ON WELL-POSEDNESS OF QUADRATIC MINIMIZATION PROBLEM ON ELLIPSOID AND POLYHEDRON

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Abstract. We consider existence of solutions for quadratic minimization problem on an ellipsoid and on a polyhedron. In the case of polyhedron, we present a necessary and sufficient conditions for Tikhonov well-posedness of the problem.

1. We consider the following extremal problem:

\[ J(u) = \|Au - f\|_F^2 \to \inf, \ u \in U, \]

where \( U \) is the ellipsoid

\[ U = \{u \in H : \|Bu\|_G \leq R\} \]

or the polyhedron

\[ U = \{u \in H : \langle c_i, u \rangle \leq \beta_i, \ i = 1, \ldots, m\}. \]

Here \( H, F, G \) are real Hilbert spaces; \( A : H \to F, B : H \to G \) are bounded linear operators; \( f \in F, c_i \in H, c_i \neq 0, i = 1, \ldots, m \) are fixed elements from the corresponding spaces; \( \beta_i, i = 1, \ldots, m \) and \( R > 0 \) are given real numbers.

The results of this paper complete the results from [1]–[3]. Namely, in the case of an ellipsoid (1), (2), we get necessary conditions for the existence of solutions and show that these conditions are sufficient for normal solvable operators \( A \) and \( B \); in the case of polyhedron (1), (3), we present the necessary and sufficient conditions for the existence of solutions as well as for the well-posedness.

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Let us introduce the following notation: $R(A)$ is the range space of the operator $A$, $AU = \{Au : u \in U\}$ is the image of $U$ under the action of $A$, $\operatorname{Ker} A$ is the kernel of $A$, $A^* : F \to H$ is the adjoint operator of $A$, $M$ is the closure of the set $M \subseteq H$ with respect to the norm of $H$, $L^\perp$ is the orthogonal complement of the subspace $L$, $P$ is the orthogonal projector of $H$ onto $R(A^*)$.

The operators $A$ and $B$ generate the following orthogonal decompositions of $H$:

$$
H = \overline{R(A^*)} \oplus \operatorname{Ker} A, \quad H = \overline{R(B^*)} \oplus \operatorname{Ker} B.
$$

An operator $A$ is called normal soluble if $R(A) = \overline{R(A)}$. This condition is equivalent to $\overline{R(A^*)} = R(A^*)$ [4].

**Lemma 1.** [5] A linear bounded operator $A : H \to F$ is normal soluble if and only if

$$
\mu := \inf\{\|Au\| : u \perp \operatorname{Ker} A, \|u\| = 1\} > 0.
$$

This lemma implies immediately

**Lemma 2.** If a linear bounded operator $A : H \to F$ is not normal soluble, then there exists a sequence $(p_n)$ such that

$$
p_n \in \overline{R(A^*)}, \quad \|p_n\| = 1, \quad p_n \to 0, \quad Ap_n \to 0 \text{ as } n \to \infty.
$$

Let us notice that the set $U$ in (2) and (3) is convex and closed with respect to the norm of $H$. If, moreover, the set $U$ in (2) is bounded, then the existence of a solution for (1), (2) for each $f \in F$ follows by Weierstrass theorem [6]. If $U$ is unbounded (that is always so for $U$ in (3) when $\dim H = \infty$), then the problems (1), (2) and (1), (3) have solutions for each $f \in F$ if and only if $AU$ is a closed set in $F$ (see [1], [2]). We will use this existence criterion repeatedly in the sequel. Now we formulate the necessary conditions for the solvability of the problem (1), (2).

**Theorem 1.** Suppose that the problem (1), (2) has a solution for each $f \in F$. Then at least one of the following conditions is satisfied:

(i) $R(A^*) \cap R(B^*B) = \{0\}$,

(ii) $\operatorname{Ker} A + \operatorname{Ker} B = \operatorname{Ker} A + \operatorname{Ker} B$.

