REGULARLY VARYING SEQUENCES
AND ENTIRE FUNCTIONS OF FINITE ORDER

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Abstract. We present a method for estimating the asymptotic behavior of:

\[ f^\alpha(x) := \sum_{n=1}^{\infty} n^\alpha l_n a_n x^n, \quad x \to \infty, \quad \alpha \in R, \]

related to a given entire function \( f(x) := \sum_{n=1}^{\infty} a_n x^n \) of finite order \( \rho, 0 < \rho < +\infty, \)
\( a_n \geq 0, \ n \in N; \) where \((l_n), \ n \in N, \) are slowly varying sequences in Karamata’s sense.

Preliminaries

A. Slowly varying functions \( l(x) \) in Karamata’s sense are defined on a positive part of real axis, positive, locally bounded and satisfy: \( \lim_{x \to \infty} \frac{l(\lambda x)}{l(x)} = 1, \) for each \( \lambda > 0. \)

The class \( R_\alpha \) of regularly varying functions (r.v.f.) with index \( \alpha \) consists of all functions \( a(x) \) which can be represented as: \( a(x) = x^\alpha l(x), \) for some \( \alpha \in R. \)

The theory of r.v.f. is very well developed and an excellent survey of results is given in [1] and [3].

Here we put special attention on a class \( SR_\alpha \subset R_\alpha \) (smoothly varying functions; [1, p. 44]) i.e., \( b(x) \in SR_\alpha \) if it is a \( C^\infty \) r.v.f. of index \( \alpha, \) satisfying
\[ x^n b^{(n)}(x)/b(x) \to \alpha(\alpha-1) \cdots (\alpha-n+1) = (\alpha)_n, \quad x \to \infty, \quad n \in N. \]

Some important properties of this class are:

If \( f \in SR_\alpha, \ g \in SR_\beta, \) then
\[ f \cdot g \in SR_{\alpha+\beta}; \ f \circ g \in SR_{\alpha\beta}; \ f' \in SR_{\alpha-1}, \ \alpha \in R^+. \]

Also, for a given \( c(x) = x^{-\alpha} l(x), \ \alpha \in R^+, \) we consider its dual \( c^*(x) \) defined by
\[ c^*(x) := \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xy} c(1/y) \frac{dy}{y} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xy} y^{\alpha-1} l(1/y) \frac{dy}{y}; \quad x, \alpha \in R^+. \]

The next proposition is of crucial importance.

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**Proposition 1.** We have: \( c^*(x) \in SR_{-\alpha}; \ c^*(x) \sim c(x), \ x \to \infty. \)

**Proof.** That \( c^*(x) \sim c(x), \ x \to \infty, \) is a consequence of Karamata’s Tauberian theorem for Laplace transforms (with \( 0^+ \) and \( \infty \) reversed) (cf. [1, p. 43]).

By the same argument
\[
(c^*)(x) = \left( \frac{-1}{\Gamma(\alpha)} \right) \int_0^\infty e^{-xy} y^{\alpha+n-1} \frac{e^{y}}{y} dy \sim \left( \frac{-1}{\Gamma(\alpha)} \right) x^{-\alpha+n} l(x), \ x \to \infty.
\]

Hence,
\[
\frac{x^n(c^*(x))}{c^*(x)} \sim \left( \frac{-1}{\Gamma(\alpha)} \right) x^{-\alpha+n} = (-\alpha)_n, \ x \to \infty;
\]
i.e., \( c^*(x) \in SR_{-\alpha}. \)

We could treat regularly varying sequences (r.v.s.) as r.v.f. defined on \( N \) (see [2]) i.e., \( (a_n) \) is a r.v.f. with index \( \alpha \) if it has the form
\[
a_n = n^\alpha l_n; \ l_n = l(n), \ n \in N, \ \alpha \in R,
\]
for some slowly varying function \( l(x) \) defined for \( x \in R^+. \)

Examples of \( l_n \) are:
\[
\ln^a 2n, \ \ln^b (\ln 3n), \ e^{(\ln n)^c}, \ e^{n^{1/\ln n}}, \ldots; \ a, b \in R, \ 0 < c < 1.
\]

**B.** Denote by \( \mathcal{G} := \{g | g : R^+ \to R^+, \ g \in C^1\}, \) and define there an operator \( \hat{g}, \)
\[
\hat{g}(x) := \frac{xg'(x)}{g(x)}.
\]

Some properties of this operator are \((g, h) \in \mathcal{G};\)

