WEAK CYLINDRIC PROBABILITY ALGEBRAS

M. Rašković, R.S. Đorđević and M. Bradić

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Abstract. We prove an analog of the Boolean representation theorem for locally finite-dimensional weak cylindric probability algebras. These algebras are designed to provide an apparatus for an algebraic study of the weak probability logic $L_{APV}$.

The notion of a weak cylindric probability algebra will be introduced as a common algebraic abstraction from the theory of deductive systems of the weak probability logic $L_{APV}$, and the geometry associated with basic set-theoretic notions.

The logic $L_{APV}$ is the minimal extension of the infinitary logic $L_A$ (see [3]) and the probability logic $L_{AP}$ (see [4]), where $A$ is a countable admissible set such that $\omega \in A$. Let $L$ be a countable $\mathcal{A}$-recursive set of finitary relation, function and constant symbols. The set Form$_L$ of all formulas of $L_{APV}$ is closed under countable disjunctions ($\lor$) and conjunctions ($\land$), negation ($\neg$), usual quantifiers ($\forall, \exists$) and probability quantifiers ($P\psi \geq r$), where $\psi$ is a finite tuple of distinct variables and $r \in A \cap [0,1]$. This set contains as distinguished elements the expressions: false ($F$), true ($T$) and $v_p = v_q$ for any $p, q < \omega$. The structure

$$\text{Form}_L = \langle \text{Form}_L, \lor, \land, \neg, F, T, \exists v, P\psi \geq r, v_p = v_q \rangle$$

is the free algebra of formulas of $L_{APV}$.

Axioms and rules of inference for $L_{APV}$ are those for $L_A$ and the weak $L_{AP}$, as listed in [3] and [4], together with the following axioms (see [6]):

(APV$_1$) $(\forall x)\varphi \rightarrow (P x \geq 1)\varphi$;

(APV$_2$) $(P x_1 \ldots x_n \geq r)\varphi \rightarrow (P x_1 \ldots x_n \geq r)\varphi$.

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where $\pi$ is a permutation of $\{1, \ldots, n\}$.

Let $\Sigma$ be any set of sentences of $L_{AP\Sigma}$. The notion of a deduction of a formula $\varphi$ from $\Sigma$ (denoted by $\Sigma \vdash \varphi$) is defined in the usual way. Let $\equiv_{\Sigma}$ be a relation on $\text{Form}_L$ defined by

$$\varphi \equiv_{\Sigma} \psi \iff \Sigma \vdash \varphi \leftrightarrow \psi.$$  

If $\Sigma \vdash \varphi \leftrightarrow \psi$, then $\Sigma \vdash (\exists x)\varphi \leftrightarrow (\exists x)\psi$ and $\Sigma \vdash (P x \geq r)\varphi \leftrightarrow (P x \geq r)\psi$. Hence the relation $\equiv_{\Sigma}$ is a congruence relation on $\text{Form}_L$. Let $\varphi^\Sigma$ be a set of all formulas $\equiv_{\Sigma}$-equivalent to $\varphi$, and let $\text{Form}_L / \equiv_{\Sigma}$ be a set of all equivalence classes $\varphi^\Sigma$, $\varphi \in \text{Form}_L$. Now, we construct the quotient algebra

$$\mathfrak{F}_{\text{Form}_L / \equiv_{\Sigma}} = \langle \text{Form}_L / \equiv_{\Sigma} \cup \{ \land, \land^\Sigma, \neg^\Sigma, F^\Sigma, T^\Sigma, (\exists x)^\Sigma, (P x \geq r)^\Sigma, (v_p=v_q)^\Sigma \rangle,$$

which will be called a weak cylindric probability algebra of formulas.

Let $\mathfrak{A} = \langle A, R_1^\mathfrak{a}, R_2^\mathfrak{a}, c_1^\mathfrak{a}, \ldots, c_n^\mathfrak{a}, \mu_n \rangle_{n < \omega}$ be a weak probability structure for $L_{AP\Sigma}$; i.e., $\langle A, R_1^\mathfrak{a}, R_2^\mathfrak{a}, c_1^\mathfrak{a}, \ldots, c_n^\mathfrak{a}, \mu_n \rangle$ is a classical first-order structure and $\mu_n$'s are finitely additive probability measures defined on the set of all definable subsets of $A^n$. By using the natural definition of the satisfaction relation, we obtain the collection $\mathbb{A}$ of all sets of the form $\varphi^\mathfrak{a} = \{ a \in A^\omega : \mathfrak{A} \models \varphi[a] \}$, $\varphi \in \text{Form}_L$. Then

$$\begin{align*}
((\exists v_i)\varphi)^\mathfrak{a} &= \{ a \in A^\omega : a \upharpoonright \omega \setminus \{ i \} = b \upharpoonright \omega \setminus \{ i \} \text{ for some } b \in \varphi^\mathfrak{a} \}, \\
((P v \geq r)\varphi)^\mathfrak{a} &= \{ a \in A^\omega : \mu_n\{ (b_{k_1}, \ldots, b_{k_n}) : b \in \varphi^\mathfrak{a}, (j \notin K \rightarrow b_j = a_j) \} \geq r \},
\end{align*}$$

where $v = v_{k_1}, \ldots, v_{k_n}$ and $K = \{ k_1, \ldots, k_n \}$. Thus we get a weak cylindric probability set algebra. As usual, a unary cylindric set operation $C_i$ is defined on the subsets of $A^\omega$ by setting, for any $X \subseteq A^\omega$,

$$C_i(X) = \{ y \in A^\omega : y \upharpoonright \omega \setminus \{ i \} = x \upharpoonright \omega \setminus \{ i \} \text{ for some } x \in X \}.$$ 