**Proof.** Assume that $R(A^*) \cap R(B^*B) \neq \{0\}$. The continuity of the operator $A$ and the closure of the set $AU$ imply that the set

$$
A^{-1}(AU) = \operatorname{Ker} A + \operatorname{Ker} B + V_R,
$$

where $V_R = \{u \in \overline{R(B^*)} : \|Bu\| \leq R\}$, is closed in the space $H$. Let $y \in R(A^*) \cap R(B^*B), y \neq 0$. Take a point $z \in H$ such that $y = B^*Bz$. Present $z$, according to
(4), in the form \( z = z_1 + z_2, \ z_1 \in \overline{R(B^*)}, \ z_2 \in \text{Ker} \ B \). Then \( y = B^*Bz_1 \), and, for the point \( x_0 = \frac{R}{\|z_1\|}z_1 \), we have

\[
x_0 \in \overline{R(B^*)}, \ B^*Bx_0 \in R(A^*) \cap R(B^*B), \ ||Bx_0||^2 = R^2,
\]

in particular, \( x_0 \in V_R \). Now take any point \( y_0 \in \text{Ker} \ A + \text{Ker} \ B \). The point \( y_0 + x_0 \) is a limit point of the closed set \( \text{Ker} \ A + \text{Ker} \ B + V_R \). Therefore, the point \( y_0 + x_0 \) is presentable as \( y_0 + x_0 = p_0 + z_0 \), where \( p_0 \in \text{Ker} \ A + \text{Ker} \ B \) and \( z_0 \in V_R \). Multiplying both sides of (6) by \( B^*Bz_0 \) and taking into account (5) and the orthogonality

\[
R(A^*) \cap R(B^*B) \perp \text{Ker} \ A + \text{Ker} \ B,
\]

we find that \( R^2 = ||Bx_0||^2 = \langle Bx_0, Bx_0 \rangle \). Since \( z_0 \in V_R \), we obtain

\[
||B(x_0 - z_0)||^2 = ||Bx_0||^2 - 2\langle Bx_0, Bz_0 \rangle + ||Bz_0||^2 \leq 0
\]

and therefore \( x_0 = z_0 \). Now we have \( y_0 = p_0 \in \text{Ker} \ A + \text{Ker} \ B \). Recalling that \( y_0 \) was an arbitrary point from \( \text{Ker} \ A + \text{Ker} \ B \), we finally get the condition (ii). This concludes the proof. \( \square \)

The following example shows that the assumptions about normal solvability of both operators \( A \) and \( B \) do not guarantee the existence of solutions of the problem (1), (2) for all \( f \in F \).

**Example.** Take \( H = F = G = l_2 \) and consider two closed subspaces of \( l_2 \):

\[
L = \{x \in l_2 : x = (0, x_2, 0, x_4, 0, x_6, 0, \ldots)\},
\]

\[
M = \{x \in l_2 : x = (0, x_2, x_2/2, x_4, x_4/4, x_6, x_6/6, \ldots)\}.
\]

Define \( A \) as the orthoprojector of \( l_2 \) onto \( L^\perp \) and \( B \) as the orthoprojector of \( l_2 \) onto \( M^\perp \). Then \( A = A^*, \ B = B^* = B^*B \), \( \text{Ker} \ A = L \), \( \text{Ker} \ B = M \), operators \( A \) and \( B \) are normal solvable but both relations (i) and (ii) from Theorem 1 are violated:

\[
x_0 = (1, 0, 0, \ldots) \in R(A^*) \cap R(B^*B) = L^\perp \cap M^\perp \neq \{0\},
\]

\[
\text{Ker} \ A + \text{Ker} \ B = L + M \neq L^\perp + M^\perp = \text{Ker} \ A + \text{Ker} \ B = \{x_0\}^\perp.
\]

It means that in this case the problem (1), (2) can not have a solution for each \( f \in l_2 \).

One can ask about additional conditions that normal solvable operators \( A \) and \( B \) should satisfy for the existence of a solution of the problem (1), (2) for each \( f \in F \). In order to answer this question, we shall prove the following

**Lemma 3.** Let \( A \) be a normal solvable operator and let \( V \subseteq H \) be a convex closed set. Then

\[
\overline{AV} = A(\overline{\text{Ker} \ A + V}).
\]
Proof. For each \( y_0 \in \overline{AV} \) there exists a sequence \( (u_n), u_n \in V \) such that the sequence \( y_n = Au_n \) converges to \( y_0 \) as \( n \to \infty \). According to (4) we can present \( u_n \) as

\[
u_n = x_n + z_n, \; x_n \in R(A^*), \; z_n \in \text{Ker} \; A.
\]

Then

\[Ax_n = Au_n = y_n \to y_0, \; n \to \infty.
\]

As the operator \( A \) is normal solvable, (7) implies that the sequence \( (x_n) \) is bounded. Therefore, \( (x_n) \) (or some of its subsequence) converges weakly to some limit \( x_0 \) and also \( x_n \in V + \text{Ker} \; A \). The set \( \text{Ker} \; A + V \) is weakly closed, thus \( x_0 \in \text{Ker} \; A + V \) and

\[
y_0 = \lim_{n \to \infty} Au_n = \lim_{n \to \infty} Ax_n = Ax_0 \in A(\text{Ker} \; A + V).
\]

