OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SOME DIFFERENCE EQUATIONS

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Abstract. We consider the oscillation and asymptotic behaviour of nonoscillatory solutions of a class of nonlinear difference equations.

1. Introduction

We consider a nonlinear difference equation

$$\Delta (r_n \Delta (u_n + p_n u_{n-k})) = q_n f(u_{n-l}), \quad n = 0, 1, 2, \ldots$$

(1)

where $\Delta$ denotes the forward difference operator, i.e., $\Delta v_n = v_{n+1} - v_n$ for any sequence $(v_n)$ of real number, $k$ and $l$ are nonnegative integers, $(p_n)$ and $(q_n)$ are sequences of real numbers with $q_n \geq 0$ eventually, $(r_n)$ is a sequence of positive numbers and

$$\sum_{n=0}^{\infty} \frac{1}{r_n} = \infty.$$  

(2)

The function $f$ is a real valued function satisfying $uf(u) > 0$ for $u \neq 0$.

By a solution of (1) we mean a sequence $(u_n)$ which is defined for $n \geq -\max\{k, l\}$ and satisfies (1) for all large $n$. A nontrivial solution $(u_n)$ of (1) is said to be oscillatory if for every positive integer $n_0$ there exists $n \geq n_0$ such that $u_n u_{n+1} \leq 0$. Otherwise it is called nonoscillatory.

Recently, there has been considerable interest in the study of oscillation and asymptotic behaviour of solutions of difference equations; see for example [2], [3], [5–15] and the references cited therein. For the general theory of difference equations one can refer to [1] and [4].

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Our purpose is to study the oscillatory and asymptotic behaviour of nonoscillatory solutions of equation (1). The obtained results extend those contained in [14].

2. Main results

Here we give some oscillatory and asymptotic properties of solution of (1).
We will need the following assumptions:

\[ f(u) \text{ is bounded away from zero if } u \text{ is bounded away from zero,} \]
\[ \sum_{n=0}^{\infty} q_n = \infty. \]

The following lemma describes some asymptotic properties of the sequence \( (z_n) \) defined as follows:

\[ z_n = u_n + p_n u_{n-k}, \]

where \( (u_n) \) is a nonoscillatory solution of (1).

**Lemma.** Assume that (3) and (4) hold and there exists a constant \( P_1 \) such that \( P_1 \leq p_n \leq 0 \).

(a) If \( (u_n) \) is an eventually positive solution of (1), then the sequences \( (z_n) \) and \( (r_n \Delta z_n) \) are eventually monotonic and either

\[ \lim_{n \to \infty} z_n = \lim_{n \to \infty} r_n \Delta z_n = \infty \]

or

\[ \lim_{n \to \infty} z_n = \lim_{n \to \infty} r_n \Delta z_n = 0, \quad \Delta z_n < 0 \text{ and } z_n > 0. \]

(b) If \( (u_n) \) is an eventually negative solution of (1), then the sequences \( (z_n) \) and \( (r_n \Delta z_n) \) are eventually monotonic and either

\[ \lim_{n \to \infty} z_n = \lim_{n \to \infty} r_n \Delta z_n = -\infty \]

or

\[ \lim_{n \to \infty} z_n = \lim_{n \to \infty} r_n \Delta z_n = 0, \quad \Delta z_n > 0 \text{ and } z_n < 0. \]

**Proof.** Let \( (u_n) \) be an eventually positive solution of (1), say \( u_{n-k} > 0 \) and \( u_{n-l} > 0 \) for \( n \geq n_0 \). From (1) we have

\[ \Delta (r_n \Delta z_n) = q_n f(u_{n-l}) \geq 0 \text{ for } n \geq n_0 \]

that is \( (r_n \Delta z_n) \) is nondecreasing, which implies that \( (\Delta z_n) \) is eventually of constant sign and in consequence \( (z_n) \) is eventually monotonic.
First suppose there exists \( n_1 \geq n_0 \) such that \( \Delta z_{n_1} \geq 0 \), then since \( q_n \equiv 0 \) eventually, there exists \( n_2 \geq n_1 \) such that \( r_n \Delta z_n \geq r_{n_2} \Delta z_{n_2} = c > 0 \) for \( n \geq n_2 \). Summing the above inequality, by (2) we have

\[
z_n \geq z_{n_2} + c \sum_{i=n_2}^{n-1} \frac{1}{r_i} \to \infty \quad n \to \infty,\]

hence \( z_n \to \infty \) as \( n \to \infty \).