Let $\langle K \rangle$ be a tuple of distinct integers corresponding to a finite subset $\{ k_1, \ldots, k_n \}$ of $\omega$. For each $\langle K \rangle$ and $r \in [0,1]$, we introduce a unary cylindric probability set operation $C_i^\langle K \rangle$ on the subsets of $A^\omega$ by setting, for any $X \subseteq A^\omega$,

$$C_i^\langle K \rangle(X) = \{ y \in A^\omega : \mu_n\{ (x_{k_1}, \ldots, x_{k_n}) : x \in X \& (j \notin K \rightarrow x_j = y_j) \} \geq r \}.$$ 

By means of $C_i^\langle K \rangle$, we obtain a cylinder generated by translating only the section of $X$ whose measure is not less than $r$ parallelly to the $(k_1, \ldots, k_n)$-axis of $A^\omega$. If $K$ is a singleton $\{ k \}$, then we write $C_i^k$ instead of $C_i^\langle (k) \rangle$. It follows from $C_i(\varphi^\mathfrak{a}) = ((\exists v_i)\varphi)^\mathfrak{a}$ and $C_i^\langle K \rangle(\varphi^\mathfrak{a}) = ((P v \geq r)\varphi)^\mathfrak{a}$ that the function $f : \text{Form}_L / \equiv_{\Sigma} \rightarrow \mathfrak{A}$ defined by $f(\varphi^\Sigma) = \varphi^\mathfrak{a}$ is a "natural" homomorphic transformation from the weak cylindric probability algebra of formulas $\mathfrak{F}_{\text{Form}_L / \equiv_{\Sigma}}$ onto the weak cylindric probability set algebra

$$\langle \mathbb{A}, \cup, \cap, \sim, 0, A^\omega, C_i, C_i^\langle K \rangle, D_{pq} \rangle.$$
where \( D_{pq} = \{ a \in A^n : a_p = a_q \} \) and, so, \( D_{pq} = (v_p = v_q)^\mathbb{N} \).

The abstract notion of a weak cylindric probability algebra is defined by equations which hold in both algebras mentioned above. We suppose in advance that a fixed indexation by hereditarily countable sets (from \( A \subseteq HC \)) is given. So let \( A = \{ x_i : i \in I \} \) and \( I \subseteq A \). We say that a Boolean algebra \( \langle A, +, \cdot, \neg, 0, 1 \rangle \) is \( A \)-complete if for any \( \{ x_j : j \in J \} \subseteq A \), where \( J \subseteq I \) and \( J \in A \), we have \( \sum_{j \in J} x_j \in A \).

Definition 1. A weak cylindric probability algebra is a structure
\[
A = \langle A, +, \cdot, \neg, 0, 1, C_i, C^r_{(K)}, d_{pq} \rangle,
\]
such that \( \langle A, +, \cdot, \neg, 0, 1 \rangle \) is an \( A \)-complete Boolean algebra, \( C_i \) and \( C^r_{(K)} \) are unary operations on \( A \) for each \( i < \omega \) and each finite \( K \subseteq \omega \), \( d_{pq} \in A \) for all \( p, q < \omega \), and the following postulates hold (by convention, let \( C^r_{(K)}x = C^1_{(K)}x \) for \( r \geq 1 \), and \( C^r_{(K)}x = C^0_{(K)}x \) for \( r \leq 0 \)).

(WCP_0) \( \langle A, +, \cdot, \neg, 0, 1, C_i, d_{pq} \rangle \) is a cylindric algebra of dimension \( \omega \).
(WCP_1) (i) \( C^r_{(K)}x = x \), (ii) \( C^r_{(K)}0 = 0 \), where \( r > 0 \).
(WCP_2) \( C^0_{(K)}x = 1 \).
(WCP_3) If \( r \geq s \), then \( C^r_{(K)}x \leq C^s_{(K)}x \).
(WCP_4) \( C^r_{(K)}(x + C^s_{(L)}y) = C^r_{(K)}x + C^s_{(L)}y \), where \( K \subseteq L \).
(WCP_5) (i) \( C^r_{(K)}x \cdot C^s_{(K)}y \leq C^r+s-1_{(K)}(x \cdot y) \),
(ii) \( C^r_{(K)}x \cdot C^s_{(K)}y \cdot C^1_{(K)} - (x \cdot y) \leq C^{r+s}_{(K)}(x + y) \).
(WCP_6) \( C^r_{(K)} - x = -\sum_{m > 0} C^1_{(K)}x \).
(WCP_7) \( C^r_{(K)}x \leq C^r_{(\pi(K))}x \), where \( \pi \) is a permutation of \( \{1, \ldots, n\} \) and \( (\pi(K)) \) is \( k_\pi 1, \ldots, k_\pi n \).
(WCP_8) \( -C^1_k - x \leq C^1_k x \).
(WCP_9) If \( i \in K \), then: (i) \( C_i C^r_{(K)}x = C^r_{(K)}x \), (ii) \( C^r_{(K)} C_i x = C^r_{(K \setminus \{i\})} C_i x \).
(WCP_10) If \( i, j \notin K \), then: (i) \( C^r_{(K \cup \{i\})} C_i (d_{ij} \cdot x) = C_i C^r_{(K \cup \{j\})} (d_{ij} \cdot x) \),
(ii) \( C^r_{(K \cup \{j\})} C_j (d_{ij} \cdot x) = C^r_{(K \cup \{j\})} C_i (d_{ij} \cdot x) \).

