ON GENERALIZATION OF FUNCTIONS $n!$ AND $!n$

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Abstract. A sequence $y_n$ is defined, and its relation to Duro Kurepa's left factorial hypothesis is discussed. Also, a generalization of functions $n!, !n$ and $y_n$, a sequence $u_{n,m}$ is defined, and a number of its properties is proved.

1. Introduction. Kurepa in [6] defined $!n$ (left factorial) by:

$$!n = 0! + 1! + 2! + \cdots + (n-1)!, \quad n \in \mathbb{N}$$  \hfill (1.1)

and stated the hypothesis that

$$(!n, n!) = 2, \quad \text{for} \quad n > 1,$$  \hfill (KH)

where $(a, b)$ denotes the greatest common divisor of integers $a$ and $b$. In [6] was proved that (KH) is equivalent to assertion that

$$!p \not\equiv 0 \pmod{p}, \quad \text{for all primes} \quad p > 2$$ \hfill (1.2)

and this is the usual form of KH.

In [8], [9], [11], [12] and [13] there are several statements equivalent to KH, which are all exposed in [5]. Here we cite, for example, the assertion proved in [14], that KH is equivalent to

$$\sum_{k=1}^{p-2} (k+1)^{p-k} k^{k-1} \not\equiv 0 \pmod{p}, \quad \text{for all primes} \quad p > 2$$ \hfill (1.3)

KH is verified in [2] for $n < 10^6$. In this paper we will try to open some new possibilities for considering KH.

AMS Subject Classification (1991): Primary 11A05
2. The sequence $y_n$. Let $f(x) = \frac{e^{-x}}{1-x}$ and $n \in N \cup \{0\}$. We define a sequence $y_n$ by:

$$y_n = f^{(n)}(0).$$  

(2.1)

It is easy to see that

$$y_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k!.$$  

(2.2)

The first few members, are: $y_0 = 1$, $y_1 = 0$, $y_2 = 1$, $y_3 = 2$, $y_4 = 9$, $y_5 = 44, \ldots$. Let us notice that the sequence $y_n$ has a combinatorial meaning to. Namely, $y_n$, $n > 0$, is the number of derangements of the set of $n$ elements, i.e. number of permutations of an $n$-element set, in which no element is fixed. Let us establish some properties of the sequence $y_n$.

**Proposition 2.1.** For every $n \in N$ we have

$$y_n = ny_{n-1} + (-1)^n,$$  

(2.3)  

$$\sum_{k=0}^{n} \binom{n}{k} y_k = n!,$$  

(2.4)  

$$\sum_{k=1}^{n} \binom{n}{k} y_{k-1} = n!,$$  

(2.5)  

$$\sum_{k=1}^{n} (-1)^{n-k} k! = y_{n-1}.$$  

(2.6)

**Proof.** Since $(1 - x)f(x) = e^{-x}$, it follows that

$$(-1)^n = [(1 - x)f(x)]_{x=0}^{[n]} = f^{[n]}(0) - n f^{[n-1]}(0) = y_n - ny_{n-1},$$

so the equality (2.3) is correct. Further we have:

$$\frac{1}{1-x} = e^x f(x) \Rightarrow n! = [e^x f(x)]_{x=0}^{[n]} = \sum_{k=0}^{n} \binom{n}{k} y_k,$$

$$y_n = \sum_{k=0}^{n-1} k! = \sum_{k=0}^{n-1} \sum_{i=0}^{k} \binom{k}{i} y_i = \sum_{i=0}^{n-1} y_i \sum_{k=0}^{n-1} \binom{k}{i} = \sum_{i=0}^{n-1} \binom{n}{i+1} y_i,$$

i.e. the equalities (2.4) and (2.5) are correct. From (2.5) it follows that:

$$y_n = u_1^{(n)}(0), \quad u_1(x) = e^x \int_0^x f(t)dt, \quad (10 = 0)$$  

(2.7)

and further $u_1(x)e^{-x} = \int_0^x f(t)dt \Rightarrow f^{(n-1)}(0) = y_{n-1} = [u_1(x)e^{-x}]_{x=0}^{[n]}$, i.e. the equality (2.6) also holds.
PROPOSITION 2.2. For every $n \in N$, and every $m \in N \cup \{0\}$ the following holds:

$$\sum_{k=0}^{n} \frac{n!}{k!} y_{k+m} = n! \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \left(\frac{n+i}{i}\right) (i!) . \quad (2.8)$$

The proof is easy and can be omitted.