Since \( u_n \geq z_n \), so \( u_n \to \infty \) as \( n \to \infty \). Then summing (10) we get

\[
r_n \Delta z_n = r_{n_2} \Delta z_{n_2} + \sum_{i=n_2}^{n-1} q_i f(u_{i-1})
\]

which in view of (3) and (4), implies that \( r_n \Delta z_n \to \infty \) as \( n \to \infty \), and thus (6) holds.

Now, if \( \Delta z_n < 0 \) for \( n \geq n_0 \), then \( r_n \Delta z_n \to L \leq 0 \) as \( n \to \infty \). Summing (10) from \( n \) to \( m \) and letting \( m \to \infty \) gives

\[
\sum_{i=n}^{\infty} q_i f(u_{i-1}) = L - r_n \Delta z_n < \infty.
\]

The last inequality together with (3) and (4) implies

\[
\lim_{n \to \infty} \inf \ u_n = 0. \tag{11}
\]

Suppose that \( L < 0 \). Then we have \( r_n \Delta z_n \leq L \) for \( n \geq n_0 \). Also, we can choose \( n_3 \geq n_0 \) such that \( z_{n_3} < 0 \). Summing the above inequality we get

\[
z_n \leq z_{n_3} + L \sum_{i=n_3}^{n-1} \frac{1}{r_i} < L \sum_{i=n_3}^{n-1} \frac{1}{r_i} \quad \text{for } n > n_3
\]

and, by assumption, we obtain

\[
P_1 u_{n-k} \leq p_n u_{n-k} < z_n < L \sum_{i=n_3}^{n-1} \frac{1}{r_i}, \quad n > n_3
\]

so

\[
u_{n-k} > \frac{L}{P_1} \sum_{i=n_3}^{n-1} \frac{1}{r_i} \to \infty \quad n \to \infty,
\]

which contradicts (11). Thus \( \lim_{n \to \infty} r_n \Delta z_n = 0 \). Next we show that \( z_n > 0 \) for \( n \geq n_0 \). If not, then there exists \( n_4 \geq n_0 \) such that \( z_{n_4} \leq 0 \), then since \( \Delta z_n < 0 \) for \( n \geq n_0 \), \( z_n < z_{n_3} < 0 \) for \( n \geq n_5 \geq n_4 \) that is

\[
u_n < z_{n_5} - p_n u_{n-k} \quad \text{for } n \geq n_5 \tag{12}
\]
By (11), there is an increasing sequence of positive integers \( (n_i) \) such that \( u_{n_i - k} \to 0 \) as \( i \to \infty \). This together with the assumption about \( (p_n) \) and (12) implies that there exists \( i_0 \) such that \( u_{n_{i_0}} < z_{n_0}/2 < 0 \), contradicting \( u_n > 0 \) eventually.

Since \( (z_n) \) is decreasing, \( z_n \to L_1 \geq 0 \). If \( L_1 > 0 \), then \( u_n \geq z_n \geq L_1 \), contradicting (11). Thus (7) holds and (a) is proved.

The proof of (b) is similar to that of (a) and hence will be omitted.

**Theorem 1.** Suppose that (3) and (4) holds. If there exists a constant \( P_2 \) such that \( P_2 \leq p_n \leq -1 \), then every nonoscillatory solution \( (u_n) \) of (1) satisfies \( |u_n| \to \infty \) as \( n \to \infty \).

**Proof.** If \( (u_n) \) is an eventually positive solution of (1) such that \( (u_n) \) does not tend to \( \infty \) as \( n \to \infty \), then (6) cannot hold since \( z_n \leq u_n \) eventually. Thus, by Lemma (a) (7) holds. Moreover, from the proof of (7) we have (11) holding. But

\[
0 < z_n = u_n + p_n u_{n-k} \leq u_n - u_{n-k},
\]

so \( u_n > u_{n-k} \) which contradicts (11). This completes the proof for \( u_n > 0 \). The proof is similar when \( (u_n) \) is eventually negative.

