ALGEBRAIC STRUCTURE COUNT OF SOME CYCLIC HEXAGONAL-SQUARE CHAINS

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Abstract. Algebraic structure count (ASC-value) of a bipartite graph \( G \) is defined by \( \text{ASC}(G) = \sqrt{\det A} \), where \( A \) is the adjacency matrix of \( G \). In the case of bipartite, plane graphs in which every face-boundary (cell) is a circuit of length \( 4s + 2 \) \( (s = 1, 2, \ldots) \), this number is equal to the number of the perfect matchings (\( K \)-value) of \( G \). However, if some of the circuits are of length \( 4s \) \( (s = 1, 2, \ldots) \), then the problem of evaluation of ASC-value becomes more complicated. In this paper the algebraic structure count of the class of cyclic hexagonal-square chains is determined. An explicit combinatorial formula for ASC is deduced in the special case when all hexagonal fragments are isomorphic.

Introduction. The algebraic structure count (ASC-value) of a bipartite graph \( G \) is defined by

\[ \text{ASC}(G) := \sqrt{\det A}, \]

where \( A \) is the adjacency matrix of \( G \). In chemistry, the thermodynamic stability of a hydrocarbon is related to the ASC-value of the graph which represents its skeleton. In recent papers \([3],[7]\) formulas for ASC for some classes of bipartite, plane graphs containing some circuits of length \( 4s \) \( (s = 1, 2, \ldots) \) are deduced.

A perfect matching (1-factor) of \( G \) is a selection of edges of \( G \) such that each vertex of \( G \) belongs to exactly one selected edge. In chemistry, perfect matchings are called the Kekulé structures of the molecule whose skeleton is represented by the graph \( G \).

By a hexagonal (unbranched) chain \( H \) we mean a finite, plane graph obtained by concatenating \( m \) \( (m \geq 1) \) circuits of length 6 which we call hexagons in such a way that any two adjacent hexagons (cells) have exactly one edge in common, each cell is adjacent to exactly two other cells, except terminal cells which are adjacent to exactly one other cell each and no one vertex belongs to more than two hexagons. Figure 1 shows one of the possible hexagonal chains consisting of 15 hexagons.

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There are several equivalent (different) explicit formulas for the number $K$ of the hexagonal chains [8–10]. For example, it is known for a long time [2] that the number of perfect matchings of the zig-zag chain of $n$ hexagons (Fig. 2a) is equal to the $(n+2)$-th Fibonacci number ($F_0 = 0, F_1 = 1; F_{k+2} = F_{k+1} + F_k,$ $k \geq 0$) and the number of perfect matchings of the linear chain of $n$ hexagons (Fig. 2b) is $n+1$.

![Fig. 1](image)

![Fig. 2](image)

The cyclic hexagonal-square chain $C_n = C_n(\mathcal{H}_1, \mathcal{H}_3, \ldots, \mathcal{H}_n)$ considered in this paper is a connected, bipartite, plane graph which consists of $n$ hexagonal unbranched chains $\mathcal{H}_1, \mathcal{H}_3, \ldots, \mathcal{H}_n$, cyclically concatenated by circuits of length 4 which we call squares (Fig. 3). Square $\alpha_i$ connects two terminal cells (hexagons) of $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ for $i = 1, 2, \ldots, n-1, n$ ($\mathcal{H}_{n+1} := \mathcal{H}_1$) in such a way that every vertex of $\alpha_i$ belongs to exactly one hexagon. Denote the edges of $\alpha_i$ belonging to $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ by $g_i$ and $f_{i+1}$ ($f_1 := f_{n+1}$) respectively and their end-vertices by $r_i, s_i$ and $p_{i+1}, q_{i+1}$ ($p_1 := p_{n+1}, q_1 := q_{n+1}$) respectively, as it is shown in Fig. 3. Note that the graph $C_n$ contains two face-boundaries which are different from squares and hexagons (the one of their regions is infinite). We call them external circuits. The vertices of $\alpha_i$ are denoted in such a way that vertices $p_i$ and $r_i$ ($i = 1, \ldots, n$) belong to the boundary of the infinite region and the vertices $s_1, q_2, s_3, \ldots, q_n, s_{n+1}, q_1$ belonging to the other external circuit are cyclically arranged in a clockwise direction.