Therefore, we have proved the inclusion \( \overline{AV} \subseteq A(\text{Ker} \; A + V) \). Conversely, for each \( y_0 \in A(\text{Ker} \; A + V) \) there exists a sequence \( u_n \in \text{Ker} \; A + V \) such that the sequence \( y_n = Au_n \to y_0 \) as \( n \to \infty \). Present the elements \( u_n \in \text{Ker} \; A + V \) in the form:

\[
u_n = z_n + x_n, \; z_n \in \text{Ker} \; A, \; x_n \in V.
\]

Since \( y_n = Au_n = Ax_n \in AV \), it follows that \( y_0 \in \overline{AV} \). Thus we have proved the inclusion \( A(\text{Ker} \; A + V) \subseteq \overline{AV} \), which completes the proof. \( \square \)

Now we show that for normal solvable operators \( A \) and \( B \) the statement of Theorem 1 can be inverted.

**Theorem 2.** Let \( A \) and \( B \) be normal solvable operators. If at least one of the conditions (i) or (ii) from Theorem 1 is satisfied, then the problem (1), (2) has a solution for each \( f \in F \).

**Proof.** First consider the case (ii) when

\[
\text{Ker} \; A + \text{Ker} \; B = \overline{\text{Ker} \; A + \text{Ker} \; B}.
\]

Using Lemma 3 for \( V = \text{Ker} \; B \), we get

\[
\overline{A(\text{Ker} \; B)} = \overline{A(\text{Ker} \; A + \text{Ker} \; B)} = A(\text{Ker} \; A + \text{Ker} \; B) = A(\text{Ker} \; B),
\]

i.e., the set \( A(\text{Ker} \; B) \) is closed. Then, by Theorem 3 in [2], it follows that the problem (1), (2) has a solution.

Now consider the case (i) when \( R(A^*) \cap R(B^*B) = \{0\} \). Since the operators \( A^*, B^*B \) are normal solvable and their ranges \( R(A^*), R(B^*B) \) are closed, we get

\[
H = \{0\}^\perp = (R(A^*) \cap R(B^*B))^\perp = \text{Ker} \; A + \text{Ker} \; B,
\]

i.e., the set \( \text{Ker} \; A + \text{Ker} \; B \) is dense in \( H \). Note that ellipsoid (2) has a nonempty interior (we consider \( R > 0 \)), therefore \( U + \text{Ker} \; A + \text{Ker} \; B = H \). On the other hand, \( U = U + \text{Ker} \; B \), hence \( U + \text{Ker} \; A = H \). Finally, we see that

\[
AU = A(U + \text{Ker} \; A) = AH = R(A),
\]
i.e., the set \( AU \) is closed. This concludes the proof. \( \Box \)

Let us consider the existence problem for (1), (3). Suppose the operator \( B : H \to \mathbb{R}^m \) is defined by \( Bu = \langle c_1, u \rangle, \langle c_2, u \rangle, \ldots, \langle c_m, u \rangle \), \( u \in H \). The operator \( B \) is normal solvable and

\[
R(B^*) = \left\{ \sum_{i=1}^{m} \lambda_i c_i : \lambda_i \in \mathbb{R}^1, i = 1, \ldots, m \right\} = \mathcal{L}(c_1, c_2, \ldots, c_m).
\]

Since \( H = R(B^*) \oplus \text{Ker} B \), the constraints (3) can be presented in the form

\[
(\text{fl}) \quad U = \text{V}_\beta \oplus \text{Ker} B,
\]

where

\[
\text{V}_\beta = \left\{ v \in R(B^*) : \langle c_i, v \rangle \leq \beta_j, j = 1, \ldots, m \right\}.
\]

**Theorem 3.** The problem (1), (3) has a solution for each \( f \in F \) if and only if the operator \( A \) is normal solvable.