From Theorem 1 we immediately obtain

**Corollary 1.** Under the assumptions of Theorem 1 all bounded solutions of (1) are oscillatory.

**Theorem 2.** Suppose that there exists a constant \( P_3 \) such that \( -1 < P_3 \leq p_n \leq 0 \) and that \( f \) is a nondecreasing continuous function such that

\[
\int_0^a \frac{du}{f(u)} < \infty, \quad a > 0.
\]

(13)

If

\[
\sum_{n=n_0}^{\infty} \frac{1}{r_{n-l}} \sum_{i=n-l}^{n} q_i = \infty,
\]

(14)

then every nonoscillatory solution \( (u_n) \) of (1) satisfies either \( |u_n| \to \infty \) or \( u_n \to 0 \) as \( n \to \infty \).

**Proof.** Assume that \( (u_n) \) is an eventually positive solution of (1) which does not satisfy our assertion. Then for \( (z_n) \) defined in (5) we see from (1), that \( \Delta(r_n \Delta z_n) \geq 0 \) eventually that is \( (r_n \Delta z_n) \) is nondecreasing and \( (z_n) \) is eventually monotonic. Now if \( (z_n) \) is eventually nonpositive, then the assumption concerning \( (p_n) \) implies \( u_n \leq -p_n u_{n-k} \leq -P_3 u_{n-k} \) so \( u_{n+k} < -P_3 u_n \) for all \( n \) sufficiently large, say for \( n \geq n_0 \). It then follows by induction that for all \( n \geq n_0 \) we have \( u_{n+i k} < (-P_3)^i u_n \) for every positive integer \( i \). Since \( 0 < -P_3 < 1 \), the last inequality implies that \( u_n \to 0 \) as \( n \to \infty \) which contradicts our assumption. Also, if there exists \( n_1 \geq n_0 \) such that \( \Delta z_{n_1} \geq 0 \), then there is \( n_2 \geq n_1 \) such that
\[ r_n \Delta z_n \geq r_{n+1} \Delta z_{n+1} > 0 \quad \text{for} \quad n \geq n_2 \quad \text{which, by (2), implies that} \quad z_n \to \infty \quad \text{as} \quad n \to \infty. \]

Since \( u_n \geq z_n \) we have \( u_n \to \infty \) as \( n \to \infty \), again a contradiction to our assumptions on \( (u_n) \).

Therefore we have \( z_n > 0 \) and \( \Delta z_n < 0 \) for \( n \geq n_0 \). Since \( 0 < z_n \leq u_n \) and \( f \)

is nondecreasing from (1) we get

\[
\Delta (r_n \Delta z_n) \geq q_n f(z_{n-1}) \quad \text{for} \quad n \geq n_1 = n_0 + 1.
\]

Summing the above inequality we obtain

\[
r_{n+1} \Delta z_{n+1} - r_{n-1} \Delta z_{n-1} \geq \sum_{i=n-1}^{n} q_i f(z_{i-1})
\]

and so

\[
\sum_{i=n-1}^{n} q_i f(z_{i-1}) \leq -r_{n-1} \Delta z_{n-1} \quad n \geq n_1.
\]

In view of monotonicity of \( (z_n) \) and \( f \) we see that

\[
f(z_{n-1}) \sum_{i=n-1}^{n} q_i \leq -\Delta z_{n-1},
\]

and further

\[
\frac{1}{r_{n-1}} \sum_{i=n-1}^{n} q_i \leq -\frac{\Delta z_{n-1}}{f(z_{n-1})} \leq \int_{z_{n+1}}^{z_{n+1}} \frac{du}{f(u)}, \quad n \geq n_1.
\]

Summing the last inequality from \( n_1 \) to \( n \) by (13) we get

\[
\sum_{j=n_1}^{n} \frac{1}{r_j} \sum_{i=n-1}^{n} q_i \leq \int_{z_{n+1}}^{z_{n+1}} \frac{du}{f(u)} < \int_{0}^{z_{n+1}} \frac{du}{f(u)} < \infty,
\]

which contradicts (14). The proof is similar when \( (u_n) \) is eventually negative.

**Corollary 2.** Under the assumptions of Theorem 2 any bounded solution of (1) is either oscillatory or tends to zero as \( n \to \infty \).

**Theorem 3.** Assume that there exist constants \( P_3 \) and \( P_4 \) such that either

\[
-1 < P_3 \leq p_n \leq 0 \quad (15)
\]

or

\[
0 \leq p_n \leq P_4 < 1. \quad (16)
\]
Then every unbounded solution \((u_n)\) of (1) is either oscillatory or satisfies \(|u_n| \to \infty\) as \(n \to \infty\).