Let us notice, that from proven equalities, one can obtain a number of congruences by module $p$, when $p \in P$, and $P$ denotes the same set as above. From (2.3) it follows

$$y_p \equiv -1 \pmod{p}, \quad (2.9)$$
$$y_{p-1} + y_{p-2} \equiv 1 \pmod{p}, \quad (2.10)$$

Also, the substitution $n = p - 1$ in (2.4) give

$$\sum_{k=0}^{p-1} (-1)^{k} y_k \equiv -1 \pmod{p}, \quad (2.11)$$

Finally, by substitution $n = p$ in (2.6) and (2.8) we obtain

$$y_{p-1} \equiv 1 \pmod{p} \quad (2.12)$$
$$y_m + y_{m+p} \equiv 0 \pmod{p}, \quad m \in N \cup \{0\} \quad (2.13)$$

Bearing in mind (1.2), one can, without great effort, formulate a number of assertions equivalent to KH. Really, according to congruences (2.9)-(2.13), KH is equivalent to every one of the following statements:

$$y_{p-1} \not\equiv 0 \pmod{p}, \quad \text{for all primes } p \geq 3, \quad (2.14)$$
$$y_{p-2} \not\equiv 1 \pmod{p}, \quad \text{for all primes } p \geq 3, \quad (2.15)$$
$$y_p \not\equiv -1 \pmod{p^2}, \quad \text{for all primes } p \geq 3, \quad (2.16)$$
$$\sum_{k=0}^{p-2} (-1)^{k} y_k \not\equiv -1 \pmod{p}, \quad \text{for all primes } p \geq 3. \quad (2.17)$$

Obviously, one can formulate a number of similar statements.

The sequence $y_n$ can be represented in another way.

PROPOSITION 2.3. For every $n \in N$

$$y_n = \left\lfloor \frac{n!}{e} \right\rfloor + \frac{1 + (-1)^n}{2}, \quad (2.18)$$

where $[x]$ denotes integer part of $x$, i.e. $[x] \in Z$ and $[x] \leq x < [x] + 1$. 
Proof. From (2.2) it follows

\[ y_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \Gamma(k + 1) = \int_{0}^{+\infty} e^{-x} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^k dx \]

\[ = \int_{0}^{+\infty} (x - 1)^n e^{-x} dx = \int_{0}^{1} (x - 1)^n e^{-x} dx + \frac{1}{e} \int_{0}^{+\infty} t^n e^{-t} dt \]

\[ = \frac{n!}{e} + \int_{0}^{1} (x - 1)^n e^{-x} dx. \]

Since we have

\[ \left| \int_{0}^{1} (x - 1)^n e^{-x} dx \right| \leq \int_{0}^{1} |x - 1| e^{-x} dx = \frac{1}{e}. \]

It follows that \( y_n = \left\lfloor \frac{n!}{e} \right\rfloor + 1 \), for \( n \) even and \( y_n = \left\lfloor \frac{n!}{e} \right\rfloor \), for \( n \) odd and thus, the equality (2.18) holds.

