TOEPLITZ OPERATORS ON
M-HARMONIC HARDY SPACE $H^p_m(S)$ WITH $0 < p \leq 1$

Miroljub Jevtić

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Abstract. Let $B^n$ be the unit ball in $\mathbb{C}^n$, $S$ is the boundary of $B^n$. Let $L^p(S)$ denote the usual Lebesgue spaces over $S$ with respect to the normalized surface measure, $H^p_m(B^n)$ is the Hardy space of $M$-harmonic functions and $H^p_{at}(S)$ denotes the atomic Hardy spaces defined in [4]. Let $P : L^2(S) \to H^p_m(B^n)$ denote the Poisson-Szegő projection. We use $M_f : L^p(S) \to L^p(S)$ to denote the multiplication operator, and we define the Toeplitz operator $T_f = PM_f$. The paper gives characterization theorems on $f$ such that the Toeplitz operator $T_f$ is bounded from $H^p_{at}(S) \to H^p_m(B^n)$ with $0 < p \leq 1$.

1. Introduction

Let $B^n$ denote the unit ball in $\mathbb{C}^n$. Set $S$ as the boundary of $B^n$. Let $\sigma$ be the normalized surface measure over $S$. Here $L^p(S)$ denotes the usual Lebesgue space over $S$ with respect to $\sigma$ for $0 < p \leq \infty$, $H^p_m(B^n)$ denotes the usual Hardy space of holomorphic functions, $H^p_{at}(B^n)$ the Hardy space of $M$-harmonic functions and $H^p_{at}(S)$ denotes the atom Hardy space of complex valued functions on $S$ defined in [4] ($H^p_{at}(S) = L^p(S)$, for $1 < p < \infty$). Also we let $BMO(S)$ denote the usual bounded mean oscillation function space with norm $\| \cdot \|_*$ and $BMOA$ be its holomorphic subspace. We define another function space $LMO(S)$. We say a function $f \in LMO(S)$ if $f \in L^1(S)$ and $\| f \|_{LMO} < \infty$ where

$$
\| f \|_{LMO} = \| f \|_{L^1(S)} + \sup_{\xi \in S, \delta > 0} \left\{ \frac{\log(\frac{2}{\sigma(B(\xi, \delta))})}{\sigma(B(\xi, \delta))} \int_{B(\xi, \delta)} |f - f_B| d\sigma \right\}.
$$

and

$$
f_B = \frac{1}{\sigma(B(\xi, \delta))} \int_{B(\xi, \delta)} f(\eta) d\sigma(\eta).
$$

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Here \( B = B(\xi, \delta) = B_{\delta}(\xi) = \{ \eta \in S : |1 - \langle \eta, \xi \rangle| < \delta \} \) is nonisotropic ball on \( S \).

Let \( C : L^2(S) \rightarrow H^1_\alpha(B^n) \) denote the orthogonal projection, i.e.

\[
C[f](z) = \int_S f(\xi)C(z, \xi)d\sigma(\xi), z \in B^n, f \in L^2(S),
\]

where \( C(z, \xi) = (1 - \langle z, \xi \rangle)^{-n} \) is the Cauchy kernel for the unit ball \( B^n \).

The Poisson–Szegö projection is an integral operator

\[
P[f](z) = \int_S f(\xi)P(z, \xi)d\sigma(\xi), z \in B^n, f \in L^2(S).
\]

The integral kernel \( P(z, \xi) = \frac{(1 - |z|^2)}{|1 - \langle z, \xi \rangle|^2} \) is called the Poisson–Szegö kernel of \( B^n \).

We use \( M_f \) to denote the multiplication operator, and we define the Toeplitz operators as

\[
T^C_f = CM_f \quad \text{and} \quad T_f = PM_f.
\]

We begin with a characterization theorem on \( f \) such that the Toeplitz operator \( T^C_f \) is bounded from \( H^1_{\alpha}(S) \) to \( H^1_{\alpha}(B^n) \).