We point out that the axioms WCP_2–WCP_6 express a well-known properties of finitely additive measures. The axioms WCP_7 and WCP_8 express the conditions (AP^T_2) and (AP^T_1) of \( L \), respectively.

Now we give some properties of the operations \( C^r_{(K)} \). The necessary properties of \( C_i \) and the substitution operation \( S^r_j \) defined by \( S^r_j x = \begin{cases} x, & \text{if } i = j \\ C_i (d_{ij} \cdot x), & \text{if } i \neq j \end{cases} \), are well-known (see [2] and [5]).

**Theorem 1.** If \( \langle A, +, \cdot, \neg, 0, 1, C_i, C^r_{(K)}, d_{pq} \rangle \) is a weak cylindric probability algebra, then:

1. \( C^r_{(K)}1 = 1 \).
(2) If $r > 0$ and $s > 0$, then $C^r_{(K)} x = x$ iff $C^s_{(K)} x = -x$.
(3) If $r > 0$ or $r = s = 0$ and $K \subseteq L$, then $C^r_{(K)} (x \cdot C^s_{(L)} y) = C^r_{(K)} x \cdot C^s_{(L)} y$.
(4) $C^r_{(K)} x \cdot -C^r_{(K)} y \leq \sum_{m > 0} C^1_{(K)} (x \cdot -y)$.
(5) If $x \leq y$, then $C^r_{(K)} x \leq C^r_{(K)} y$.
(6) $C^r_{(K)} x + C^r_{(K)} y \leq C^r_{(K)} (x + y)$.
(7) $C^r_{(K)} (x \cdot y) \leq C^r_{(K)} x \cdot C^r_{(K)} y$.
(8) $C^1_{(K)} x \cdot C^1_{(K)} y = C^1_{(K)} (x \cdot y)$.
(9) $C^1_{(K)} x = x$ iff $C^1_{(K)} x = x$.
(10) If $K = \{k_1, \ldots, k_n\}$ and $r > 0$, then $C^r_{(K)} x \leq C_{k_1} \cdots C_{k_n} x$.
(11) $C^1_{(K)} d_{pq} = d_{pq}$, where $p, q \notin K$.
(12) If $i \in K$, then: (a) $S_j^i C^r_{(K)} x = C^r_{(K)} x$, (b) $S_j^i S^m_i C^r_{(K)} x = S^m_i C^r_{(K)} x$.
(13) If $i, j \notin K$, then: (a) $S_j^i C^r_{(K)} x = C^r_{(K)} S_j^i x$,
(b) $C^r_{(K \cup \{i\})} S_j^i x = C^r_{(K \cup \{j\})} S_j^i x$.

**Proof.** (1) It follows from WCP$_1$ (ii) and WCP$_6$ that

$$C^1_{(K)} 1 = \sum_{m > 0} C^{1-r+1/m}_r 0 = 1.$$ 

(2) If $C^r_{(K)} x = x$, then

$$C^r_{(K)} x - x = - \sum_{m > 0} C^{1-r+1/m}_r x \quad \text{by WCP}_6$$

$$= - \sum_{m > 0} C^{1-r+1/m}_r x \quad \text{by assumption}$$

$$= - \sum_{m > 0} C^r_{(K)} x \quad \text{by WCP}_4 \text{ (putting } x = 0) \text{ and WCP}_1$$

$$= -x \quad \text{by WCP}_6.$$ 

The converse follows by symmetry.

(3) It follows from (2), WCP$_4$ and WCP$_6$ that, for $r > 0$, we have:

$$C^r_{(K)} (x \cdot C^s_{(L)} y) = C^r_{(K)} x \cdot (-x + -C^s_{(L)} y)$$

$$= - \sum_{m > 0} C^{1-r+1/m}_r (x \cdot -C^s_{(L)} y)$$

$$= - \left( \left( \sum_{m > 0} C^{1-r+1/m}_r -x \right) \cdot -C^s_{(L)} y \right)$$

$$= \left( \sum_{m > 0} C^{1-r+1/m}_r -x \right) \cdot C^s_{(L)} y$$

$$= C^r_{(K)} x \cdot C^s_{(L)} y.$$ 

(4) We have:

$$C^r_{(K)} x \cdot -C^r_{(K)} y = C^r_{(K)} x \cdot \sum_{m > 0} C^{1-r+1/m}_r y \quad \text{by WCP}_6$$

$$= \sum_{m > 0} C^r_{(K)} x \cdot C^{1-r+1/m}_r y \quad \text{by WCP}_6$$

$$\leq \sum_{m > 0} C^{1/m}_r (x \cdot -y) \quad \text{by WCP}_5 \text{ (i).}$$
(5) If $x \leq y$, then $x \cdot -y = 0$. So, $C^r_{(K)} x \cdot -C^r_{(K)} y = 0$ from (4) and WCP$_1$; i.e., $C^r_{(K)} x \leq C^r_{(K)} y$.