Bearing in mind properties of the sequence \( y_n \) from (2.18) one immediately obtains

\[ \left\lfloor \frac{n!}{e} \right\rfloor = \left\lfloor \frac{e^{-x} - e^{x} + e^{-x}}{1 - x} \right\rfloor_{x=0}^{\left\lfloor n \right\rfloor} \]  \hspace{1cm} (2.19)

Also, according to Proposition 2.1, it follows that for every \( n \in N \), the following equalities hold:

\[ \left\lfloor \frac{n!}{e} \right\rfloor = n \left\lfloor \frac{(n-1)!}{e} \right\rfloor + \frac{1 - (-1)^n}{2} (n-1), \]  \hspace{1cm} (2.20)

\[ \sum_{k=0}^{n} \binom{n}{k} \left\lfloor \frac{k!}{e} \right\rfloor = n! - 2^{n-1}, \] \hspace{1cm} (2.21)

\[ \sum_{k=1}^{n} \binom{n}{k} \left\lfloor \frac{(k-1)!}{e} \right\rfloor = n! - 2^{n-1}. \] \hspace{1cm} (2.22)

Bearing in mind (2.9)–(2.13), it follows that for every prime \( p \geq 3 \), the following congruences hold:

\[ \left\lfloor \frac{p!}{e} \right\rfloor \equiv -1 \pmod{p}, \] \hspace{1cm} (2.23)

\[ \left\lfloor \frac{(p-1)!}{e} \right\rfloor + \left\lfloor \frac{(p-2)!}{e} \right\rfloor \equiv 0 \pmod{p}, \] \hspace{1cm} (2.24)
\[
\sum_{k=0}^{p-1} (-1)^k \left[ \frac{k!}{e} \right] \equiv \frac{p-3}{2} \pmod{p}, \tag{2.25}
\]
\[
\left[ \frac{(p-1)!}{e} \right] \equiv \left(1 - p \right) \pmod{p}, \tag{2.26}
\]
\[
\left[ \frac{m!}{e} \right] + \left[ \frac{(m + p)!}{e} \right] \equiv -1 \pmod{p}, \quad m \in \mathbb{N} \cup \{0\}. \tag{2.27}
\]

The statements (2.14)–(2.17) all equivalent to KH, could be reformulated similarly. From the discussion above, it is clear, that the sequence \(y_n\) is closely related to the functions \(n!\) and \(\ln\).

3. The sequence \(u_{n,m}\). Let \(f(x) = \frac{e^{-x}}{1 - x}\) and let \(u_m(x), m \in \mathbb{Z}\), be the sequence of functions defined by:

\[
u_m(x) = \begin{cases} 
\frac{e^x}{m!} \int_0^x \int_0^{t_1} \cdots \int_0^{t_{m-1}} f(t_m) dt_m, & m > 0 \\
\frac{e^x}{m!} f^{(-m)}(x), & m \leq 0
\end{cases}
\tag{3.1}
\]

The sequence of numbers \(u_{n,m}, n \in \mathbb{N} \cup \{0\}\), is defined by

\[u_{n,m} = u_{n,0}^{(m)}(0).\tag{3.2}\]

Let us first notice, that the sequence \(u_{n,m}\), in some special cases represents the functions \(n!\), \(\ln\) and \(y_n\).

**Proposition 3.1.** For every \(n \in \mathbb{N} \cup \{0\}\) following equalities hold:

\[
u_{n,0} = n!,
\]
\[
u_{n,1} = \ln, \quad (|0 = 0),
\]
\[
u_{0,-n} = y_n.
\tag{3.5}
\]

**Proof.** Referring to (2.1) and (3.1) and bearing in mind (2.7), it follows immediately that

\[
u_{n,0} = \left[ \frac{1}{1 - x} \right]_{x=0}^{(n)} = n!,
\]
\[
u_{n,1} = \left[ \frac{e^x}{1 - x} \int_0^x f(t) dt \right]_{x=0}^{(n)} = \ln,
\]
\[
u_{0,-n} = \left[ \frac{e^x f^{(n)}(x)}{1 - x} \right]_{x=0} = f^{(n)}(0) = y_n,
\]

which proves the assertion.
Considering further properties of the sequence $u_{n,m}$, let us show, that the sequence $u_{n,m}$ has some properties similar to those of binomial coefficients.