**Theorem 0.** Let \( f \in L^2(S) \). Then the following statements are equivalent:

(i) \( f \in L^\infty(S) \cap \text{LMO}(S) \),

(ii) The Toeplitz operators \( T^C_f \) and \( T_f \) are bounded from \( H^1_{\alpha}(S) \) to \( H^1_{\alpha}(B^n) \),

(iii) \( M_f : \text{BMO}(S) \rightarrow \text{BMO}(S) \) is bounded,

(iv) \( M_f : \text{BMOA} \rightarrow \text{BMO}(S) \) is bounded,

(v) The Toeplitz operator \( T^C_f \) is bounded from \( H^1_{\alpha}(S) \) to \( H^1_{\alpha}(B^n) \).

This is Theorem 1 in [7]. We note that in the proof of the implication (iv) \( \Rightarrow \) (i) it is not shown that \( M_f : \text{BMOA} \rightarrow \text{BMO}(S) \) implies \( f \in L^\infty \). Here is a simple proof.

For \( \xi_0 \in S \) any Lebesgue point of \( f \), we consider \( g_{\xi_0}(z) = \log(1 - \langle z, \xi_0 \rangle) \in \text{BMOA} \) with \( \|g_{\xi_0}\|_* \leq C \), where \( C \) is a constant depending only on \( n \). By assumption we have \( \|f g_{\xi_0}\| \leq C \) (In this paper constants are denoted by \( C \) which may indicate a different constant from one occurrence to the next). Therefore, for any nonisotropic ball \( B_\delta(\xi_1) \subset S \) we have

\[
\frac{1}{\sigma(B_\delta(\xi_1))} \int_{B_\delta(\xi_1)} |f(\eta)g_{\xi_0}(\eta)| \ d\sigma(\eta) \leq C \log \frac{1}{\sigma(B_\delta(\xi_1))}
\]

(see [7, Lemma 1]).

In particular, if we let \( \xi_1 = \xi_0 \) and \( 0 < \delta < 1/4 \), we have

\[
C |g_{\xi_0}(\eta)| \geq \log \frac{1}{\sigma(B_\delta(\xi_0))}, \quad \eta \in B_\delta(\xi_0)
\]
Thus
\[
C \log \frac{1}{\sigma(B_\delta(\xi_0))} \geq \frac{1}{\sigma(B_\delta(\xi_0))} \int_{B_\delta(\xi_0)} |f(\eta)g_{\xi_0}(\eta)| \, d\sigma(\eta)
\]
\[
\geq C^{-1} \left( \log \frac{1}{\sigma(B_\delta(\xi_0))} \right) \frac{1}{\sigma(B_\delta(\xi_0))} \int_{B_\delta(\xi_0)} |f(\eta)| \, d\sigma(\eta)
\]
Therefore
\[
\frac{1}{\sigma(B_\delta(\xi_0))} \int_{B_\delta(\xi_0)} |f(\eta)| \, d\sigma(\eta) \leq C^2
\]
for all \( \delta > 0 \). Since \( \xi_0 \) is Lebesgue point, we have \( |f(\xi_0)| \leq C^2 \). By Lebesgue theorem, we have \( |f(\xi)| \leq C^2 \) for a.e. \( \xi \in S \). This completes the proof of Theorem 0.

The main purpose of this paper is to give characterization theorems on \( f \) such that the Toeplitz operator \( T_f \) is bounded from \( H^p_{al}(S) \to H^p_{al}(B^n) \) with \( 0 < p \leq 1 \). More precisely we prove the following:

**Theorem 1.** Let \( f \in L^2(S) \). Then the following statements are equivalent:
(i) \( f \in L^\infty(S) \cap \text{LMO}(S) \)
(ii) The Toeplitz operator \( T_f \) is bounded from \( H^1_{al}(S) \) to \( H^1_{al}(B^n) \).
(iii) \( M_f : \text{BMO}(S) \to \text{BMO}(S) \) is bounded.