(6) Immediate by (5) and $x \leq y$, $y \leq x + y$, $x \cdot y \leq x$, $x \cdot y \leq y$.

(8) By WCP$_5$ (i) we have $C^1_{(K)} x \cdot C^1_{(K)} y \leq C^1_{(K)} (x \cdot y)$. The reverse inequality is an instance of (7).

(9) If $C^1_k x = z$, then $C_k x = C_k C^1_k x = C^1_k x = x$ from WCP$_9$ (i). It follows from WCP$_3$, WCP$_6$ and WCP$_8$ that $C^1_k x \leq \sum_{m>0} C^1_k x = -C^1_k x \leq C_k x$. Hence, if $C_k x = x$, then $C^1_k x \leq x$ and $x = -C_k x \leq C_k x$ by WCP$_0$ and WCP$_8$; i.e., $x = C^1_k x$. Now, by induction, it follows from WCP$_9$ that $C^1_{(K)} x = x$ if and only if $C_{(K)} x = x$.

(10) First, we prove $-C^1_{(K)} x \leq C_{k_1} \ldots C_{k_n} x$ by induction on $|K|$. The inequality is clear if $K = \emptyset$. Suppose that $K = \{k_1, \ldots, k_{n+1}\}$. Now $x \leq C_{k_{n+1}} x$, so $-C_{k_{n+1}} x \leq -x$, and hence $C^1_{(K)} x \leq C_{k_{n+1}} x \leq C^1_{(K)} x$, and so

$$-C^1_{(K)} x = -C^1_{(K)} x = \sum_{m>0} C^1_{(K)} x$$

Finally, choose $p > 0$ so that $1/p < r$. Then

$$C^r_{(K)} x \leq C^1_{(K)} x \leq \sum_{m>0} C^1_{(K)} x = -C^1_{(K)} x \leq C_{k_1} \ldots C_{k_n} x.$$

(11) Immediate by (9) and $C_{(K)} d_{pq} = d_{pq}$, where $p, q \notin K$.

(12) Assuming $i \neq j$ and $i \in K$, we have:

$$S^i_j C^r_{(K)} x = C_i (d_{ij} \cdot C^r_{(K)} x)$$

and $S^i_j C^m_{(K)} x = S^i_j S^m_j C^r_{(K)} x = S^m_j S^i_j C^r_{(K)} x = S^m_j C^r_{(K)} x$ by WCP$_0$.

(13) Assuming $i \neq j$ and $i, j \notin K$, we have:

$$S^i_j C^r_{(K)} x = C_i (d_{ij} \cdot C^r_{(K)} x)$$

and

$$C^r_{(K \cup \{i\})} S^i_j x = C^r_{(K \cup \{i\})} C_j (d_{ij} \cdot x) = C^r_{(K \cup \{j\})} C_i (d_{ij} \cdot x) = C^r_{(K \cup \{j\})} S^i_j x.$$
The algebraic notion of an ideal in a weak cylindric probability algebra can be modified using specific properties of these algebras.

**Definition 2.** An ideal in a cylindric probability algebra \( A \) is a nonempty set \( \mathcal{I} \subseteq A \) such that the following conditions hold:

1. \( \mathcal{I} \) is a Boolean ideal of \( A \); i.e.,
   - (a) \( 0 \in \mathcal{I} \),
   - (b) If \( \{a_j : j \in J\} \subseteq \mathcal{I} \) and \( J \in A \), then \( \sum_{j \in J} a_j \in \mathcal{I} \),
   - (c) If \( x \in \mathcal{I} \) and \( y \leq x \), then \( y \in \mathcal{I} \);
2. For all \( i < \omega \), if \( x \in \mathcal{I} \), then \( C_i x \in \mathcal{I} \).

It follows from Definition 2 and (10) of Theorem 1 that, for any finite \( K \subseteq \omega \) and \( r \in (0, 1] \), if \( x \in \mathcal{I} \), then \( C^r_{\mathcal{K}} x \in \mathcal{I} \). An ideal \( \mathcal{I} \) determines the relation \( \sim = \{(x, y) : x \cdot -y + y \cdot -x \in \mathcal{I}\} \). As usual, if \( x \sim y \), then \( C_i x \sim C_i y \). For \( r > 0 \) and \( x, y \in A \), we have

\[
C^r_{\mathcal{K}} x \cdot -C^r_{\mathcal{K}} y + C^r_{\mathcal{K}} y \cdot -C^r_{\mathcal{K}} x \leq \sum_{m \geq 0} C^{1/m}_{\mathcal{K}} (x \cdot -y) + \sum_{m \geq 0} C^{1/m}_{\mathcal{K}} (y \cdot -x)
\]

by (4) of Theorem 1. So, if \( x \sim y \), then \( C^r_{\mathcal{K}} x \sim C^r_{\mathcal{K}} y \). Hence, \( \sim \) is a congruence relation of \( A \). We define a new algebra \( A/\mathcal{I} = \langle A/\mathcal{I}, +, \cdot, \leq, \wedge, \vee, 0, \hat{1}, \hat{\alpha}, C^0_{\mathcal{K}}, \hat{d}_{pq} \rangle \) as usual. It is not difficult to see that \( A/\mathcal{I} \) is a weak cylindric probability algebra, and that there is a "natural" homomorphism from \( A \) onto \( A/\mathcal{I} \).