*Proof.* The implication normal solvability \( \Rightarrow \) existence was proved in [1, p. 12]. Let us prove the converse implication. First observe that (fl) implies \( AU = AV_\beta + A(\text{Ker} B) \). We claim that \( AU = AV_\beta + \overline{A(\text{Ker} B)} \). Since by assumption the set \( AU \) is closed, we see that any point \( y \in AV_\beta + \overline{A(\text{Ker} B)} \) as a limit point of \( AU \) belongs to \( AU \). So, we have obtained that \( AV_\beta + \overline{A(\text{Ker} B)} \subseteq AU \). It is obvious that the inverse inclusion is valid. Therefore

\[
AV_\beta + \overline{A(\text{Ker} B)} = AV_\beta + \overline{A(\text{Ker} B)}
\]

is really true. Adding \( A(R(B^*)) \) to both sides, by the inclusion \( \text{V}_\beta \subset R(B^*) \), we get

\[
R(A) = A(R(B^*)) + A(\text{Ker} B) = A(R(B^*)) + \overline{A(\text{Ker} B)}.
\]

To conclude the proof, it remains to note that the set \( R(A) \) is closed as a sum of the finite-dimensional subspace \( A(R(B^*)) \) and the closed subspace \( \overline{A(\text{Ker} B)} \). \( \Box \)

2. Consider the question of well-posedness for the problem (1), (3) in Tikhonov sense.

**Definition.** [1] The problem (1) is well-posed in the space \( H \) in Tikhonov sense if the following three conditions hold: 1) \( J_* = \inf \{ J(u) : u \in U_* \} > -\infty \); 2) \( U_* = \left\{ u \in U : J(u) = J_* \right\} \neq \emptyset \); 3) each minimizing sequence \( (u_n) \) of the problem (1) converges strongly in \( H \) to the solution set \( U_* \), i.e.,

\[
d(u_n, U_*) = \inf \{ ||u_n - u|| : u \in U_* \} \to 0 \text{ as } n \to \infty.
\]

If at least one of the conditions 1), 2), 3) is not valid, then the problem is called *ill-posed.*
THEOREM 4. The problem (1), (3) is well-posed in the sense of Tikhonov if and only if the operator A is normal solvable.

Proof. Let A be a normal solvable operator and let \( u_n \) be an arbitrary minimizing sequence of the problem (1), (3). Present the elements \( u_n \) in the form

\[
\|Pu_n - Pu_*\| \to 0 \text{ as } n \to \infty,
\]

where \( u_* \in U_* \) is a solution (for instance, normal) of the problem (1), (3). Consider the sequence \( v_n = Pu_* + (I - P)u_n \). Then

\[
J(v_n) = J(Pu_*) = J(u_*) = J_*
\]

and

\[
\langle c_i, v_n \rangle = \langle c_i, Pu_* \rangle + \langle c_i, (I - P)u_n \rangle = \langle c_i, u_n \rangle + \langle c_i, Pu_* - Pu_n \rangle, \quad i = 1, 2, \ldots, m.
\]

Let us introduce the notation \( \alpha_{in} = \langle c_i, Pu_* - Pu_n \rangle \). The last relation implies that

\[
\langle c_i, v_n \rangle \leq \beta_i + \alpha_{in},
\]

and, moreover, according to (2)

\[
\alpha_{in} \to 0 \text{ as } n \to \infty, \quad i = 1, 2, \ldots, m.
\]

Present the set \( U_* \) in the form: \( U_* = (Pu_* + \text{Ker } A) \cap U \) and notice that \( v \in U_* \) if and only if \( v = Pu_* + (I - P)v \) and

\[
\beta_i \geq \langle c_i, v \rangle = \langle c_i, Pu_* \rangle + \langle c_i, (I - P)v \rangle, \quad i = 1, \ldots, m.
\]

Take the finite-dimensional subspace

\[
L = \mathcal{L}\{(I - P)c_1, (I - P)c_2, \ldots, (I - P)c_m\}
\]

and denote by \( Q \) the orthogonal projector of \( H \) onto \( L \). Then we have

\[
\langle (I - P)c_i, (I - P)v \rangle = \langle (I - P)c_i, Qv \rangle, \quad i = 1, \ldots, m.
\]

According to (5) and (6), we get that for each \( v \in U_* \)

\[
\langle (I - P)c_i, Qv \rangle \leq \beta_i - \gamma_i, \quad i = 1, 2, \ldots, m,
\]

where \( \gamma_i = \langle c_i, Pu_* \rangle \). Using (3), (7), we obtain

\[
\langle (I - P)c_i, Qv_n \rangle \leq \beta_i - \gamma_i + \alpha_{in}, \quad i = 1, 2, \ldots, m, \quad n = 1, 2, \ldots
\]
In the subspace $L$ define the set $W$ by

\[(\text{f9}) \quad W = \{ w \in L : \langle (I - P)c_i, w \rangle \leq \beta_i - \gamma_i, \; i = 1, 2, \ldots, m \}. \]