**Theorem 2.** Let \( f \in L^2(S) \), and \( 0 < p < 1 \). Then the following statements are equivalent:
(i) \( f \in (H^p_{al}(S))^* \)
(ii) \( T_f : H^p_{al}(S) \to H^p_{al}(B^n) \) is bounded.

2. Some basic notations and known results

Let \( \tilde{\Delta} = (1 - |z|^2)^2 \sum_{j,k} (\delta_{jk} - z_j z_k)D_j D_k \) be the invariant or Bergman laplacian. The functions annihilated by \( \tilde{\Delta} \) are called \( M \)-harmonic functions, \( f \in M \), (see [8, Chapter 4], for general properties of these functions).

We will also use the following expressions, defined for a smooth function \( u \) in \( B^n \):
(a) The radial maximal function
\[
u^+(\xi) = \sup\{|u(r\xi)|; 0 \leq r < 1\}.
\]
(b) The admissible maximal function
\[
M_\delta[u](\xi) = M[u](\xi) = \sup \left\{ |u(z)| : z \in D_\delta(\xi) \right\}.
\]
(c) The admissible area function

\[ S[u](\xi) = \left\{ \int_{D_\delta(\xi)} \| \nabla_{B^*} u(z) \|_{B^*}^2 \, d\tau(z) \right\}^{1/2}. \]

Here, in (b) and (c), \( D_\delta(\xi) = D(\xi) \) is the admissible approach region given by

\[ D_\delta(\xi) = \{ z \in B^n : |1 - \langle z, \xi \rangle| < \delta(1 - |z|^2) \}, \]

\[ d\tau(z) = \frac{1}{(1 - |z|^2)^{n+1}} dV(z), \]

d\( V \) denoting Lebesgue measure, and \( \| \nabla_{B^*} u \|_{B^*} \) is the Bergman length of the Bergman gradient given in coordinates by

\[ \| \nabla_{B^*} u \|_{B^*}^2 = \]

\[ (1 - |z|^2) \left\{ \sum_{i=1}^n |D_i u|^2 - \left| \sum_{i=1}^n z_i D_i u \right|^2 + \sum_{i=1}^n |\overline{D_i} u|^2 - \left| \sum_{i=1}^n z_i \overline{D_i} u \right|^2 \right\}. \]

A function \( f \in M \) is said to belong \( H^p_m(B^n) \), \( 0 < p < \infty \), if \( M_2[f] \in L^p(S) \). For the proof of Theorem 1 and Theorem 2 the following two lemmas will be needed.

**Lemma 2.1.** [1] Let \( u \in M \). Then the following are equivalent:

(i) \( u \in H^p_m(B^n) \)

(ii) The radial maximal function \( u^+ \in L^1(S) \).

(iii) The area function \( S[u] \in L^1(S) \).

(iv) There exists \( f \in H^1_m(S) \) such that \( u = P[f] \).

**Lemma 2.2.** [1], [4], [5], [8] The following statements hold:

(i) \( P : H^p_m(S) \rightarrow H^p_m(B^n), \ 0 < p < \infty \) is bounded and onto.

(ii) The dual space of \( H^p_m(S), \ 0 < p < 1, \) is \( \mathcal{L}^\gamma(S), \ \gamma = n(1/p - 1) \).

(iii) The dual space of \( H^1_m(S) \) is \( \text{BMO}(S) \).

See [4] and [7] for the definition of \( \mathcal{L}^\gamma(S) \) spaces.

### 3. Proof of Theorem 1

First, we prove (ii) \( \Rightarrow \) (iii).