The dimension set \( \Delta x \) of an element \( x \in A \) is introduced by \( \Delta x = \{k : C_k x \neq x\} \). It follows from the clause (9) of Theorem 1 that \( \Delta x = \{k : C^1_k x \neq x\} \), i.e., the coordinates in which \( x \) is not a cylinder can be obtained also by applying probability cylindrications of the form \( C^1_k \).

**Definition 3.** A weak cylindric probability algebra \( A \) is locally finite-dimensional if \( \Delta x \) is finite for all \( x \in A \).

Every formula \( \varphi \) of \( L_{APV} \) has only finitely many free variables. If \( v_i \) is a variable not occurring in \( \varphi \), then \( \models (\exists v_i) \varphi \iff \varphi \) and \( \models (Pv_i > 0) \varphi \iff \varphi \). So, for any given set \( \Sigma \) of sentences of \( L_{APV} \), there are at most finitely many indices \( i < \omega \) such that \( \varphi \) is not equivalent under \( \Sigma \) neither to \( (\exists v_i) \varphi \) nor to \( (Pv_i > 0) \varphi \); hence, \( \forall \text{form}_v /_{\Xi} \) is locally finite-dimensional weak cylindric probability algebra.

The following theorem gives some elementary properties of \( \Delta \).

**Theorem 2.** If \( \langle A, +, \cdot, -, 0, 1, C, C^r_{\mathcal{K}}, d_{pq} \rangle \) is a weak cylindric probability algebra, then:

1. \( \Delta 0 = \Delta 1 = \emptyset \);
2. \( \Delta (\sum_{j \in J} x_j) \subseteq \bigcup_{j \in J} \Delta x_j, \ J \in A; \)
3. \( \Delta (\prod_{j \in J} x_j) \subseteq \bigcup_{j \in J} \Delta x_j, \ J \in A; \)
4. \( \Delta -x = \Delta x; \)
5. \( \Delta d_{pq} = \{p, q\}; \)
6. \( \Delta C_i x \subseteq \Delta x \setminus \{i\}; \)
7. \( \Delta S^i_j x \subseteq (\Delta x \setminus \{i\}) \cup \{j\}; \)
8. \( \Delta C^r_{\mathcal{K}} x \subseteq \Delta x \setminus K. \)
Proof. The clauses (1)-(7) are well-known properties of $\Delta$ from the classical theory of cylindric algebras.

(8) Let $i$ be any integer such that $i \notin \Delta x \setminus K$. If $i \in K$, then $C_i C_{(K)}^r x = C_{(K)}^r x$ by WCP$_\emptyset$ (i). If $i \notin \Delta x \cup K$, then

$$C_i C_{(K)}^r x = C_i C_{(K)}^r C_i x = C_i C_{(K \cup \{i\})}^r C_i x \quad \text{by WCP$_\emptyset$ (ii)}$$

$$= C_{(K \cup \{i\})}^r C_i x = C_{(K)}^r x \quad \text{by WCP$_\emptyset$ (i)}.$$

So, $i \notin \Delta C_{(K)}^r x$. $\square$

The main result of this paper is the following analog of the Boolean representation theorem from the classical theory of cylindric algebras.

**Theorem 3.** If $A$ is a locally finite-dimensional weak cylindric probability algebra and $|A| > 1$, then there is a homomorphism from $A$ onto a weak cylindric probability set algebra.

Proof. We prove that $A$ is isomorphic to a weak cylindric probability algebra of formulas $\mathfrak{Form}_L / \equiv_\Sigma$ for some $L$ and $\Sigma$.

Let $R_a$ be an $n$-ary relation symbol corresponding to $a$ for each $a \in A$, where the integer $n$ is obtained from $\Delta a \subseteq \{1, \ldots, n\}$. Fix the language $L = \{R_a : a \in A\}$. By induction on the complexity of formulas of the logic $L_{APV}$ we define a function $f: \mathfrak{Form}_L \rightarrow A$ satisfying: if $\vdash \varphi$, then $f(\varphi) = 1$ as follows:

1. Let $\varphi$ be an atomic formula $R_a(v_1, \ldots, v_n)$ and let $j_1, \ldots, j_n$ be the first $n$ integers in $\omega \setminus \{1, \ldots, n, k_1, \ldots, k_m\}$. Then

$$f(\varphi) = S_{k_1}^{j_1} \cdots S_{k_m}^{j_m} S_{j_1} \cdots S_{j_n} a;$$

2. $f(v_{i_1} = v_{i_2}) = d_{m_{i_1}}$;

3. $f(\neg \varphi) = -f(\varphi)$;

4. $f(\bigwedge \varphi) = \prod_{\varphi \in \mathfrak{A}} f(\varphi), \quad \Phi \in \mathfrak{A}$;

5. $f(\bigvee \varphi) = \bigvee_{\varphi \in \mathfrak{A}} f(\varphi), \quad \Phi \in \mathfrak{A}$;

6. $f((\exists v_1) \varphi) = C_{i_1} f(\varphi)$;

7. $f((Pv \geq r) \varphi) = C_{i_1} f(\varphi)$,

where $v = v_{k_1}, \ldots, v_{k_m}$ and $K = \{k_1, \ldots, k_m\}$.