**Proposition 3.2.** For every $n \in N \cup \{0\}$ we have:

\[
\begin{align*}
    u_{n,m} + u_{n,m+1} &= u_{n+1,m+1}, \quad m \in Z, \quad (3.6) \\
    m > n &\Rightarrow u_{n,m} = 0, \quad m \in N, \quad (3.7) \\
    u_{n,n} &= 1, \quad (3.8) \\
    u_{n,n-1} &= n. \quad (3.9)
\end{align*}
\]

**Proof.** Referring to (3.1) we have

\[
u_m(x) = e^x (u_{m+1}(x) e^{-x})' = u_{m+1}'(x) - u_{m+1}(x) \Rightarrow
\]

\[
u_{n,m} = u_{m}^{(n)}(0) = [u_{m+1}'(x) - u_{m+1}(x)]^{(n)}_{x=0} = u_{n+1,m+1} - u_{n,m+1},
\]

and thus, the equality (3.6) holds.

Let $m > n$, then

\[
u_{m}^{(n)}(x) = e^x \sum_{k=0}^{n} \binom{n}{k} \left[ \int_{0}^{x} dt \int_{0}^{t_1} dt_2 \cdots \int_{0}^{t_{m-1}} f(t_m) dt_m \right]^{[k]}
\]

\[
= e^x \sum_{k=0}^{n} \binom{n}{k} \int_{0}^{x} dt_{k+1} \cdots \int_{0}^{t_{m-1}} f(t_m) dt_m,
\]

and after substituting $x = 0$, we obtain (3.7).

Further, from (3.6) and (3.7) it follows

\[
u_{n,n} = u_{n-1,n-1} + u_{n-1,n} = u_{n-1,n-1} = \cdots = u_{0,0} = 1
\]

\[
u_{n,n-1} = u_{n-1,n-2} + u_{n-1,n-1} = u_{n-1,n-2} + 1 = \cdots = u_{1,0} + (n-1) = n
\]

and thus, the assertion is proved.

We can easily calculate members of the sequence $u_{n,m}$. For example, for $|m| < 5$ and $n < 6$, we have:

\[
\begin{array}{cccccccccccc}
\cdot \cdot \cdot & 4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdot \\
0 & 9 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \\
1 & 53 & 11 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & \\
2 & 362 & 64 & 14 & 4 & 2 & 2 & 1 & 0 & 0 & \\
3 & 2790 & 426 & 78 & 18 & 6 & 4 & 3 & 1 & 0 & \\
4 & 24024 & 3216 & 504 & 96 & 24 & 10 & 7 & 4 & 1 & \\
5 & 229080 & 27240 & 3720 & 600 & 120 & 34 & 17 & 11 & 5 & \\
\end{array}
\]
PROPOSITION 3.3. For every \( n \in N \), and every \( m \in Z \), the following equalities hold:

\[
\sum_{k=m}^{n} (-1)^{k-m} u_{n,k} = u_{n-1,m-1},
\]
(3.10)

\[
\sum_{k=0}^{n-1} u_{k,m} = u_{n,m+1} - u_{0,m+1}.
\]
(3.11)

Proof. Referring to (3.6) and (3.7) we obtain

\[
\sum_{k=m}^{n} (-1)^{k-m} u_{n,k} = \sum_{k=m}^{n} (-1)^{k-m} (u_{n-1,k-1} - u_{n-1,k})
\]

\[
= u_{n-1,m-1} + \sum_{k=m+1}^{n} (-1)^{k-m} u_{n-1,k-1} + (-1)^{n-m} u_{n-1,n} + \sum_{k=m}^{n-1} (-1)^{k-m} u_{n-1,k}
\]

\[
= u_{n-1,m-1} + \sum_{k=m}^{n-1} (-1)^{k-m+1} u_{n-1,k} + \sum_{k=m}^{n-1} (-1)^{k-m} u_{n-1,k} = u_{n-1,m-1}.
\]

Also

\[
\sum_{k=0}^{n-1} u_{k,m} = \sum_{k=0}^{n-1} (u_{k+1,m+1} - u_{k,m+1}) = \sum_{k=0}^{n-1} u_{k,m+1} - \sum_{k=0}^{n-1} u_{k,m+1} = u_{n,m+1} - u_{0,m+1}.
\]

and that proves the proposition.