The graphs $\mathcal{H}_i$ and $\mathcal{H}_j$ are said to be isomorphic if there is a $(1,1)$-mapping $y = \varphi(x)$ of the vertex set of the graph $\mathcal{H}_i$ onto the vertex set of the graph $\mathcal{H}_j$ such
that: (i) two vertices \( x \) and \( x' \) are adjacent in \( H_i \) iff \( \varphi(x) \) and \( \varphi(x') \) are adjacent in \( H_j \); (ii) \( \varphi(p_i) = p_j, \varphi(q_i) = q_j, \varphi(r_i) = r_j \) and \( \varphi(s_i) = s_j \).

In what follows we denote the subgraph obtained from \( G \) by deleting the edge \( e \) by \( G - e \) and the subgraph obtained from \( G \) by deleting both the edge \( e \) and its terminal vertices by \( G - (e) \).

Fig. 3

Our aim is to prove the following result:

**Theorem 1.** If all the hexagonal chains \( H_i \) \((i = 1, 2, \ldots, n)\) in the graph \( C_n \) are mutually isomorphic, then

\[
\text{ASC}\{C_n\} = \begin{cases} 
((L - D)^n + (L + D)^n)/2^n, & \text{if } n \text{ is odd;} \\
((L - D)^n + (L + D)^n)/2^n - 2, & \text{if } n \text{ is even}
\end{cases}
\]

where \( L = K_2 + K_3 + K_4, \ D = \sqrt{L^2 + 4(K_1K_4 - K_2K_3)} \) and:

\[
K_1 = K\{H_i - (f_i) - (g_i)\} \\
K_2 = K\{H_i - (f_i) - g_i\} \\
K_3 = K\{H_i - f_i - (g_i)\} \\
K_4 = K\{H_i - f_i - g_i\}.
\]
**Preliminaries.** All the graphs considered are assumed to be connected, planar, bipartite graphs whose all circuits are of even length. Define a binary relation $\rho$ in the set of all perfect matchings of $G$ in the following way.

**Definition 1.** Two perfect matchings $P_1$ and $P_2$ are $\rho$-related iff the union of the sets of edges of $P_1$ and $P_2$ forms an even number of circuits of length $4s$ ($s = 1, 2, \ldots$).

It can be proved that $\rho$ is an equivalence relation and subdivides the set of perfect matchings into two equivalence classes $[2]$. In [2] this relation is called “being of the same parity” and the numbers of elements of these classes are denoted by $K_+$ and $K_-$. We have the following theorem (Dewar and Longuet-Higgins [2]).

**Theorem 2.** For the determinant of the adjacency matrix $A$ of the graph $G$ we have $\det A = (-1)^n(K_+ - K_-)^2$.

This theorem yields the following corollary

**Corollary 1.** For the algebraic structure count of the graph $G$ there holds

$$\text{ASC}\{G\} = |K_+ - K_-|.$$ 

In the case of graphs in which every cell is a circuit of length of the form $4s + 2$ ($s = 1, 2, \ldots$) (for example molecular graphs of benzenoid hydrocarbons), all perfect matchings are in the same class $[2]$. This implies

$$\text{ASC}\{G\} = K\{G\},$$

where $K\{G\}$ is the number of perfect matchings of $G$. The enumeration of $K$-values for these graphs is well-known problem $[1]$.

If some cells are allowed to be of length $4s$ ($s = 1, 2, \ldots$) (non-benzenoid hydrocarbons) then (1) need not be true. In such a case the following theorem, which follows directly from Definition 1, can be useful for evaluating the ASC-value.