By Lemma 2.1 every function in the space \( H^p_m(B^n) \) is the Poisson integral of a function in the space \( H^1_m(S) \) and so the hypotheses that \( T_f \) is bounded from \( H^1_m(S) \) to \( H^1_m(B^n) \) is equivalent to the hypotheses that \( M_f \) is bounded from \( H^1_m(S) \)
to itself. But then by duality \((H^1_{\text{at}}(S))^* = \text{BMO}(S)\) (Lemma 2.2) \(M_f\) is bounded from \(\text{BMO}(S)\) to itself.

By Theorem 0 we have that (iii) \(\iff\) (i),

The proof of the implication (i) \(\Rightarrow\) (ii).

By the atomic decomposition theorem [4], [7], it suffices to show that for every atom \(a\) on \(S\) with support \(B_0 = B(\xi_0, \delta)\), we have

\[
\|T_f(a)\|_{H^1_{\text{at}}(B^n)} \leq C(\|f\|_{\text{LMO}} + \|f\|_{\infty}).
\]

Now we let \(B_1 = 2B_0 = B(\xi_0, 2\delta)\). Since \(f \in L^\infty(S), T_f : L^2(S) \to H^2_{\text{at}}(B^n)\) is bounded (Lemma 2.2). So we have that

\[
\int_{B_1} \left( \sup_{z \in D(\eta)} \int_{B_0} |a(\xi)| |f(\xi)| P(z, \xi) d\sigma(\xi) \right) d\sigma(\eta)
\leq \|f\|_{\infty} \left( \int_{B_1} \left( \sup_{z \in D(\eta)} \int_{B_0} |a(\xi)| P(z, \xi) d\sigma(\xi) \right)^2 d\sigma(\eta) \right)^{1/2} \sigma(B_1)^{1/2}
\leq \|f\|_{\infty} \|a\|_{L^2(S)} (\sigma(B_1))^{1/2} \leq C\|f\|_{\infty}
\]

(Here we used Lemma 2.2 and the estimate \(|a(\xi)| \leq (\sigma(B_0))^{-1}, \xi \in B_0\). Let \(\eta \in S \setminus B_1\) and \(z \in D(\eta)\). Then

\[
T_f(a)(z) = \int_{B_0} a(\xi)f(\xi)P(z, \xi) d\sigma(\xi)
= \int_{B_0} a(\xi)(f(\xi) - f_{B_0})P(z, \xi) d\sigma(\xi) + f_{B_0} \int_{B_0} a(\xi)P(z, \xi) d\sigma(\xi)
= I_1(z) + I_2(z)
\]

Now

\[
\int_S \sup_{B_1 \setminus B_0} |I_2(z)| d\sigma(\eta) \leq |f_{B_0}| \int_S \left( \sup_{z \in D(\eta)} \int_S |a(\xi)| P(z, \xi) d\sigma(\xi) \right) d\sigma(\eta)
\leq C|f_{B_0}| \|a\|_{H^1_{\text{at}}(S)} \leq C\|f\|_{\infty}, \text{ by Lemma 2.1 (or Lemma 2.2)}.
\]

We have

\[
I_1(z) = \int_{B_0} (f(\xi) - f_{B_0})a(\xi)P(z, \xi) d\sigma(\xi)
= \int_{B_0} (f(\xi) - f_{B_0})a(\xi)[P(z, \xi) - P(z, \xi_0)] d\sigma(\xi)
+ \int_{B_0} (f(\xi) - f_{B_0})a(\xi)P(z, \xi_0) d\sigma(\xi) = I_{11}(z) + I_{12}(z)
\]
So
\[
\int_{S\setminus B_1} \left( \sup_{z \in D_\delta(\eta)} |I_{12}(z)| \right) d\sigma(\eta)
\]
\[
\leq \int_{S\setminus B_1} \left( \sup_{z \in D_\delta(\eta)} \int_{B_\delta} |f(\xi) - f_{B_\delta}| |a(\xi)| |P(z, \xi_0) d\sigma(\xi) \right) d\sigma(\eta)
\]
\[
\leq C \frac{\|f\|_{L^\infty}}{\log \frac{1}{\delta}} \int_{S\setminus B_1} \left( \sup_{z \in D_\delta(\eta)} P(z, \xi_0) \right) d\sigma(\eta)
\]
\[
\leq C \frac{\|f\|_{L^\infty}}{\log \frac{1}{\delta}} \int_{S\setminus B_1} \frac{d\sigma(\eta)}{|1 - \langle \eta, \xi_0 \rangle|^p} \leq C \|f\|_{L^\infty}.
\]