PROPOSITION 3.4. For every \( k, n \in N \cup \{0\} \) and every \( m \in Z \), the following equalities hold:

\[
\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} u_{k+i,m} = u_{k,m-n},
\]
(3.12)

\[
\sum_{i=0}^{n} \binom{n}{i} u_{k,m-i} = u_{n+k,m}.
\]
(3.13)

Proof. According to (3.1), we have

\[
u_{m-n}(x) = e^x (u_m(x) e^{-x})^{(n)} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} u_{m-i}^{(i)}(x).
\]

Differentiating the last equality \( k \) times and substituting \( x = 0 \) we obtain (3.12).

Further we have

\[
u_{m}^{(n)}(x) = \sum_{i=0}^{n} \binom{n}{i} u_{m-i}(x) \Rightarrow u_{m+i}^{(n+k)}(x) = \sum_{i=0}^{n} \binom{n}{i} u_{m-i}^{(k)}(x),
\]

and by substituting \( x = 0 \) we obtain (3.13).
Let us notice, that by substituting \( k = 0 \) in (3.13) and considering (3.5) and (3.7) we obtain

\[
u_{n,m} = \sum_{i=0}^{n} \binom{n}{i} u_{0,m-i} = \sum_{i=0}^{n} \binom{n}{i} y_{i-m} \quad (m \leq 0), \quad \text{i.e.}
\]

\[
u_{n,m} = \sum_{i=0}^{m-1} \binom{n}{i} u_{0,m-i} + \sum_{i=m}^{n} \binom{n}{i} u_{0,m-i} = \sum_{i=m}^{n} \binom{n}{i} y_{i-m}, \quad (0 < m \leq n).
\]

We can conclude that, for every \( n \in \mathbb{N} \cup \{0\} \) and every \( m \in \mathbb{Z} \), \( m \leq n \) holds

\[
u_{n,m} = \sum_{i=s}^{n} \binom{n}{i} y_{i-m}, \quad s = \max(0,m).
\] (3.14)

Referring to (2.8), substitution \( m = -k \), \( k \in \mathbb{N} \cup \{0\} \) in (3.14), gives

\[
u_{n,-k} = \sum_{i=0}^{n} \binom{n}{i} y_{i+k} = n! \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \binom{n+i}{i} i!.
\] (3.15)

Especially, for \( k = 1 \) we obtain

\[
u_{n,-1} = n \cdot n!.
\] (3.16)

It is obvious that for \( m \leq 0 \)

\[
u_{n,m} \equiv 0 \pmod{n!}.
\] (3.17)

Substitution \( n = p \), \( p \in \mathbb{P} \) in (3.13), gives us

\[
u_{k,m} + \nu_{k,m-p} \equiv \nu_{p+k,m} \pmod{p}.
\] (3.18)

According to (3.17) and (3.18), for \( m \leq p \leq k \), we obtain

\[
u_{k,m} \equiv \nu_{p+k,m} \pmod{p}.
\] (3.19)

Let us notice, that for \( n = p + 1 \), \( p \in \mathbb{P}, \ p > 2 \), in (3.14), referring to (2.10) we obtain:

\[
u_{p+1,2} \equiv 1 \pmod{p}.
\] (3.20)

At last, let us prove two recurrent formulas for the sequence \( \nu_{n,m} \).