Let $\varphi$ be a formula of $L_{APV}$ and let $\varphi^*$ be a formula obtained by the substitution of some free variables $v_{k_1}, \ldots, v_{k_n}$ of $\varphi$ with $v_{m_1}, \ldots, v_{m_n}$, respectively. By induction on complexity of formulas of $L_{APV}$, we prove the following substitution property:

(S) $f(\varphi) = S_{k_1}^{j_1} \cdots S_{k_n}^{j_n} S_{j_1} \cdots S_{j_n} f(\varphi^*)$,

where $j_1, \ldots, j_n$ are some distinct integers in $\omega \setminus \{1, \ldots, n, k_1, \ldots, k_n, m_1, \ldots, m_n\}$.

Suppose $\varphi$ is $R_a(v_{k_1}, \ldots, v_{k_n})$. Let $p_1, \ldots, p_m, q_1, \ldots, q_n$ be distinct integers in $\omega \setminus \{1, \ldots, n, k_1, \ldots, k_n, m_1, \ldots, m_n\}$. For some distinct integers $j_1, \ldots, j_n$ in the
set \( \omega \setminus \{1, \ldots, n, k_1, \ldots, k_n, m_1, \ldots, m_n\} \), we have:

\[
S_{k_1}^{j_1} \cdots S_{k_n}^{j_n} S_{j_1}^{m_1} \cdots S_{j_n}^{m_n} f(\varphi^*) = S_{k_1}^{j_1} S_{k_2}^{j_2} S_{k_3}^{j_3} \cdots S_{k_n}^{j_n} S_{j_1}^{m_1} \cdots S_{j_n}^{m_n} f_a = S_{k_1}^{j_1} S_{k_2}^{j_2} S_{k_3}^{j_3} \cdots S_{k_n}^{j_n} S_{j_1}^{m_1} \cdots S_{j_n}^{m_n} S_{q_1} \cdots S_{q_n} a
\]

by WCP\(_0\) (see [2] or [5]).

Let \( \varphi \) be \( v_{k_1} = v_{k_2} \). We may suppose \( k_1 \neq k_2 \). It follows from WCP\(_0\) that

\[
f(\varphi) = d_{k_1, k_2} = S_{k_1}^{j_1} S_{k_2}^{j_2} S_{j_1}^{m_1} S_{j_2}^{m_2} d_{m_1, m_2} = S_{k_1}^{j_1} S_{k_2}^{j_2} S_{j_1}^{m_1} S_{j_2}^{m_2} f(\varphi^*).
\]

The steps \( \neg \psi, \lor \Phi \) and \( \land \Phi \) in the inductive proof of (S) are easy using appropriate properties of \( S_j^i \) (see [2] and [5]).

Let \( \varphi \) be \((\exists v_i)(v_{k_1}, \ldots, v_{k_n}, v_i)\) and \( i \notin \{k_1, \ldots, k_n, m_1, \ldots, m_n\} \). For some distinct integers \( j_1, \ldots, j_n \) in \( \omega \setminus \{1, \ldots, n, k_1, \ldots, k_n, m_1, \ldots, m_n, i\} \) we have:

\[
f(\varphi) = C_i S_{k_1}^{j_1} \cdots S_{k_n}^{j_n} S_{j_1}^{m_1} \cdots S_{j_n}^{m_n} f(\varphi^*) \quad \text{by induction assumption}
\]

\[
= S_{k_1}^{j_1} \cdots S_{k_n}^{j_n} S_{j_1}^{m_1} \cdots S_{j_n}^{m_n} C_i f(\varphi^*) \quad \text{by WCP\(_0\)}
\]

\[
= S_{k_1}^{j_1} \cdots S_{k_n}^{j_n} S_{j_1}^{m_1} \cdots S_{j_n}^{m_n} f(\varphi^*).
\]

Suppose \( \varphi \) is \((P v_1, \ldots, v_m = r)(v_{k_1}, \ldots, v_{k_n}, v_{l_1}, \ldots, v_{l_m})\), \( L = \{l_1, \ldots, l_m\} \) and \( L \cap \{m_1, \ldots, m_n, k_1, \ldots, k_n\} = \emptyset \). For some distinct integers \( j_1, \ldots, j_n \) in \( \omega \setminus \{1, \ldots, n, k_1, \ldots, k_n, m_1, \ldots, m_n, l_1, \ldots, l_n\} \) we have:

\[
f(\varphi) = C_{r(L)} S_{k_1}^{j_1} \cdots S_{k_n}^{j_n} S_{j_1}^{m_1} \cdots S_{j_n}^{m_n} f(\varphi^*) \quad \text{by induction assumption}
\]

\[
= S_{k_1}^{j_1} \cdots S_{k_n}^{j_n} S_{j_1}^{m_1} \cdots S_{j_n}^{m_n} C_{r(L)} f(\varphi^*) \quad \text{by (13) (a) of Theorem 1}
\]

\[
= S_{k_1}^{j_1} \cdots S_{k_n}^{j_n} S_{j_1}^{m_1} \cdots S_{j_n}^{m_n} f(\varphi^*).
\]

Next, by induction on the complexity of formulas of the logic \( L_{AP^\forall} \), we prove the following dimension property:

\[
(D) \quad \text{if } v_i \text{ does not occur free in } \varphi, \text{ then } i \notin \Delta f(\varphi).
\]

We point out only the case of the probability quantification, because other cases are easy using appropriate parts of Theorem 2. So, let \( \varphi \) be the formula
\[(P \psi_{i_1}, \ldots, \psi_{i_m} \geq r) \psi(\psi_{k_1}, \ldots, \psi_{k_n}, \psi_{i_1}, \ldots, \psi_{i_m})\text{ such that } \psi_i \text{ does not occur free in } \varphi, \text{ i.e., } i \notin \{k_1, \ldots, k_n\}.\text{ Then}

\[
\Delta f(\varphi) \subseteq \Delta f(\psi) \setminus \{l_1, \ldots, l_m\} \quad \text{by (8) of Theorem 2}
\]

\[
\subseteq \{k_1, \ldots, k_n\} \quad \text{by induction hypothesis,}
\]
i.e., \(i \notin \Delta f(\varphi)\).