1. \( \hat{g}_y = \hat{g}, \ c \in R^+; \)
2. \( \hat{a}^a = a, \ a \in R; \)
3. \( \hat{g} + \hat{h} \leq \max|\hat{g}|, |\hat{h}|; \)
4. \( \hat{g} \cdot \hat{h} = \hat{g} + \hat{h}; \)
5. \( \hat{g} \cdot h^b = a \hat{g} + b \hat{h}, \ a, b \in R; \)
6. \( \hat{g} \circ \hat{h} = (\hat{g} \circ h) \cdot \hat{h}; \)
7. \( \hat{g} \in \mathcal{G} \Rightarrow g \uparrow, \ x \in R^+; \)
8. \( \hat{g}(x) \to \alpha, \ x \to \infty, \alpha \in R \Rightarrow g \in R_\alpha. \)

We also consider a set of entire functions \( F(z) = \sum_{k=0}^{\infty} a_k z^k, \) with non-negative coefficients and of finite order \( \rho, \ 0 < \rho < \infty. \) By definition:
\[
\rho = \lim_{x \to \infty} \sup \left( \frac{\ln \ln M_F(x)}{\ln x} \right)
\]
where \( M_F(x) \) denotes the maximum modulus of \( F(z) \) on the circle \( |z| = x. \)

In our case we have:
\[
M_F(x) = \max_{|z|=x} |F(z)| = \max_{|z|=x} \left| \sum_{k=0}^{\infty} a_k z^k \right| = \sum_{k=0}^{\infty} a_k x^k = F(x), \ x \in R^+.
\]

Let us denote: \( f(x) := M_F(x) = (F(x), \ x \in R^+). \) Hence:
\[
\lim_{x \to \infty} \sup \left( \frac{\ln \ln f(x)}{\ln x} \right) = \rho, \ \rho \in R^+, \ \rho \in C^\infty; \ f^{(n)} \in \mathcal{G}; \ \hat{f} \in \mathcal{G}.
\]
Proposition 2. We have \( \hat{f} \in G \).

Proof. Taking into account properties (1) and (2), we have:

\[
\hat{f} = x f' - \hat{f} = \frac{1}{f} \left( \frac{x(xf')'}{f} - \hat{f}^2 \right).
\]

Since

\[
x(xf')' = \sum_{k=0}^{\infty} k^2 a_k x^k; \quad \hat{f} = \frac{1}{f} \sum_{k=0}^{\infty} k a_k x^k,
\]

we obtain:

\[
\hat{f} = \frac{1}{f f} \sum_{k=0}^{\infty} (k - \hat{f})^2 a_k x^k > 0, \quad x \in R^+;
\]

and the proof is over.

Corollary 1. The function \( \hat{f} \) is monotone increasing on \( R^+ \).

Proposition 3. We have: \( \lim_{x \to \infty} \frac{\ln \hat{f}(x)}{\ln x} = \rho \).

Proof. Let

\[
\delta := \lim_{x \to \infty} \frac{\ln \hat{f}(x)}{\ln x}, \quad \delta \in R^+.
\]

From (2) it follows that, for each positive \( \epsilon \) and large enough \( x \):

\[
\ln f(x) < x^{\rho + \epsilon}, \quad x > x_0.
\]

Corollary 1 gives

\[
\ln f(ex) - \ln f(x) = \int_{x}^{ex} \frac{f'(t)}{f(t)} dt = \int_{x}^{ex} \hat{f}(t) \cdot \frac{dt}{t} > \hat{f}(x) \int_{x}^{ex} \frac{dt}{t} = \hat{f}(x),
\]

hence, for \( x > x_0 \), we get:

\[
\hat{f}(x) < \ln f(ex) < (ex)^{\rho + \epsilon},
\]

i.e.,

\[
\frac{\ln \hat{f}(x)}{\ln x} < (\rho + \epsilon) (1 + 1/\ln x), \quad x > x_0.
\]

Since \( \epsilon \) is arbitrarily small, we conclude \( \delta \leq \rho \).

From the other side \( \ln \hat{f}(x) < (\delta + \epsilon) \ln x \) for \( x > x_1 \), i.e.,

\[
\hat{f}(x) < x^{\delta + \epsilon}, \quad \frac{f'(x)}{f(x)} < x^{\delta - 1 + \epsilon}, \quad x > x_1.
\]
It follows that
\[
\ln f(x) = \int_{x_1}^{x} \frac{f'(t)}{f(t)} \, dt + \ln f(x_1) < \frac{x^{\delta + \epsilon}}{\delta + \epsilon} + O(1), \quad x > x_1;
\]
i.e.,
\[
\frac{\ln \ln f(x)}{\ln x} < \delta + \epsilon + o(1), \quad x \to \infty;
\]
i.e.,
\[
\rho \leq \delta.
\]
Thus, we conclude:
\[
\limsup_{x \to \infty} \frac{\ln \hat{f}(x)}{\ln x} = \limsup_{x \to \infty} \frac{\ln f(x)}{\ln x} = \rho.
\]

**Corollary 2.** The function \(\hat{f}(x)\) is strictly increasing on \(R^+\) and \(\lim_{x \to \infty} \hat{f}(x) = +\infty\).