**Theorem 3.** Two perfect matchings are in distinct classes (of opposite parity) if one is obtained from the other by cyclically rearranging of an even number edges within a single circuit.

Consider the graph $C_n$. Let $m_i$ be the number of hexagons in $H_i$. Note that the number of vertices in $H_i$ and $C_n$ are equal to $4m_i + 2$ and $\sum_{i=1}^{n}(4m_i + 2) = 2n + 4\sum_{i=1}^{n}m_i$, respectively. Denote lengths of external circuits by $c_1$ and $c_2$ (the order is not important). The requirement for the graph $C_n$ to be bipartite implies that the numbers $c_1$ and $c_2$ must be even. Note that the union of the external circuits represents the spanning subgraph of $C_n$, so:

$$c_1 + c_2 = 2n + 4\sum_{i=1}^{n}m_i$$
In order to distinguish edges of $\alpha_i$ we can represent them graphically by two vertical and two horizontal lines as in Fig. 3. Consider now a perfect matching of $C_n$. Observe that the edges belonging to the perfect matching can be arranged in and around a square in seven different ways ($modes$ 1–7), as it is shown in Fig. 4 (these edges are marked by double lines).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig4.png}
\caption{Fig. 4}
\end{figure}

**Definition 2.** The arrangement word of a perfect matching of the graph $C_n$ is the word $u = u_1 u_2 \ldots u_n$ from $\{1, 2, \ldots, 6, 7\}^n$, where $u_i$ is the mode (1–7) of the arrangement of edges of the perfect matching in and around the square $\alpha_i$ for $i = 1, \ldots, n$.

For example, the arrangement words of the perfect matchings represented in Figures 5a and 5b are $u = 21322$ and $u = 77777$ respectively. (We can imagine that our position of observation of squares is inside the finite region whose boundary is the external circuit containing edges represented by lower horizontal lines and our motion (rotation) is in a clockwise direction.)

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig5.png}
\caption{Fig. 5}
\end{figure}

The modes 4 and 5 (Fig. 4) are interconverted by rearranging two (an even number) edges of the considered perfect matching. Therefore, using Theorem 3, it follows that the perfect matchings of $C_n$, with arrangement of edges in and around a square $\alpha_i$ ($1 \leq i \leq n$) of modes 4 and 5, can be divided into pairs of opposite sign. It implies, by Corollary 1, that the perfect matchings in which the mode 4 or 5 appears for any $\alpha_i$ ($1 \leq i \leq n$) can be excluded from the consideration, when the algebraic structure count is evaluated.
**Good perfect matchings.** Definition 3. The perfect matchings are called good if their arrangement words belong to \(\{1, 2, 3\}^n\).

Note that the edge of a square which belongs to one of the external circuits (horizontal lines in Fig. 4) is never in a good perfect matching. This means that every good perfect matching of the graph \(C_n\) induces in every hexagonal chain \(H_i\) \((i = 1, \ldots, n)\) a perfect matching of \(H_i\) i.e., the edges of a good perfect matching of \(C_n\) can be rearranged only within each fragment \(H_i\) \((i = 1, 2, \ldots, n)\). Hence all good perfect matchings are of equal parity.

In order to determine the value \(\text{ASC}\{C_n\}\), determine at first the number of all good perfect matchings of \(C_n\) using the so-called transfer matrix method [2]. Denote the graphs \(H_i - (f_i) - (g_i), H_i - (f_i) - g_i, H_i - f_i - (g_i)\) and \(H_i - f_i - g_i\) (Fig. 3) by \(H_{i,1}, H_{i,2}, H_{i,3}, H_{i,4}\) and their \(K\)-values by \(K_{i,1}, K_{i,2}, K_{i,3}\) and \(K_{i,4}\), respectively. Observe that \(K_{i,1}\) is the number of all perfect matchings of \(H_i\) which contain both edges \(f_i\) and \(g_i\); \(K_{i,2}\) is the number of all perfect matchings of \(H_i\) which contain \(f_i\) and do not contain \(g_i\); \(K_{i,3}\) is the number of all perfect matchings of \(H_i\) which contain \(g_i\) and do not contain \(f_i\); \(K_{i,4}\) is the number of all perfect matchings of \(H_i\) which do not contain any of edges \(f_i\) and \(g_i\). In this way, the set of all perfect matchings of \(H_i\) is divided into four disjointed classes. These classes (i.e., their elements) are said to be assigned to the corresponding graphs \(H_{i,j}\) \((j = 1, \ldots, 4)\).

