GENERALIZED LINE GRAPHS WITH THE SECOND LARGEST EIGENVALUE AT MOST 1

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Abstract. All connected generalized line graphs whose second largest eigenvalue does not exceed 1 are characterized. Besides, all minimal generalized line graphs with second largest eigenvalue greater than 1 are determined.

1. Introduction

In this paper we consider simple graphs with (0,1) adjacency matrix. The eigenvalues of a graph are denoted by \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \).

The second largest eigenvalue \( \lambda_2(G) \) of a graph \( G \) has attracted much attention in literature (see, for example, [3] and [6]).

Graphs with \( \lambda_2(G) \leq 1 \) have been studied in 1982 by Cvetković [2]. It turned out that some of these graphs are the complements of the graphs whose least eigenvalue is greater than or equal to \(-2\), while, on the other hand, the complement of a graph whose least eigenvalue is not less than \(-2\) always has \( \lambda_2 \leq 1 \). A representation of graphs with \( \lambda_2(G) = 1 \) in the Lorentz space is given in 1983 by Neumaier and Seidel [8]. Bipartite graphs with \( \lambda_2(G) \leq 1 \) have been characterized in 1991 by Petrović [9]. In particular, trees with second largest eigenvalue less than 1 were treated by Neumaier [7]. Line graphs whose second largest eigenvalue does not exceed 1 have been studied in 1998 by Petrović and Milekić [10].

The exact characterization of graphs with second largest eigenvalue around 1 still remains an interesting open question in the spectral theory of graphs.

In this paper we explicitly characterize all connected generalized line graphs with property \( \lambda_2(G) \leq 1 \). We prove that a connected generalized line graph \( G \) has this property if and only if \( G \) is an induced subgraph of any of 11 graphs displayed in Fig. 2. We note that one of the mentioned 11 graphs represents in fact a class of graphs.

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In this paper we also determine all minimal generalized line graphs with the property \( \lambda_2(G) > 1 \). There are exactly 21 such graphs (see Fig. 3).

Throughout this paper \( H \subset G \) will denote that \( H \) is an induced subgraph of a graph \( G \).

We now recall to some known classes of graphs.

**Definition 1.** The cocktail party graph on \( 2n \) vertices denoted by \( CP(n) \), is the regular graph on \( 2n \) vertices of degree \( 2n - 2 \).

**Definition 2.** A generalized line graph, denoted by \( L(H; a_1, \ldots, a_n) \), is constructed from a graph \( H \) with \( n \) vertices \( v_1, \ldots, v_n \) and nonnegative integers \( a_1, \ldots, a_n \) in the following way: it consists of disjoint copies of \( L(H) \) and \( CP(a_i) \) \( (i = 1, \ldots, n) \), with additional lines joining a vertex in \( L(H) \) with a vertex in \( CP(a_i) \) if the vertex in \( L(H) \) corresponds to a line in \( H \) that has \( v_i \) as an end point.

Special cases include an ordinary line graph \( (a_1 = \cdots = a_n = 0) \) and the cocktail party graph \( CP(m) \) \( (n = 1 \) and \( a_1 = m \).

**Definition 3.** A generalized cocktail party graph \( (GCP) \) is a graph obtained by deletion of independent edges from the complete graph \( K_n \). Any vertex of degree \( n - 1 \) is said to be of \( l \)-type, while the other are said to be of \( a \)-type.

D. Cvetković, M. Doob and S. Simić characterized generalized line graphs by showing that there are exactly 31 minimal nongeneralized line graphs.

![Fig. 1](image_url)
2. Main results

Let \( F_1, \ldots, F_{11} \) denote the generalized line graphs displayed in Fig. 2. Here the line between \([CP(r)]\) and \([K_{s+2t}]\) denotes the join of graphs \( CP(r) \) and \( K_{s+2t} \), i.e. all possible edges between the graphs \( CP(r) \) and \( K_{s+2t} \) are present. The graph \( F_{11} \) is a graph with \( 2r + 3s + 3t \) vertices which contains the generalized cocktail party graph (GCP) with \( 2r a \)-type vertices and \( (s + 2t) \) \( l \)-type vertices as an induced subgraph \( (r \geq 1, s \geq 1, t \geq 1) \).