We also consider the set of entire functions \(\{f_m\}\) generated from \(f\) by the recurrence relation
\[
f_m(x) := x f_{m-1}'(x), \quad f_0(x) = f(x), \quad m \in N.
\]
They are of the same order \(\rho\) and evidently satisfy:

**Proposition 4.** \(f_m(x) = \sum_k k^{m} a_k x^k; \quad \hat{f}_m(x) \uparrow \infty; \quad \hat{f}_m(x) > 0\)

\[
f_m = f_{m-1} \hat{f}_{m-1} = f \prod_{k=1}^{m} \hat{f}_{k-1}; \quad \hat{f}_m = \hat{f}_{m-1} + \hat{f}_{m-1} = \hat{f} + \sum_{k=1}^{m} \hat{f}_{m-1}.
\]

**Main results**

Now we come to our main subject, i.e., the investigation of the asymptotic behavior concerning functions \(f^\alpha(x) := \sum_{k=0}^{\infty} c_k a_k x^k\), related to a given entire function \(f(x) := \sum_{k=0}^{\infty} a_k x^k\) considered before and where \((c_k), (\alpha) := 1\) is any regularly varying sequence of index \(\alpha\).

It is not difficult to prove that \(\{f^\alpha(x)\}\) are also entire functions of the same order \(\rho\) as \(f(x)\) (using, for example, the relation: \(\rho = \limsup_{n \to \infty} \frac{\ln n}{\ln|a_n|}\)).

The main idea of our method is to replace sequences \((c_k)\) with asymptotically equivalent \((c_k^*)\) achieving thus an integral representation for \(f^\alpha(x)\) (see also [5]). Then, using analytic properties of \(c^*(x)\) and \(f(x)\), we establish the required asymptotic behavior in an almost elementary way.
**THEOREM A.** If \( \hat{f}(x) \) is bounded from above, then:

\[
\frac{f^{\alpha}(x)}{f(x)} \sim c_{[f(x)]}, \quad x \to \infty,
\]

for any regularly varying sequence \((c_k)\) of index \(\alpha\), \(\alpha < 0\).

As we already explained, we first prove the theorem for a subclass of r.v.s. generated by \(c^*(x) \in SR_\alpha\), i.e.,

**PROPOSITION A1.** Theorem A is valid for sequences \((c_k^*)\) defined by

\[
c_{k-1}^* := c^*(k), \quad k \in \mathbb{N}; \quad c^*(x) := \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xt}t^{\alpha-1}l(1/t) \, dt, \quad x \in \mathbb{R}^+.
\]

**Proof.** With: \(u(\alpha, t) := \frac{1}{\Gamma(\alpha)} t^{\alpha-1}l(1/t)\) we produce an integral representation for:

\[
f^{(\alpha)}(x) := \sum_{k=0}^\infty c_k^* a_k x^k = \sum_{k=0}^\infty a_k \int_0^\infty e^{-t}u(\alpha, t)(xe^{-t})^k \, dt = \int_0^\infty e^{-t}u(\alpha, t) \left( \sum_{k=0}^\infty a_k (xe^{-t})^k \right) dt = \int_0^\infty e^{-t}u(\alpha, t)f(xe^{-t}) \, dt.
\]

The interchanging of the sum and the integral is justified since both converge for \(x \in \mathbb{R}^+\). Now:

\[
\frac{f^{(\alpha)}(x)}{f(x)} = \left( \int_0^\xi + \int_\xi^\infty \right) \left( e^{-t}u(\alpha, t) f(xe^{-t}) \frac{f(x)}{f(x)} \right) dt = T_1 + T_2
\]

where \(\xi = \xi(x) := \hat{f}(x)^{-1/2}\).

For estimating \(T_1\) we use the following identity:

\[
\ln \frac{f(xe^{-t})}{f(x)} + t\hat{f}(x) = \int_0^t w\hat{f}(a)\hat{f}(a) \, dw, \quad a := xe^{-t}.
\]

Taking into account Proposition 2 and condition from Theorem A, we have:

\[
0 < \hat{f}(a) \leq M < +\infty,
\]

where the constant \(M\) does not depend on \(a\).