Now we turn to estimate
\[
|I_{11}(z)| \leq \frac{C}{\sigma(B_0)} \int_{B_\delta} |f(\xi) - f_{B_\delta}| |P(z, \xi) - P(z, \xi_0)| d\sigma(\xi)
\]
\[
\leq \frac{C}{\sigma(B_0)} \int_{B_\delta} |f(\xi) - f_{B_\delta}| \frac{\delta^{1/2}}{|1 - \langle z, \xi_0 \rangle|^{n+1/2}} d\sigma(\xi)
\]
\[
\leq C \|f\|_* \frac{\delta^{1/2}}{|1 - \langle \eta, \xi_0 \rangle|^{n+1/2}}.
\]

(See [8]) (Note that \( z \in D_\delta(\eta) \) and \( \eta \in S\setminus B_1 \)).

Therefore
\[
\int_{S\setminus B_1} \sup_{z \in D_\delta(\eta)} |I_{11}(z)| d\sigma(\eta) \leq C \|f\|_* \leq C \|f\|_\infty.
\]

4. Proof of Theorem 2

First, we prove that (ii) \( \Rightarrow \) (i). Let \( g \in H^p_{at}(S) \cap L^2(S) \). Then
\[
\left| \int_{S} f(\xi)g(\xi) d\sigma(\xi) \right| = |P[f,g](0)| = |T_f(g)(0)|
\]
\[
\leq \|T_f\| \|g\|_{H^p_{at}(S)} \leq \|T_f\| \|g\|_{H^p_{at}(S)}
\]

Since \( H^p_{at}(S) \cap L^2(S) \) is dense in \( H^p_{at}(S) \), then we have that
\[
\left| \int_{S} f(\xi)g(\xi) d\sigma(\xi) \right| \leq \|T_f\| \|g\|_{H^p_{at}(S)}
\]
for all \( g \in H^p_{at}(S) \). Therefore \( f \in (H^p_{at}(S))^* \), i.e. (i) holds.

Now we prove that (i) \( \Rightarrow \) (ii). By the atom decomposition theorem, we need only to show that
\[
\|T_f(a)\|_{H^p_{at}(S)} \leq C(f),
\]
for all \( p \)-atom \( a \) will support \( B(\xi_0, \delta) \subset S \) for some \( \xi_0 \in S \) and \( 0 < \delta < 1 \).
Now we proceed as in the proof of the implication (i) $\Rightarrow$ (iv), Theorem 2 [7]. The contributions of $B(\xi_0, 2\delta)$ to the integral $\int_S (\sup_{z \in D(\eta)} |T_f(a)(z)|)^p \, d\sigma(\eta)$ is estimated using Hölder’s inequality and the boundedness of the map $P : L^2(S) \to H^2_m(B^n)$.

For points $\eta \notin B(\xi_0, 2\delta)$ one uses the cancellation of the atom and boundedness $P : H^p_m(S) \to H^p_m(B^n)$ (Lemma 2.2) to obtain the desired estimate.

References

7. Song-Ying Li, Toeplitz Operators on Hardy Space $H^p(S)$ with $0 < p \leq 1$, Integral Equat. Oper. Theory 15 (1992), 807–823.

Matematički fakultet
Studenčki trg 16 (Received 08 11 1996)
11001 Beograd, p.p. 550
Yugoslavia (Revised 14 07 1997)