**Proof.** Let \((u_n)\) be an unbounded solution of (1) which is eventually positive, say \(u_{n-k} > 0\) and \(u_{n-i} > 0\) for \(n \geq n_0\). Then as before we have \(\Delta(r_n \Delta z_n) \geq 0\) for \(n \geq n_0\), so \((r_n \Delta z_n)\) is nondecreasing and hence \((z_n)\) is monotonic.

First assume that (15) holds. Then it follows that \(z_n > 0\) for \(n \geq n_1 \geq n_0\). Otherwise, there exists \(n_2 \geq n_1\) such that \(u_n + P_n u_{n-k} = z_n \leq 0\) for \(n \geq n_2\) and (15) implies that \(u_n \leq -P_3 u_{n-k} \leq u_{n-k}\). This implies that \((u_n)\) is bounded, a contradiction.

Further we claim that \((\Delta z_n)\) is eventually positive. Otherwise, \((z_n)\) is decreasing and hence is bounded from above, say \(0 < z_n \leq M\) for some constant \(M\). Therefore \(u_n = z_n - P_n u_{n-k} \leq M - P_3 u_{n-k}\). Since \((u_n)\) is unbounded there is an increasing sequence of positive integers \((n_i)\) such that \(u_{n_i} \to \infty\) as \(i \to \infty\) and \(u_{n_i} = \max_{n_1 \leq n \leq n_i} u_n\). Then we have

\[
u_{n_i} \leq M - P_3 u_{n_i-k} \leq M - P_3 u_{n_i},
\]

so \((1 + P_3) u_{n_i} \leq M\) for all \(i\) which is impossible in view of (15).

Finally, observe, as in the proof of Lemma, that \((r_n \Delta z_n)\) nondecreasing and \((\Delta z_n)\) eventually positive implies that \(z_n \to \infty\) as \(n \to \infty\) and hence \(u_n \to \infty\) as \(n \to \infty\) since \(u_n \geq z_n\).

Now assume that (16) holds. Then it is clear that \(z_n > 0\) for \(n \geq n_0\). Also we see that \((\Delta z_n)\) is eventually positive. In fact, if not, then \((z_n)\) is decreasing and so is bounded from above and since \(z_n \geq u_n\) \((u_n)\) is bounded, a contradiction.

As previously we conclude that \(z_n \to \infty\) as \(n \to \infty\). Since \(z_n \leq u_n + P_3 z_{n-k} \leq u_n + P_4 z_n\) we have \((1 - P_3) z_n \leq u_n\) which in view of (16), implies \(u_n \to \infty\) as \(n \to \infty\).

A similar argument treats the case of eventually negative solution.

**Theorem 4.** Suppose that there exist constants \(P_3\) and \(P_4\) such that \(P_3 \leq P_n \leq P_4 < -1\) and \(f\) is a nondecreasing continuous function such that

\[
\int_{\epsilon}^{\infty} \frac{du}{f(u)} < \infty, \quad \int_{-\epsilon}^{\infty} \frac{du}{f(u)} < \infty, \quad \epsilon > 0.
\]

If

\[
\sum_{n=n_0}^{\infty} \frac{1}{r_{n-l}} \sum_{i=n-k+1}^{\infty} q_i = \infty \quad \text{when } l \geq k,
\]

or

\[
\sum_{n=n_0}^{\infty} \frac{1}{r_{n-l}} \sum_{i=n}^{\infty} q_i = \infty \quad \text{when } l < k
\]

then all bounded solutions of (1) are oscillatory.
Proof. Assume that there exists a bounded nonoscillatory solutions \( (u_n) \) of (1) and let \( u_n > 0 \) eventually, say \( u_{n-k-1} > 0 \) for \( n \geq n_0 \). Then as before for the sequence \( (z_n) \) defined in (5) it follows that \( (r_n \Delta z_n) \) is a nondecreasing sequence and in consequence \( (z_n) \) is eventually monotonic. We show first that \( (z_n) \) is eventually negative. If there exists \( n_1 \geq n_0 \) such that \( z_{n_1} > 0 \), then by the assumptions we get \( u_{n_1} = z_{n_1} - p_{n_1} u_{n_1-k} > -P_0 u_{n_1-k} \). Then it follows by induction that \( u_{n_1+ik} > (-P_0)^i u_{n_1} \), which implies \( u_{n_1+ik} \to \infty \) as \( i \to \infty \) contradicting the boundedness of \( (u_n) \). Therefore \( (z_n) \) is eventually negative, say for \( n \geq n_0 \). Now we observe that \( \Delta z_n < 0 \) for \( n \geq n_0 \). If not, then a similar argument as in the proof of Lemma leads to the fact that \( z_n \to \infty \) contradicting \( z_n < 0 \) for \( n \geq n_0 \). By assumption, we have \( P_5 u_{n-k} \leq p_n u_{n-k} < z_n < 0 \), which implies that \( 0 < z_{n+k}/P_5 < u_n \) for \( n \geq n_0 \).