Also, since \(a \leq x\), Corollary 1 gives \(\hat{f}(a) \leq \hat{f}(x)\), i.e.,

\[
0 < \int_0^t w\hat{f}(a)\hat{f}(a) \, dw \leq M\hat{f}(x) \int_0^t w \, dw = \frac{M}{2}\hat{f}(x)t^2.
\]
Hence,
\[ \ln \frac{f(xe^{-t})}{f(x)} = \hat{f}(x)(-t + O(t^2)), \quad x \in R^+, \quad t \geq 0 \]
where the constant in $O$ is independent of $x$ or $t$. From here it follows
\[
T_1 = \int_0^\xi e^{-t}u(\alpha, t) \exp \left( \ln \frac{f(xe^{-t})}{f(x)} \right) dt = \int_0^\xi e^{-t}u(\alpha, t)e^{-t\hat{f}(x)} e^{O(\hat{f}(x)t^2)} dt.
\]
Since, for any $B \in R^+, e^B = 1 + O(Be^B)$ and, for $t \in (0, \xi), \quad \hat{f}(x)t^2 = O(1)$, we obtain:
\[
T_1 = \int_0^\xi e^{-t}u(\alpha, t)e^{-t\hat{f}(x)} dt + \int_0^\xi e^{-t}u(\alpha, t)O(\hat{f}(x)t^2) dt = \int_0^\xi e^{-t(1+\hat{f}(x))}u(\alpha, t) dt - \int_\xi^\infty e^{-t}u(\alpha, t)e^{-t\hat{f}(x)} dt
\]
\[+ O(\hat{f}(x)) \int_0^\infty t^2u(\alpha, t)e^{-t(1+\hat{f}(x))} dt = T_{11} + T_{12} + T_{13}. \]

Now:
\[ T_{11} = c^*(\hat{f}(x) + 1) \sim c^*_{\hat{f}(x)}, \quad x \to \infty; \]
\[ |T_{12}| = O(e^{-\xi\hat{f}(x)} \int_0^\xi e^{-t}u(\alpha, t) dt) = O(e^{-\xi\hat{f}(x)}); \]
\[ T_{13} = O(\hat{f}(x)) \cdot \frac{d^2c^*(s)}{ds^2}_{[s=1+\hat{f}(x)]}
\]
\[ = O(\hat{f}(x)) \cdot O(c^*(s))_{[s=1+\hat{f}(x)]} = O\left(\frac{c^*(\hat{f}(x))}{\hat{f}(x)}\right), \]
since $c^*(s) \in SR_{\alpha}$. Hence, we conclude that: $T_1 \sim c^*_{\hat{f}(x)}, \quad x \to \infty$.

For the estimation of the integral $T_2$ the next lemma is necessary.

**Lemma A1.** Under the condition of Theorem A, i.e., sup $\hat{f}(x) \leq M < +\infty$, for each $x, t \in R^+$:
\[ \frac{f(xe^{-t})}{f(x)} \leq \exp\left(\frac{e^{-Mt} - 1}{M}\hat{f}(x)\right). \]

**Proof.** Write the condition as
\[ D(\ln \hat{f}(s)) \leq M D(\ln s), \quad s > 0. \quad (A1.1) \]
Integrating $(A1.1)$ over $[xe^{-u}, x], \quad u \geq 0$, we obtain
\[ \hat{f}(xe^{-u}) \geq \hat{f}(x) \cdot e^{-Mu}. \quad (A1.2) \]
Integrating (A1.2) for \( u \in [0,t] \), we come to the conclusion from the lemma. Therefore,

\[
T_2 = \int_\xi^\infty e^{-t} u(\alpha, t) \frac{f(xe^{-t})}{f(x)} \, dt \leq \int_\xi^\infty e^{-t} u(\alpha, t) \exp\left( -\frac{M}{M - 1} \hat{f}(x) \right) \, dt
\]

\[
< \exp\left( -\frac{1}{M} \right) \int_0^\infty e^{-t} u(\alpha, t) \, dt;
\]

i.e.,

\[
T_2 = O(e^{-\hat{f}(x)^{1/2}}), \quad x \to \infty;
\]

so, Proposition A1 is proved.