Now we shall prove that each logical axiom of \(L_{AP}\) is in the set

\[
\Gamma = \{\varphi \in \text{Form}_L : f(\varphi) = 1\}.
\]

\[(A) \quad \text{All axioms of } L_A \text{ (see [3]):}
\]
It follows from the classical theory of cylindric algebras that each logical axiom of \(A \cap L_{w}^w\) is in \(\Gamma\). Suppose \(\varphi \in \bigwedge \Psi \rightarrow \psi\), where \(\psi \in \Psi\). Then

\[
f(\varphi) = -\prod_{\xi \in \Psi} f(\xi) + f(\psi) \geq -f(\psi) + f(\psi) = 1.
\]

Similarly, if \(\varphi \in \neg \bigwedge \Psi \leftrightarrow \bigvee_{\psi \in \Psi} \neg \psi\), then \(f(\varphi) = 1\).

\[(AP) \quad \text{All axioms of the weak logic } L_{AP} \text{ (see [4]):}
\]
Monotonicity: Let \(\varphi \in (P \psi \geq r) \psi \rightarrow (P \psi \geq s)\psi\), where \(r \geq s\). Then for \(\psi = \psi_{k_1}, \ldots, \psi_{k_n}\) and \(K = \{k_1, \ldots, k_n\}\), we have \(f(\varphi) = 0\).

Non-negativity: If \(\varphi \in (P \psi \geq 0)\psi\), then \(f(\varphi) = 0\).

Similarly, if \(\varphi \in (P \psi \geq 0)\psi\), then \(f(\varphi) = 1\) by WCP_2.

Let \(\varphi \in (P \psi \geq 0)\psi\), where \(\theta_1 = (P \psi_{k_1}, \ldots, \psi_{k_n} \geq r) \psi_{k_1}, \ldots, \psi_{k_n}\) and \(\theta_2 = (P \psi_{i_1}, \ldots, \psi_{i_m} \geq r) \psi_{i_1}, \ldots, \psi_{i_m}\). Let \(K = \{k_1, \ldots, k_n\}\) and \(L = \{l_1, \ldots, l_m\}\). We may assume that \(L \cap K = \emptyset\). Let \(m_1, \ldots, m_n\) be distinct integers in the set \(\omega \setminus \{k_1, \ldots, k_n, l_1, \ldots, l_n\}\). For some distinct integers \(j_1, \ldots, j_n\) taken from the set \(\omega \setminus \{1, \ldots, n, k_1, \ldots, k_n, l_1, \ldots, l_n, m_1, \ldots, m_n\}\) we have:

\[
f(\theta_1) = C_{(K)}^{\tau_k} \phi_{k_1} \ldots \phi_{k_n} \phi_{j_1} \ldots \phi_{j_1} \phi_{l_1} \ldots \phi_{l_m} f(\psi^*) \quad \text{by (S)}
\]

\[
= C_{(K)}^{\tau_k} \phi_{k_1} \ldots \phi_{k_n} \phi_{j_1} \ldots \phi_{j_1} \phi_{l_1} \ldots \phi_{l_m} f(\psi^*) \quad \text{by WCP_0}
\]

\[
= C_{(L)}^{\tau_1} \phi_{l_1} \ldots \phi_{l_m} f(\psi^*) \quad \text{by (13) (b) of Theorem 1}
\]

\[
= C_{(L)}^{\tau_1} \phi_{l_1} \ldots \phi_{l_m} f(\psi^*) \quad \text{by WCP_0}
\]

\[
= f(\theta_2) \quad \text{by (S)}
\]

so, \(f(\varphi) = 1\).

Finite additivity: (i) If \(\varphi \in (P \psi \leq r) \psi \land (P \psi \leq s) \theta \rightarrow (P \psi \leq r + s) (\psi \vee \theta)\), then

\[
f(\varphi) = -(C_{K}^{\tau_k} - f(\psi) \cdot C_{K}^{\tau_k} \phi_{l_1} \ldots \phi_{l_m} f(\theta)) + C_{K}^{\tau_k} \phi_{l_1} \ldots \phi_{l_m} f(\theta)
\]

\[
\geq -(C_{K}^{\tau_k} - f(\psi) \cdot f(\theta)) + C_{K}^{\tau_k} \phi_{l_1} \ldots \phi_{l_m} f(\theta) \quad \text{by WCP_5 (i)}
\]

\[
= 1.
\]
(ii) If $\varphi$ is $(P\varphi \geq r)\psi \land (P\varphi \geq s)\theta \land (P\varphi \leq 0)(\psi \land \theta) \rightarrow (P\varphi \geq r + s)(\psi \lor \theta)$, then

\[
f(\varphi) = -\left(C_{(K)}^{r}f(\psi) \cdot C_{(K)}^{s}f(\theta) \cdot C_{(K)}^{1} - (f(\psi) \cdot f(\theta))\right) + C_{(K)}^{r+s}(f(\psi) + f(\theta))
\geq -C_{(K)}^{r+s}(f(\psi) + f(\theta)) + C_{(K)}^{r+s}(f(\psi) + f(\theta)) \quad \text{by WCP}_5 \ (ii)
\]

\[
= 1.
\]