Associate with each good perfect matching of \(C_n\) a word \(j_1j_2\ldots j_n\) of the alphabet \(\{1, 2, 3, 4\}\) in the following way: If the considered perfect matching induces in \(H_i\) a perfect matching assigned to the graph \(H_{i,j}\), then \(j_i = j\). For example, the word \(j_1j_2\ldots j_n\) for the perfect matching represented in Fig. 5a, is 44144. Note that by choosing the edges of a perfect matching of \(C_n\) in \(H_i\) and \(H_{i+1}\) \((i = 1, \ldots, n; H_{n+1} := H_1)\) we must not generate one of the modes 4 and 5 of arrangements of the perfect matching in the square between \(H_i\) and \(H_{i+1}\) i.e., the subwords \(j_ij_{i+1}\) \((i = 1, \ldots, n - 1)\) and subword \(j_n\) must not belong to the set \(\{11, 12, 31, 32\}\). According to the foregoing we obtain the following statement.

**Lemma 1.** If we denote the number of all good perfect matchings of \(C_n\) by \(\kappa\{C_n\}\), then

\[
\kappa\{C_n\} = \sum_{\substack{j_1j_2\ldots j_n \in \{1, 2, 3, 4\}^n \\ j_1, j_2, j_n \in \{11, 12, 31, 32\} \\ 1 \leq i \leq n-1}} K_{i,1}K_{i,2}K_{i,3}K_{i,4} \quad \square
\]

Let

\[
M_i = \begin{bmatrix}
0 & 0 & K_{i,1} & K_{i,3} \\
K_{i,1} & K_{i,2} & K_{i,3} & K_{i,4} \\
0 & 0 & K_{i,2} & K_{i,3} \\
K_{i,3} & K_{i,4} & K_{i,4} & K_{i,4}
\end{bmatrix}
\]

where

\[
K_{i,1} = K\{H_i - (f_i) - (g_i)\} \\
K_{i,2} = K\{H_i - (f_i) - g_i\} \\
K_{i,3} = K\{H_i - f_i - (g_i)\} \\
K_{i,4} = K\{H_i - f_i - g_i\}.
\]

Then the previous lemma can be written in the following form.
Lemma 2. The number of good perfect matchings of the graph $C_n$ is equal to the sum of entries of the main diagonal of the matrix $M_1 \cdot M_2 \cdots M_n$, i.e.,

$$\kappa\{C_n\} = \text{tr}(M_1 \cdot M_2 \cdots M_n)$$

Determination of the ASC-value for an arbitrary cyclic hexagonal-square chain. In order to determine ASC$\{C_n\}$ we shall consider the remaining perfect matchings of $C_n$ i.e., the ones whose arrangement words contain 6 and/or 7.

Definition 4. The edges of the graph $C_n$ are called internal if they do not belong to the external circuits.

Lemma 3. Let the word $u \equiv u_1u_2\ldots u_n$ be the arrangement word of a perfect matching $M$ and $u_i \in \{6, 7\} \ (1 \leq i \leq n)$. Then no internal edge of $C_n$ is in $M$.