The assertion of Theorem A follows using the fact \( c_n \sim c_n^* \), \( n \to \infty \) and a variant of Toeplitz’s Limit Preservation Theorem (cf. [8, p. 36]) which says:

Let \( \{\phi_k(x)\}, k = 0, 1, 2, \ldots \), be a set of non-negative functions defined on \( R^+ \), satisfying \( \sum_k \phi_k(x) = 1 \), and let \( (s_k), k = 0, 1, 2, \ldots \) be any convergent sequence of positive reals, \( \lim s_k = s \).

Then a necessary and sufficient condition for \( \sum_k s_k \phi_k(x) \to s \), \( x \to \infty \), is \( \lim_{x \to \infty} \phi_k(x) = 0 \), for each fixed \( k \in N \).

We are going to use this proposition by putting:

\[
\phi_k(x) := \frac{c_k^* a_k x^k}{f^*(x)}; \quad s_k := \frac{c_k}{c^*}, \quad k = 0, 1, 2, \ldots
\]

Then,

\[
\sum_k \phi_k(x) = 1; \quad s = 1; \quad \sum_k s_k \phi_k(x) = \frac{f^{(\alpha)}(x)}{f^*(x)};
\]

and all we have to prove is \( \lim_{x \to \infty} \phi_n(x) = 0 \) for fixed \( n \).

For \( a_n \neq 0 \) (otherwise, there is nothing to prove) write

\[
\phi_n(x) = \frac{c_n^* a_n x^n}{f^*(x)} = c_n^* \left( \frac{a_n x^n}{f^{(2)}(x)} \right)^2 \left( \frac{f(x)}{f^*(x)} \right)^2.
\]

From Proposition A1:

\[
\frac{f(x)}{f^*(x)} \sim 1/c^* \hat{f}(x) = O(\hat{f}(x)^2|x|).
\]

Lemma A1, for \( t = \ln 2 \), gives

\[
\frac{f(x/2)}{f(x)} \leq \exp\left( -1 - \frac{2M}{M} \hat{f}(x) \right), \quad M > 0;
\]
and, evidently: \( f(x/2) > a_n(x/2)^n \). Hence,

\[
\phi_n(x) = O\left( \hat{f}(x)^{2|\alpha|} \exp\left(-\frac{1 - 2^{-M}}{M} \hat{f}(x)\right) \right) = o(1), \quad x \to \infty, \quad \hat{f}(x) \uparrow \infty,
\]

i.e.,

\[
f^{(\alpha)}(x) \sim f^{(*)}_{\mu}(x) \sim f(x) \quad \forall \, 0 < \alpha < 1; \quad x \to \infty; \quad \alpha < 0;
\]

therefore, Theorem A is valid.

Our task now is to extend the validity of Theorem A to non-negative indexes of r.v.s. \((\epsilon_k)\). First of all, we prove

**Proposition 5.** Under conditions of Theorem A, for any \( \alpha \geq 0 \) we have

\[
\lim \inf_{x \to \infty} \frac{f^{(\alpha)}(x)}{(f(x))^{\alpha q} l(f(x)) f(x)} \geq 1.
\]

**Proof.** We use a form of Hölder’s inequality:

\[
\sum_{k} u_k w_k \geq \left( \sum_{k} u_k^p \right)^{1/p} \left( \sum_{k} w_k^q \right)^{1/q}, \quad u_k, w_k \in R^+; \quad \frac{1}{p} + \frac{1}{q} = 1, \quad q < 0. \quad (7.1)
\]

Putting there

\[
u_k = k^{1}(i_k a_k x_k)^{1/p}; \quad w_k = \begin{cases} k^{\alpha + 1}(i_k a_k x_k)^{1/q}, & a_k > 0; \\ 0, & a_k = 0 \end{cases}, \quad k \in N, \]

we obtain:

\[
f^{(\alpha)}(x) \geq (f^{-p}(x))^{1/p} (f(q^{(\alpha+1)})(x))^{1/q}.
\]

Theorem A gives

\[
\lim \inf_{x \to \infty} \frac{f^{(\alpha)}(x)}{(f(x))^{\alpha q} l(f(x)) f(x)} \geq
\]

\[
f \geq \lim_{x \to \infty} \left( \frac{f^{-p}(x)}{(f(x))^{-p l(f(x)) f(x)}} \right)^{1/p} \lim_{x \to \infty} \left( \frac{f^{(q^{(\alpha+1)})(x)}}{(f(x))^{q^{(\alpha+1)} l(f(x)) f(x)}} \right)^{1/q} = 1.
\]