According to (f7), $Qv \in W$ for all $v \in U_*$. By virtue of (f4), (f8), and Hoffman’s lemma [7] we derive

\[(\text{f10}) \quad d(Qv_n, W) = \inf \{ \|Qv_n - w\| : w \in W \} \to 0, \; n \to \infty. \]

Note that in (f10) the infimum is achievable for each $n = 1, 2, \ldots$ and take the elements $w_n \in W$ so that $d(Qv_n, W) = \|Qv_n - w_n\|$. Furthermore, consider the sequence $y_n = Pu_* + (I - Q)(I - P)u_n + w_n, \; n = 1, 2, \ldots$. Then, for all $n = 1, 2, \ldots, J(y_n) = J(Pu_*) = J(u_*) = J$, and using (f9) we get

\[
\langle c_i, y_n \rangle = \langle c_i, Pu_* \rangle + \langle c_i, (I - Q)(I - P)u_n \rangle + \langle c_i, w_n \rangle \\
= \gamma_i + \langle c_i, (I - P)u_n \rangle - \langle Qc_i, (I - P)u_n \rangle + \langle c_i, w_n \rangle \\
= \gamma_i + \langle (I - P)c_i, (I - P)u_n \rangle - \langle (I - P)c_i, (I - P)u_n \rangle + \langle (I - P)c_i, w_n \rangle \\
\leq \gamma_i + \beta_i - \gamma_i = \beta_i.
\]

This means that $(y_n)$ is a minimizing sequence for the problem (1), (3) (moreover, $y_n \in U_*$). Let us now note that

\[
\|v_n - y_n\| = \|Pu_* + (I - P)u_n - Pu_* - (I - Q)(I - P)u_n - w_n\| \\
= \|Q(I - P)u_n - w_n\| = \|Qv_n - w_n\|.
\]

Finally, by (f10), we obtain

\[
d(u_n, u_*) \leq \|u_n - y_n\| \leq \|u_n - v_n\| + \|v_n - y_n\| \\
= \|Pu_n - Pu_*\| + \|Qv_n - w_n\| \to 0, \; n \to \infty,
\]

hence, the well-posedness of the problem (1), (3) is proved.

Suppose conversely, that the problem (1), (3) is well-posed in the sense of Tikhonov. It is necessary to prove that the operator $A$ is normal solvable. Let us suppose conversely that $R(A^*) \neq \bar{R(A^*)}$. Then, according to Lemma 2, there exists a sequence $p_n$ such that

\[(\text{f11}) \quad p_n \in \bar{R(A_*)}, \; \|p_n\| = 1, \; p_n \to 0, \; Ap_n \to 0, \; n \to \infty. \]

Let $c_1, \ldots, c_k$ be some base of the system $c_1, \ldots, c_m$. Define the sequences $(\lambda_{n_1}), \ldots, (\lambda_{n_k})$ so that for the elements

\[
v_n = u_* + p_n + \sum_{i=1}^{k} \lambda_n c_i
\]
we have
\[ \langle v_n, c_i \rangle = \langle u_*, c_i \rangle, \quad i = 1, \ldots, m. \]

These relations form a system of linear equations
\[ \lambda_{n_1} \langle c_1, c_i \rangle + \lambda_{n_2} \langle c_2, c_i \rangle + \cdots + \lambda_{n_k} \langle c_k, c_i \rangle = -\langle p_n, c_i \rangle, \quad i = 1, \ldots, m. \]

This system is equivalent to the shortened system
\[ (f12) \quad \lambda_{n_1} \langle c_1, c_i \rangle + \lambda_{n_2} \langle c_2, c_i \rangle + \cdots + \lambda_{n_k} \langle c_k, c_i \rangle = -\langle p_n, c_i \rangle, \quad i = 1, \ldots, k. \]

The system (f12) has a unique solution \( \lambda_{n_1}, \ldots, \lambda_{n_k} \); moreover, by virtue of (f11), we have
\[ \lim_{n \to \infty} \lambda_{n_i} = 0, \quad i = 1, \ldots, k. \]

Thus we see that \( (v_n) \) is a minimizing sequence; however, by (f11), we derive that
\[ d^2(v_n, U_*) \geq ||p_n||^2 - \sum_{i=1}^{k} \lambda_{n_i}^2 ||c_i||^2 \to 1 \text{ as } n \to \infty. \]

Therefore, we have constructed a minimizing sequence \( (v_n) \) that does not converge to the solution set \( U_* \), but this is impossible under the above assumption of the well-posedness of the problem (1), (3). This completes the proof. \( \square \)

References