**Proposition 3.5.** For every \( n \in \mathbb{N} \) and every \( m \in \mathbb{Z} \) following equality holds

\[(m-1)\nu_{n,m} = (n-m+1)\nu_{n,m-1} - \nu_{n,m-2} + a(n,m), \] (3.21)
where

\[ a(n, m) = \begin{cases} \binom{n}{m-2}, & m \geq 2 \\ 0, & m < 2 \end{cases} \]

Proof. For \( m = 1 \), the equality holds, since (3.21) reduces to (3.16). For \( m < 1 \), according to (2.3), (3.6) and (3.14), we obtain

\[
u_{m-1} + u_{m, m-2} = u_{n+1, m-1} = \sum_{i=0}^{n+1} \binom{n + 1}{i} y_{i-1}
\]

\[
= \sum_{i=0}^{n+1} \binom{n + 1}{i} (i - m + 1) y_{i-1} + (-1)^{1-m} \sum_{i=0}^{n+1} \binom{n + 1}{i} (-1)^i
\]

\[
= (n + 1) u_{n, m-1} - (m - 1)(u_{n,m} + u_{n,m-1}).
\]

and, after putting this in order, we obtain (3.21).

For \( m > 1 \), we have

\[
u_{n+1, m-1} = \sum_{i=0}^{n+1} \binom{n + 1}{i} y_{i-1}
\]

\[
= \binom{n+1}{m-1} + \sum_{i=m}^{n+1} \binom{n+1}{i} y_{i-1} + (-1)^{1-m} \sum_{i=m}^{n+1} \binom{n+1}{i} (-1)^i
\]

\[
= \binom{n+1}{m-1} + (n+1) \sum_{i=m}^{n+1} \binom{n}{i-1} y_{i-1} - (m - 1)(n+1) \sum_{i=m}^{n+1} \binom{n+1}{i} y_{i-1}
\]

\[
+ (-1)^m \sum_{i=0}^{m-1} \binom{n+1}{i} (-1)^i
\]

\[
= \binom{n+1}{m-1} + (n+1) \sum_{i=m}^{n} \binom{n}{i} y_{i-1} - (m - 1) u_{n+1, m} - \binom{n}{m-1}
\]

\[
= \binom{n}{m-2} + (n+1) u_{n, m-1} - (m - 1)(u_{n,m} + u_{n,m-1}),
\]

and, after putting this in order, we obtain (3.21).

Proposition 3.6. For every \( n \in N \cup \{0\} \) and every \( m \in Z \), the following equality holds:

\[
u_{n+2, m} = (n - m + 3) u_{n+1, m} - (n + 1) u_{n,m} + a(n, m).\] (3.22)
Proof. According to (3.6), (3.12) and (3.21), we obtain
\[ u_{n+2,m} - 2u_{n+1,m} + u_{n,m} = u_{n,m-2} \]
\[ = (n - m + 1)u_{n,m-1} - (m - 1)u_{n,m} + a(n,m) \]
\[ = (n-m+1)(u_{n+1,m}-u_{n,m}) - (m-1)u_{n,m} + a(n,m), \]
and, after putting this in order, we get (3.22).

Notice that, according to (3.4) and (3.6), after substituting \( m = 2 \) in (3.22), we obtain
\[ u_{n+2,2} = (n + 1)(\ln) + 1. \]  
(3.23)

From properties of the sequence \( u_{n,m} \), it is clear that one can state number of assertions equivalent to KH. For example KH is equivalent to all following assertions:

\[ (\exists k)(k \geq p \land u_{k,2} \neq 1 \pmod{p}), \quad \text{for all primes } p > 2, \]  
(3.24)

\[ u_{p-1,2} \neq 0 \pmod{p}, \quad \text{for all primes } p > 2, \]  
(3.25)

\[ u_{p-2,2} \neq 0 \pmod{p}, \quad \text{for all primes } p > 2, \]  
(3.26)

\[ u_{p+1,2} \neq p + 1 \pmod{p^2}, \quad \text{for all primes } p > 2. \]  
(3.27)

Really, (3.24) and (3.27) are direct corollaries of (1.2) and (3.23), while (3.6) implies (3.25) and (3.26).

Bibliography


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(Received 23 05 1996)