An extension of Theorem A is the following

**Theorem A’.** For any \( \epsilon > 0 \), Theorem A is valid for \( \alpha \leq 2 - \epsilon \).

**Proof.** Put in (7.1):

\[
u_k^p = k^{2-\epsilon} i_k a_k x_k; \quad w_k^q = \begin{cases} k^{2} a_k x_k^k, & a_k > 0; \\ 0, & a_k = 0 \end{cases}, \quad k \in N.
\]
Since we could restrict $0 < \epsilon \leq 2$, by taking $p = \epsilon/3$, it follows that
\[
\sum_k k^{2-\epsilon} l_k a_k x^k \leq f^{2/p}_2 \left( \sum_k k^{-1} l_k^{1/p} a_k x^k \right)^p, \quad k \in \mathbb{N}.
\]
But
\[
f_2 = f \hat{f} f_1 = f \hat{f} (\hat{f} + \hat{f}) \sim f \hat{f}^2, \quad x \to \infty,
\]
and
\[
\sum_k k^{-1} l_k^{1/p} a_k x^k \sim \hat{f}^{-1} (l(\hat{f}))^{1/p}, \quad x \to \infty.
\]
Hence,
\[
\limsup_{x \to \infty} \frac{f^{(2-\epsilon)}(x)}{(\hat{f}(x))^{2-\epsilon} l(\hat{f}(x)) f(x)} \leq \lim_{x \to \infty} \frac{(f(x) \hat{f}(x)^2 - p/q(f(x) \hat{f}(x)^{1/(l(\hat{f}(x)))^{1/p}} f(x))^p}{(\hat{f}(x))^{2-\epsilon} l(\hat{f}(x)) f(x)} = 1.
\]
This together with Proposition 5 proves the theorem.

Therefore we see that the Theorems A and A' provide the required asymptotic behavior of $f^{(\alpha)}(x)$ for all regularly varying sequences $(c_k)$ with index less than 2.

**Commentaries**

The condition $\sup \hat{f} < +\infty$ seems a little ambiguous but is not very restrictive, as we are going to show.

The explicit representation
\[
\hat{f}(x) = f(1) \exp \left( \int_1^x \frac{\hat{f}(t)}{t} dt \right); \quad 0 < \hat{f}(t) \leq M,
\]
means that $\hat{f}$ belongs to the class ER (Extended Variation, see [1, p. 74]).

Moreover, from $\hat{f}(x) = \hat{f}_1(x) - \hat{f}(x)$, we see that $\hat{f}$ is of bounded variation (as a difference between two monotone increasing functions), i.e., bounded on finite intervals. Therefore, condition in Theorem A could be replaced by
\[
\limsup_{x \to \infty} \hat{f}(x) < +\infty.
\]

Strengthening this a bit, we obtain:

**Proposition 6.** If there exist $\lim_{x \to \infty} \hat{f}(x) = \delta$, then $\delta = \rho$ and
\[
\lim_{x \to \infty} \frac{\ln \ln f(x)}{\ln x} = \lim_{x \to \infty} \frac{\hat{f}(x)}{\ln x} = \lim_{x \to \infty} \hat{f}(x) = \rho.
\]
Moreover, in this case $\hat{f} \in R_\rho$.

**Proof.** This follows from Proposition 2 and
\[
\hat{f}(x) = \frac{D(\ln \hat{f}(x))}{D(\ln x)} = \frac{D(x D(\ln f(x)))}{D(\ln f(x))}.
\]
The second part is Property 8 from (1) (cf. [1, p. 59]).

At this point we could connect asymptotically $\hat{f}$ with $\ln f$; namely:
PROPOSITION 7. The following are equivalent:

(i) \( \ln f(x) \sim a x^\rho b(x) \);
(ii) \( \hat{f}(x) \sim a \rho x^\rho b(x), \quad x \to \infty; \quad b(x) \in R_0, \quad a, \rho \in R^+ \).

Proof. (i) \( \Rightarrow \) (ii): Since \( \ln f(x) = \int_1^x \hat{f}(t) \cdot dt/t + O(1) \), the statement follows from r.v.f. Integration Theorem [3], i.e.,
\[
\ln f(x) \sim a \rho b(x) \int_1^x t^\rho \, dt/t + O(1) \sim a x^\rho b(x), \quad x \to \infty.
\]

