proof of Theorem 1. That (ii) implies (iii) is obvious. To prove that (i) 
implies (ii) let $f \in H^1$ and for fixed $n \geq 2$ and $w \in D$ define the function $g \in H^1$ by 
$$g(z) = (1 - rz)^{-1} f(rwz) \quad (|z| \leq 1),$$
where $r = 1 - 1/n$. We have $g(z) = \sum_{j=0}^{\infty} s_j f(w) r^j z^j$. Applying Hardy’s inequality 
(cf. [1]) we get 
$$\sum_{j=0}^{\infty} \frac{1}{j + 1} |s_j f(w)| r^j \leq \sum_{j=0}^{\infty} \frac{1}{j + 1} |\hat{g}(j)| \leq \pi \|g\|.$$ 
Since $r^j = (1 - 1/n)^j \geq c$ for $0 \leq j \leq n$, where $c > 0$ is an absolute constant, we have 
$$\sum_{j=0}^{n} \frac{1}{j + 1} |s_j f(w)| \leq (\pi/c) \|g\| = (1/2c) \int_0^{2\pi} |1 - re^{it}|^{-1} |f(re^{it})| dt.$$
Integrating this inequality over the circle $|w| = 1$ we find

$$
\sum_{j=0}^{n} \frac{1}{j+1} ||s_j f|| \leq (1/2c)||f|| \int_0^{2\pi} |1 - re^{it}|^{-1} dt,
$$

where we have used Fubini’s theorem. Finally, using the familiar estimate

$$
\int_0^{2\pi} |1 - re^{it}|^{-1} dt \leq C \log \frac{1}{1 - r} = C \log n,
$$

we see that (2) holds and therefore we have proved that (i) implies (ii).

Let $f$ be analytic in $D$. From the uniform convergence of $s_n f$ on compact sets it follows that $P_n f \to f$ ($n \to \infty$) uniformly on compact subsets of $D$. Assuming that $||P_n f|| \leq 1$ for each $n$ we have $I(P_n f, r) \leq 1$ for all $n$ and $r < 1$. This implies, via the uniform convergence of $P_n f$ on the circle $|z| = r$, that $I(f, r) \leq 1$ for every $r < 1$, which means that $||f|| < 1$. Thus we have proved that (iii) implies (i), and this completes the proof. □

REFERENCES


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