In view of monotonicity of \( f \) from (1) we see that

\[
\Delta (r_n \Delta z_n) \geq q_n f \left( \frac{z_{n+k-l}}{P_5} \right) \quad \text{for} \quad n \geq n_1 = n_0 + l \tag{20}
\]

Summing (20) from \( n-k \) to \( m > n-k \) we obtain

\[
r_{m+1} \Delta z_{m+1} - r_{n-k} \Delta z_{n-k} \geq \sum_{i=n-k}^{m} q_i f \left( \frac{z_{i+k-l}}{P_5} \right).
\]

After letting \( m \to \infty \), we have

\[
-r_{n-k} \Delta z_{n-k} \geq \sum_{i=n-k}^{\infty} q_i f \left( \frac{z_{i+k-l}}{P_5} \right) \geq \sum_{i=n-k+1}^{\infty} q_i f \left( \frac{z_{i+k-l}}{P_5} \right),
\]

from which we get

\[
-r_{n-k} \Delta z_{n-k} \geq f \left( \frac{z_{n+1-l}}{P_5} \right) \sum_{i=n-k+1}^{\infty} q_i \tag{21}
\]

Since \( (r_n \Delta z_n) \) is nondecreasing, for \( l \geq k \) we have \( r_{n-l} \Delta z_{n-l} \leq r_{n-k} \Delta z_{n-k} \) and further from (21) we obtain

\[
\frac{1}{r_{n-l}} \sum_{i=n-k+1}^{\infty} q_i \leq -\frac{\Delta z_{n-l}}{f \left( \frac{z_{n+1-l}}{P_5} \right)} \quad \text{for} \quad n \geq n_1 \tag{22}
\]

In view of monotonicity of \( (z_n) \) and \( f \) for \( z_{n-l}/P_5 \leq u \leq z_{n+1-l}/P_5 \) we have

\[
\frac{1}{f(u)} \geq \frac{1}{f \left( \frac{z_{n+1-l}}{P_5} \right)}
\]
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and so

\[ \int_{z_{n-1}/P_b}^{z_{n+1-i}/P_b} \frac{du}{f(u)} \geq \frac{1}{P_b} \frac{\Delta z_{n-l}}{f\left(\frac{z_{n+1-l}}{P_b}\right)} \quad \text{for } n \geq n_1. \tag{23} \]

Now using (23) in (22) and summing both sides from \( n_1 \) to \( n \) we get

\[ \sum_{j=n_1}^{n} \frac{1}{r_j} \sum_{l=j-k+1}^{\infty} q_l \leq -P_b \int_{z_{n-1}/P_b}^{z_{n+1-i}/P_b} \frac{du}{f(u)}, \quad n \geq n_1 \]

which in view of (17) contradicts the condition (18).

If \( l < k \), then summing (20) from \( n \) to \( m > n \) and letting \( m \to \infty \) we obtain

\[ -r_n \Delta z_n \geq \sum_{i=n}^{\infty} q_i f\left(\frac{z_i+k-l}{P_b}\right) \geq f\left(\frac{z_{n+k-l}}{P_b}\right) \sum_{i=n}^{\infty} q_i. \tag{21} \]

Since \( n+k-l \geq n+1 \), it follows that

\[ f\left(\frac{z_{n+1}}{P_b}\right) \leq f\left(\frac{z_{n+k-l}}{P_b}\right). \]

Therefore from (24) we get

\[ \frac{1}{r_n} \sum_{i=n}^{\infty} q_i \leq -\frac{\Delta z_n}{f\left(\frac{z_{n+1}}{P_b}\right)} \quad \text{for } n \geq n_1 \]

and the rest of the proof follows analogously to that as above.

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