(i) \( \Rightarrow \) (ii): Corollary 2 gives, for \( x \geq y > 0 \):
\[
\ln f(x) - \ln f(y) = \int_y^x \hat{f}(t) \, dt/t \begin{cases} \leq \hat{f}(x) \ln x/y, \\ \geq \hat{f}(y) \ln x/y, \end{cases} \tag{5.1}
\]
Putting in (5.1) \( x = \lambda y, \lambda > 1 \) and \( y = \lambda x, \lambda < 1 \), we get
\[
\hat{f}(x) \begin{cases} \leq \frac{\ln f(\lambda x) - \ln f(x)}{\ln \lambda}, \quad \lambda > 1 \\ \geq \frac{\ln f(x) - \ln f(\lambda x)}{\ln 1/\lambda}, \quad 0 < \lambda < 1. \end{cases}
\]
Therefore,
\[
\limsup_{x \to \infty} \frac{\hat{f}(x)}{a x^\rho b(x)} \leq \frac{1}{\ln \lambda} \left( \lim_{x \to \infty} \frac{\ln f(\lambda x)}{a x^\rho b(x)} - \lim_{x \to \infty} \frac{\ln f(x)}{a x^\rho b(x)} \right) = \frac{\lambda^\rho - 1}{\ln \lambda}, \quad \lambda > 1; \tag{5.2}
\]
and analogously:
\[
\liminf_{x \to \infty} \frac{\hat{f}(x)}{a x^\rho b(x)} \geq \frac{1 - \lambda^\rho}{\ln 1/\lambda}, \quad 0 < \lambda < 1. \tag{5.3}
\]
Since the right-hand side does not depend on \( x \), putting \( \lambda \downarrow 1 \) in (5.2) and \( \lambda \uparrow 1 \) in (5.3), we obtain the statement from Proposition 7.

Further extension needs some smoothness condition on \( f \), i.e.,

**Theorem B.** If \( \ln f(x) \in SR_\rho \), then
\[
\frac{f^{(\beta)}(x)}{f(x)} \sim \rho^\beta c_{\ln f(x)}, \quad x \to \infty;
\]
for any regularly varying sequence \( (c_k) \) of arbitrary index \( \beta \in R \).

For justification of the condition from this theorem we cite an adapted version of Valiron’s Proximate Order Theorem i.e., (cf. [1, p. 311]):

If \( f \) is an entire function of finite order \( \rho \), then there always exists a \( g \in SR_\rho \), with:
\[
\limsup_{x \to \infty} \frac{\ln f(x)}{g(x)} = 1.
\]

We prove first:
Proposition B1. If \( \ln f(x) \in SR_\rho \), then \( \ln f_m(x) \in SR_\rho \) and \( \lim_{x \to \infty} \frac{x^n(\ln f_{m-1}(x))}{\ln f_m(x)} = (\rho)_n \), \( x \to \infty \).

Proof. Suppose that \( \ln f_{m-1} \in SR_\rho \), i.e.,

\[
\frac{x^n(\ln f_{m-1})}{\ln f_{m-1}(x)} \to (\rho)_n, \ x \to \infty.
\]

Using properties of the class \( SR \) and Proposition 4 (see Preliminaries), we have \( D(\ln f_{m-1}(x)) \in SR_{\rho-1} \), i.e., \( \hat{f}_{m-1}(x) = xD(\ln f_{m-1}(x)) \in SR_\rho \). Since \( \ln x \in SR_0 \), we have \( \ln \hat{f}_{m-1}(x) \in SR_0 \). Hence

\[
x^n(\ln \hat{f}_{m-1}(x)) \sim o(\ln \hat{f}_{m-1}(x)), \ x \to \infty,
\]

and

\[
\frac{x^n(\ln f_m(x))}{\ln f_m(x)} = \frac{x^n(\ln f_{m-1}(x) + \ln \hat{f}_{m-1}(x))}{\ln f_{m-1}(x) + \ln \hat{f}_{m-1}(x)} \sim (\rho)_n, \ x \to \infty.
\]

Therefore, \( \ln f_m(x) \in SR_\rho \) and, analogously, \( \hat{f}_m(x) \in SR_\rho \). Also

\[
\hat{f}_m(x) = \frac{x(\hat{f}_m(x))'}{\hat{f}_m(x)} \to \rho, \ x \to \infty;
\]

and, since \( \ln f_0(x) = \ln f(x) \in SR_\rho \), the proof is finished by induction.