![Fig. 6](image)

Proof. Consider an internal edge $p'q'$ of $C_n$ and denote the part of $C_n$ between the edges $p_{i+1}q_{i+1}$ and $p'q'$ (one of two possible hexagonal-square chains which contain both edges $p_{i+1}q_{i+1}$ and $p'q'$) by $L'$ (Fig. 6). Note that exactly one of the vertices $p_{i+1}$ and $q_{i+1}$ is connected in $M$ to a vertex which does not belong to the subgraph $L'$. Since the number of vertices in $L'$ is even, exactly one of the vertices $p'$ and $q'$ is connected in $M$ to a vertex which does not belong to $L'$. Consequently, the edge $p'q'$ cannot be in $M$. □

Lemma 4. If an arrangement word $u$ contains 6 and/or 7, then this word belongs to $\{6, 7\}^n$ i.e., all its letters are 6 and/or 7. There are exactly two perfect matchings of $C_n$ with such arrangement words.

Proof. The proof of Lemma 3 implies the first part of the lemma. Moreover, if the colours of the vertices $p_{i+1}$ and $r_{i+1}$ (Fig. 6) are different, then $u_{i+1} = u_i$ i.e., $u_iu_{i+1} \in \{66, 77\}$; if the colours of the vertices $p_{i+1}$ and $r_{i+1}$ are identical, then $u_iu_{i+1} \in \{67, 76\}$. Further, for each arrangement of the word $u \equiv u_1u_2\ldots u_n$ there is another one $\overline{u} \equiv \overline{u}_1\overline{u}_2\ldots \overline{u}_n$ which is “complementary” in the sense that

$$\overline{u}_i = \begin{cases} 6, & \text{if } u_i = 7 \\ 7, & \text{if } u_i = 6 \end{cases}$$

for $i = 1, \ldots, n$.

For each of these two only possible arrangement words ($u$ and $\overline{u}$) from the set $\{6, 7\}^n$ there exists exactly one perfect matching because no internal edge of $C_n$
can be in it (Lemma 3). Consequently, there are exactly two perfect matchings (we shall denote them by $\mathcal{U}$ and $\overline{\mathcal{U}}$) with arrangement words from the set $\{6,7\}^n$. □

In order to examine the parities of $\mathcal{U}$ and $\overline{\mathcal{U}}$ (refer to Definition 1) note that we can obtain one of them from the other by cyclically rearranging the edges at first within one of external circuits and then within the other external circuit.

Further, if the number $n$ is odd, then the length of one of external circuits is $\equiv 0 \pmod{4}$ and the length of the other one is $\equiv 2 \pmod{4}$. According to the foregoing and Theorem 3 we obtain that the perfect matchings $\mathcal{U}$ and $\overline{\mathcal{U}}$ are of opposite parity. Consequently, $\text{ASC}\{C_n\} = \kappa\{C_n\}$. A graph $C_n$ presented in Fig. 5 is an example of such a case.

Consider now the case when the number $n$ is even. The length of both of the external circuits is $c_1 \equiv c_2 \equiv 0 \pmod{4}$ or $c_1 \equiv c_2 \equiv 2 \pmod{4}$. For both of these two subcases we obtain that $\mathcal{U}$ and $\overline{\mathcal{U}}$ are of equal parity. We distinguish two possibilities.

**Possibility I:** words $u$ and $\overline{\pi}$ consist of the same letters i.e., $u = 66 \ldots 6$ and $\overline{\pi} = 77 \ldots 7$. If we cyclically rearrange the edges of $\mathcal{U}$ within one of the external circuits which contains vertices $p_1, r_1, p_2, r_2, \ldots, p_n, r_n$ (Figure 9), we obtain a good perfect matching. The number of rearranged edges is even for the case $c_1 \equiv c_2 \equiv 0 \pmod{4}$ and odd for the case $c_1 \equiv c_2 \equiv 2 \pmod{4}$. This implies (Theorem 3) that the perfect matchings $\mathcal{U}$ and $\overline{\mathcal{U}}$ are of opposite parity with good perfect matchings of $C_n$ in the first case and of the equal parity with good perfect matchings of $C_n$ in the second case. So we obtain

$$\text{ASC}\{C_n\} = \begin{cases} \kappa\{C_n\} - 2, & \text{if } c_1 \equiv c_2 \equiv 0 \pmod{4}; \\ \kappa\{C_n\} + 2, & \text{if } c_1 \equiv c_2 \equiv 2 \pmod{4} \end{cases} \quad (3)$$