The Archimedean property: If $\varphi$ is $(P\varphi > r)\psi \leftrightarrow \bigvee_{m>0}(P\varphi \geq r + 1/m)\psi$, then

\[
f((P\varphi > r)\psi) = -C_{(K)}^{1-r}f(\psi) = \sum_{m>0} C_{(K)}^{r+1/m}f(\psi) \quad \text{by WCP}_6
\]

so, $f(\varphi) = 1$.

(APV$_1$) Let $\varphi$ be $(\forall u_{i})\psi \rightarrow (P\psi_{i} \geq 1)\psi$. Then

\[
f(\varphi) = -C_{i} - f(\psi) + C_{i}^{1}f(\psi)
\geq C_{i} - f(\psi) + -C_{i} - f(\psi) \quad \text{by WCP}_8
\]

\[
= 1.
\]

(APV$_2$) Let $\varphi$ be $(P\psi_{k_{1}} \cdots \psi_{k_{n}} \geq r)\psi \rightarrow (P\psi_{k_{1}} \cdots \psi_{k_{n}} \geq r)\psi$. Then

\[
f(\varphi) = -C_{(K)}^{r}f(\psi) + C_{(K)}^{r}(f(\psi))
\geq -C_{(K)}^{r}(f(\psi)) + C_{(K)}^{r}(f(\psi)) \quad \text{by WCP}_7
\]

\[
= 1.
\]

Finally, we shall prove that each logical theorem of $L_{APV}$ is in $\Gamma$. Obviously $\Gamma$ is closed under Modus Ponens and under Conjunction rule. We have two Generalization rules.

If $\varphi \rightarrow \psi(u_{i}) \in \Gamma$ and $u_{i}$ is not free in $\varphi$, then

\[
f(\varphi \rightarrow (\forall u_{i})\psi) = -f(\varphi) + -C_{i} - f(\psi)
\]

\[
= -(C_{i}f(\varphi) \cdot C_{i} - f(\psi)) \quad \text{by (D)}
\]

\[
= -C_{i}(C_{i}f(\varphi) \cdot f(\psi)) \quad \text{by WCP}_0
\]

\[
= 1 \quad \text{by assumption}.
\]

So, $\varphi \rightarrow (\forall u_{i})\psi \in \Gamma$.

If $\varphi \rightarrow \psi(u_{k_1}, \ldots, u_{k_n}) \in \Gamma$ and $u_{k_1}, \ldots, u_{k_n}$ are not free in $\varphi$, then

\[
f(\varphi \rightarrow (P\psi \geq 1)\psi) = -f(\varphi) + C_{(K)}^{1}f(\psi)
\]

\[
= -C_{(K)}^{1}f(\varphi) + C_{(K)}^{1}f(\psi) \quad \text{by (D) and (11) of Theorem 1.}
\]

\[
= C_{(K)}^{1}((-f(\varphi) + f(\psi)) \quad \text{by WCP}_4 \text{ and (2) of Theorem 1.}
\]

\[
= 1 \quad \text{by assumption}.
\]
So, \( \varphi \to (Pv \geq 1)\psi \in \Gamma \).

It follows that \( \vdash \varphi \leftrightarrow \psi \) implies \( f(\varphi) = f(\psi) \). So, we introduce a well-defined function \( g: \text{Form}_L/\equiv_0 \to A \) by \( g(\varphi^0) = f(\varphi) \). It is easy to see that \( g \) is a homomorphism from \( \text{Form}_L/\equiv_0 \) onto \( A \) such that \( g(R^A(v_1, \ldots, v_n)^0) = a \). Let \( \mathcal{I} = \{ \varphi^0 : g(\varphi^0) = 0 \} \) be a subset of \( \text{Form}_L/\equiv_0 \), and let \( \Sigma \) be a set of all sentences \( \varphi \) of \( L_{\text{APV}} \) such that \( (\neg \varphi)^0 \in \mathcal{I} \). Then \( \mathcal{I} \) is an ideal in \( \text{Form}_L/\equiv_0 \) and

\[
A \cong (\text{Form}_L/\equiv_0) / \mathcal{I} \cong \text{Form}_L/\equiv_\Sigma .
\]

Moreover, \( \Sigma \) is consistent, since \( |A| > 1 \). Let \( \mathfrak{A} \) be a weak probability model of \( \Sigma \) (see [6]). Then we have a "natural" homomorphism from \( \text{Form}_L/\equiv_\Sigma \) onto the weak cylindric probability set algebra

\[
\langle \{ \varphi^A : \varphi \in \text{Form}_L \}, \cup, \cap, \sim, 0, A^w, C_1, C^r_{\langle K \rangle}, D_{\mu_A} \rangle.
\]

This completes the proof. \( \square \)

References


Prirodno-matematički fakultet
34000 Kragujevac, p.p. 50
Yugoslavia

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