This proposition and our former considerations show that we could apply Theorem A to \( f_m(x) \) for some fixed \( m \in \mathbb{N} \). We obtain:

\[
f_m^{(\alpha)}(x) = f^{(m+\alpha)}(x) \sim f_m(x) \cdot (\hat{f}_m(x))^{\alpha}l(\hat{f}_m(x)), \ \alpha < 0, \ x \to \infty. \tag{B.1}
\]

But (Proposition 4),

\[
\hat{f}_k(x) = \hat{f}(x) + \sum_{1}^{k} \hat{f}_{i-1}(x) = \hat{f}(x) + O(k\rho) \sim \hat{f}(x), \ k = 1, 2, \ldots, m;
\]

and

\[
f_m(x) = f(x) \prod_{1}^{m} \hat{f}_{i-1}(x) = f(x) \prod_{1}^{m} (\hat{f}(x) + O(1)) \sim f(x)(\hat{f}(x))^m, \ x \to \infty.
\]

From (B.1) and Proposition 7 it follows that:

\[
f^{(m+\alpha)}(x) \sim f(x)(\hat{f}(x))^{m+\alpha}l(\hat{f}(x)) \sim \rho^{m+\alpha}[l(\ln f(x))]^{m+\alpha}l(\ln f(x)), \ \alpha < 0, \ x \to \infty.
\]
Putting $m + \alpha = \beta$ we see that Theorem B is valid for $\beta < m$. Since $m$ is an arbitrary integer, the proof is done.

Finally, for an illustration of our results, we give two characteristic examples.

**Example 1.** Consider an entire function $g$ of integer order $p$ in the form:

$$g(x) := \exp P_p(x) = \sum_k a_k x^k,$$

where $P_p(x) := b_p x^p + \cdots$, $b_p > 0$, is a polynomial with nonnegative coefficients. Since $\ln g(x) = P_p(x) \in SR_p$, applying Theorem B we obtain:

**Proposition 8.** $e^{-P_p(x)} \sum_k c_k a_k x^k \sim (p b_p)^\beta c_k x^\beta$, $x \to \infty$, for any r.v.s. $(c_k)$ of index $\beta \in R$.

**Example 2.** Let $h(x) := \sum_k b_k x^k$, $h(0) = 1$ be an entire function of order $\rho$, $0 < \rho < 1$ with negative zeros only. According to Hadamard’s Factorization Theorem, we have the representation

$$h(x) = \prod_k \left(1 + \frac{x}{r_k}\right), \quad \sum_k \frac{1}{r_k} < \infty;$$

where $\{-r_k\}$ are zeros of $h(x)$ in decreasing order.

Denoting by $n(x)$ zero-counting function of $h$, we get

$$\hat{h}(x) = x D(\ln h(x)) = \sum_k \frac{x}{x + r_k} = \int_0^\infty \frac{x}{x + t} dn(t),$$

and

$$x D(\hat{h}(x)) = \int_0^\infty \frac{xt}{(x + t)^2} dn(t) < \int_0^\infty \frac{x}{x + t} dn(t) = \hat{h}(x),$$

i.e., $\hat{h}(x) < 1$. So, we can apply Theorem A to $h(x)$.

There is more if we notice that the zeros of $h(x)$ are separated by the zeros of $h'(x)$; hence, all zeros of $h_1(x)/x$ are negative and, by induction, the same is valid for $h_n(x)/x$, $n \in N$. Therefore,

$$\hat{h}_n(x) < \left(\frac{h_n(x)}{x}\right) < 1$$

and, reproducing the proof of Theorem B, we come to:

**Proposition 9.** If $h(x)$ is defined as before then, without any condition,

$$\sum_k c_k b_k x^k \sim c_{[\hat{h}_1]} h(x), \quad x \to \infty,$$

for r.v.s. $(c_k)$ of arbitrary index.

More precisely, supposing the regular distribution of zeros, we get:
Proposition 10. If \( n(x) \in R_\rho \) then:

\[
\sum_k c_k b_k x^k \sim \left( \frac{\pi \rho}{\sin \pi \rho} \right)^\beta c_{[n(x)]} h(x), \quad x \to \infty;
\]

for any regularly varying sequence \( (c_k) \) of index \( \beta \in R \).

Proof. As we already showed,

\[
\hat{h}(x) = x \int_0^\infty \frac{dn(t)}{x + t}.
\]

Karamata’s Tauberian Theorem for the Stieltjes transform [1, p. 40] gives:

For \( 0 < \rho \leq 1 \); \( n(x) \sim x^\rho l(x), \quad x \to \infty \) if and only if

\[
\int_0^\infty \frac{dn(t)}{x + t} \sim \Gamma(1 - \rho) \Gamma(1 + \rho) x^{\rho - 1} l(x), \quad x \to \infty.
\]

Hence \( n(x) \in R_\rho \) implies \( \hat{h}(x) \sim \frac{\pi \rho}{\sin \pi \rho} n(x), \quad x \to \infty \).

The rest is Proposition 9.

References