**Possibility II:** Both the letters 6 and 7 appear in the word $u$. Let $i_1, i_2, \ldots, i_k$ ($i_1 \leq i_2 \leq \ldots \leq i_k$) be indices of letters 7 in arrangement word $u$. Observe the external circuit which contains vertices $p_1, r_1, p_2, r_2, \ldots, p_n, r_n$. If we remove edges $r_ip_{i_j+1}$ for $j = 1, \ldots, k$ $(p_{n+1} := p_1)$ and add edges $r_is_{i_j}, s_{i_j}q_{i_j+1}$ and $q_{i_j+1}p_{i_j+1}$ $(j = 1, \ldots, k)$, then we obtain a new circuit of length $c + 2k$ (indicated by bold lines in Fig. 7 and Fig. 8). Rearranging edges of the perfect matching $\mathcal{U}$ in this circuit we obtain a good perfect matching. The number $(c + 2k)/2$ of rearranged edges can be even, as in example in Fig. 7 or odd, as in example in Fig. 8. This implies

$$\text{ASC}\{C_n\} = \begin{cases} \kappa\{C_n\} - 2, & \text{if } c + 2k \equiv 0 \pmod{4}; \\ \kappa\{C_n\} + 2, & \text{if } c + 2k \equiv 2 \pmod{4} \end{cases} \quad (4)$$

Note that the relation (3) is just a special case of (4) for $k = 0$. According to the foregoing we can state the following theorem.

**Theorem 4.** We have

$$\text{ASC}\{C_n\} = \begin{cases} \kappa\{C_n\}, & \text{if } n \text{ is odd}; \\ \kappa\{C_n\} - 2, & \text{if } n \text{ is even and } c + 2k \equiv 0 \pmod{4}; \\ \kappa\{C_n\} + 2, & \text{if } n \text{ is even and } c + 2k \equiv 2 \pmod{4} \end{cases}$$
where \( c \) is the length of an external circuit; \( k \) is the number of letters \( 6 \) (or \( 7 \)) in the arrangement word \( u \) (or \( \overline{u} \)) and \( \kappa\{C_n\} \) is determined by Lemma 2.

Fig. 7

**Proof of Theorem 1.** Let now all hexagonal chains \( H_i \) be mutually isomorphic in such a way that all edges \( f_i, i = 1, \ldots, n \) are mutually corresponded and all edges \( g_i, i = 1, \ldots, n \) are mutually corresponded too. We introduce the following notions:

\[
\begin{align*}
m &:= m_1 = m_2 = \ldots = m_n \\
M &:= M_1 = M_2 = \ldots = M_n \\
K_j &:= K_{1,j} = K_{2,j} = \ldots = K_{n,j}, \quad j = 1, \ldots, 4.
\end{align*}
\]

We can obtain both a recurrence relation and an explicit formula for the number of good perfect matchings of \( C_n \).

**Lemma 5.** In the case of isomorphic hexagonal chains \( H_i \) \((i = 1, \ldots, n)\) we have

\[
\kappa\{C_n\} = (K_2 + K_3 + K_4)\kappa\{C_{n-1}\} + (K_1 K_4 - K_2 K_3)\kappa\{C_{n-2}\}
\]
with initial conditions

\[ \kappa \{ C_1 \} = K_2 + K_3 + K_4 \]
\[ \kappa \{ C_2 \} = (K_2 + K_3 + K_4)^2 + 2(K_1 K_4 - K_2 K_3) \]

Proof. The characteristic equation of \( M \) is

\[ \lambda^4 - (K_2 + K_3 + K_4)\lambda^3 + (K_2 K_3 - K_1 K_4)\lambda^2 = 0. \]  