![Fig 2](image)

**Theorem 1.** Graphs \( F_1 \sim F_{11} \) from Fig. 2 have the property \( \lambda_2(F_i) \leq 1 \) \( (i = 1, \ldots, 11) \).

**Proof.** We easily get by computer that \( \lambda_2(F_i) \leq 1 \) \( (i = 1, \ldots, 10) \).

Let \( A \) be the adjacency matrix of the graph \( F_{11} \), let \( \lambda \) be an eigenvalue of \( F_{11} \) distinct from \( \pm 1, -2 \) and 0, and let \( x \) be an eigenvector of \( F_{11} \) belonging to the eigenvalue \( \lambda \). From equality

\[
Ax = \lambda x,
\]

we get that \( x = (x, \ldots, x, y, \ldots, y, z, \ldots, z, 2z, \ldots, 2z) \) and all eigenvalues of the graph \( F_{11} \) distinct from \( \pm 1, -2 \) and 0 are determined by equation

\[
P(\lambda) = C_0 \lambda^3 + C_1 \lambda^2 + C_2 \lambda + C_3 = 0,
\]
where
\[
C_0 = 1, \\
C_1 = -(2r + s + 2t - 3), \\
C_2 = -2(r + s + 2t), \\
C_3 = 4(r - 1).
\]

For \( r = 1 \), in the sequence \((C_0, C_1, C_2, C_3)\) there is exactly one sign change, and for \( r > 1 \) there are exactly two sign changes in this sequence. Since \( P(0) = 4(r - 1) \geq 0 \) and \( P(1) = -3(s + 2t) < 0 \) we conclude that equation (1) has exactly one root greater than 1. It follows that \( \lambda_2(F_{11}) \leq 1. \]

In the sequel, we shall determine all connected generalized line graphs \( G \) with the property

(2) \[ \lambda_2(G) \leq 1. \]

The property (2) is hereditary because, whenever \( G \) satisfies (2) and \( H \subset G \), it follows that \( H \) also satisfies (2). The hereditary property (2) implies that there are minimal generalized line graphs that do not satisfy (2); such graphs are called forbidden subgraphs.

In the set of all generalized line graphs with at most 7 vertices, there are exactly 21 forbidden subgraphs (18 connected and 3 disconnected); see Fig. 3. Exactly 5 of these graphs are not line graphs: \( G_2, G_4, G_{10}, G_{13} \) and \( G_{18} \). We use them in the proofs of Lemmas 2 and 3. The remaining graphs from the set \( \{G_1, \ldots, G_{21}\} \) are taken from the results in [10].

Now, let \( \mathcal{L} \) denote the set of all connected generalized line graphs \( G \) such that \( G \) contains as an induced subgraph neither of the graphs \( G_1-G_{21} \) in Fig. 3. Clearly, since the complement of a generalized cocktail party graph \( G \) is a graph with the least eigenvalue \(-1\) (in fact, it is a line graph), its second largest eigenvalue is less than 1 and it belongs to \( \mathcal{L} \). Denote by \( \mathcal{L}_0 \) the set of all other members of \( \mathcal{L} \) distinct from generalized cocktail party graphs.

Let \( G = L(H; a_1, \ldots, a_n) \), where \( a_1 = \cdots = a_n = 0 \). Then \( G \) is a line graph and the following lemma holds.

**Lemma 1** [10]. If \( G = L(H; a_1, \ldots, a_n) \in \mathcal{L}_0 \), \( a_1 = \cdots = a_n = 0 \), then \( G \) is an induced subgraph of some of the graphs \( F_1-F_8 \) and \( F_{11} \) displayed in Fig. 2.

Now, let \( G = L(H; a_1, \ldots, a_n) \in \mathcal{L}_0 \), where \( V(H) = \{v_1, \ldots, v_n\} \), \( a_1 \geq \cdots \geq a_n \) and \( a_1 > 0 \).