Using the Cayley-Hamilton theorem we obtain

\[ M^n - (K_2 + K_3 + K_4)M^{n-1} + (K_2 K_3 - K_1 K_4)M^{n-2} = 0 \]

for \( n \geq 2 \). Consequently,

\[ \text{tr}(M^n) - (K_2 + K_3 + K_4)\text{tr}(M^{n-1}) + (K_2 K_3 - K_1 K_4)\text{tr}(M^{n-2}) = 0 \]

for \( n \geq 2 \). Using Lemma 2 we obtain the desired recurrence relation for \( \kappa \{ C_n \} \). \( \Box \)

![Diagram](image-url)

Fig. 8

The eigenvalues of the matrix \( M \) are \( \lambda_1 = \lambda_2 = 0, \lambda_3 = (L - D)/2 \) and \( \lambda_4 = (L + D)/2 \), where

\[ L = K_2 + K_3 + K_4 \] \[ D = \sqrt{L^2 + 4(K_1 K_4 - K_2 K_3)} \]

Since \( \text{tr}(M^n) = \sum_{i=1}^{4} \lambda_i^n \) we obtain the following statement.
**Lemma 6.** For $\kappa \{ C_n \}$ we have $\kappa \{ C_n \} = \left[ (L - D)^n + (L + D)^n \right]/2^n$, where $L$ and $D$ are given by (6). □

![Diagram](image1.png)

**Fig. 9**

**Example 1.** For the graph $C_5$ from Fig. 5 we obtain $K_1 = 1$, $K_2 = K_3 = 0$, $K_4 = 1$. Using Lemma 6 we get $\kappa \{ C_5 \} = 11$.

**Example 2.** The graph $C_6$ from Fig. 9 consists of six isomorphic hexagonal zig-zag chains for which $K_1 = F_3 = 2$, $K_2 = F_2 = 1$, $K_3 = F_4 = 3$, $K_4 = F_3 = 2$. Using Lemma 6 we get $\kappa \{ C_6 \} = \left( (6 - 2\sqrt{10})^6 + (6 + 2\sqrt{10})^6 \right)/2^6 = 54758$. 

![Diagram](image2.png)

**Fig. 10**
Example 3. The graph $C_4$ from Fig. 10 consists of four isomorphic linear chains for which $K_1 = 2$, $K_2 = K_3 = 1$, $K_4 = 0$. Using Lemma 6 again, we get $\kappa\{C_4\} = 2$.

In order to complete the proof of Theorem 1 consider the perfect matchings $\mathcal{U}$ and $\mathcal{U}$ again. The graph $C_n$ is bipartite in the case when all hexagonal chains are isomorphic is equivalent to the following one: If $p_i$ and $r_i$ are of the same colour, then $n$ must be even. In the case when $p_i$ and $r_i$ are of the same colour (Fig. 10) we have $u = 6767 \ldots 67$ and $n = 2k$. The number of edges of the external circuit from $p_i$ to $r_i$ is even; so we obtain $c + 2k \equiv 0 \pmod{4}$. In another case, when $p_i$ and $r_i$ are of different colours (Fig. 9) we get $u = 66 \ldots 6$ and $k = 0$. The number of edges in the external circuit from $p_i$ to $r_i$ is odd. So, if $n$ is even, then we obtain again that $c + 2k \equiv 0 \pmod{4}$. Using Theorem 4 and Lemma 6 we obtain the assertion of Theorem 1. $\Box$

Example 4. The graph in Fig. 5 (Example 1) is an example for the case $n$-odd and the graphs in Fig. 9 and Fig. 10 (Examples 2 and 3) are examples for the case $n$-even. In the first case we get $ASC\{C_5\} = \kappa\{C_5\} = 11$ and in the second two cases $ASC\{C_6\} = \kappa\{C_6\} - 2 = 54756$ and $ASC\{C_4\} = \kappa\{C_4\} - 2 = 2 = 0$.

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