Denote by \( G_0 \) generalized cocktail party graph induced by vertices of the graph \( CP(a_1) \) and vertices of the graph \( L(H) \) which correspond to lines in \( H \) that have \( v_1 \) as an end point. Let \( \{x_1, \ldots, x_m\} \) be the set of all \( L \)-type vertices of the graph \( G_0 \). Then \( m \geq 1 \) (in the opposite case we would have that \( G \) is disconnected graph, what is a contradiction).
Denote by $T$ the set $V(G) \setminus V(G_0)$. By Definition 2 we have that the vertices from $T$ are not adjacent to $a$-type vertices of $G_0$. Also, they can be adjacent to at most two vertices from the set $\{x_1, \ldots, x_m\}$ (in the opposite case we would have $H_{29} \subset G$, what is a contradiction*). Hence we have

$$T = T_0 \cup T_1 \cup T_2,$$

where $T_0$ is the set of vertices which are not adjacent to vertices from $\{x_1, \ldots, x_m\}$, $T_1$ is the set of vertices which are adjacent to exactly one vertex from $\{x_1, \ldots, x_m\}$, and $T_2$ is the set of vertices which are adjacent to exactly two vertices from $\{x_1, \ldots, x_m\}$. Also, we have

$$T_1 = T_{x_1} \cup \cdots \cup T_{x_m},$$

and

$$T_2 = T_{x_1x_2} \cup \cdots \cup T_{x_{m-1}x_m},$$

where $T_{x_i}$ is the set of vertices from $T_1$ which are adjacent to a vertex $x_i$, and $T_{x_ix_j}$ is the set of vertices from $T_2$ which are adjacent to vertices $x_i$ and $x_j$ of the set $\{x_1, \ldots, x_m\}$.

*To be short, we shall often reduce the mentioned sentence simply by "$H_{29} \subset G"
Lemma 2. If \( G = L(H; a_1, \ldots, a_n) \in \mathcal{L}_0, a_1 \geq \cdots \geq a_n \) and \( a_1 = 1 \), then \( G \) is an induced subgraph of some of the graphs \( F_0, F_0^1 \) and \( F_{11}^0 \) displayed in Fig. 2.

Proof. In the proof we distinguish the following four cases:

\( (A) \ m = 1; \quad (B) \ m = 2; \quad (C) \ m = 3; \quad (D) \ m \geq 4. \)

Case A. In this case we have

\[ T = T_0^0 \cup T_2. \]

Each vertex from the set \( T_2 \) can be nonadjacent to at most one vertex from this set \( (H_1 \subset G \cap H_2^{10} \subset G) \). It follows that the graph induced by vertices of the set \( T_2 \) is generalized cocktail party graph. Denote by \( \{y_1, \ldots, y_p\} \) the set of all \( l \)-type vertices of this graph.

The vertices from \( T_0^0 \) are not adjacent to \( a \)-type vertices of generalized cocktail party graph induced by vertices of \( T_2 \) \( (H_2^{10} \subset G \cap H_2^{10} \subset G) \), and they can be adjacent to at most two vertices from the set \( \{y_1, \ldots, y_p\} \) \( (H_{11} \subset G) \). Hence we have

\[ T_0^0 = T_0^0 \cup T_1^0 \cup T_2^0, \]

where \( T_0^0 \) is the set of vertices which are not adjacent to vertices from \( \{y_1, \ldots, y_p\} \), \( T_1^0 \) is the set of vertices which are adjacent to exactly one vertex from \( \{y_1, \ldots, y_p\} \), and \( T_2^0 \) is the set of vertices which are adjacent to exactly two vertices from \( \{y_1, \ldots, y_p\} \). Also, we have

\[ T_1^0 = T_0^0 \cup \cdots \cup T_0^0 \]

and

\[ T_2^0 = T_0^0 \cup \cdots \cup T_0^0, \]

where \( T_{yi}^0 \) is the set of vertices from \( T_1^0 \) which are adjacent to a vertex \( y_i \), and \( T_{yiy_j}^0 \) is the set of vertices from \( T_2^0 \) which are adjacent to vertices \( y_i \) and \( y_j \) of the set \( \{y_1, \ldots, y_p\} \).

The vertices of the set \( T_0^0 \) have the following properties:

1. \( T_0^0 = \emptyset \) \( (G_2 \subset G) \);
2. The set \( T_{yi}^0 \) does not contain adjacent vertices \( (G_4 \subset G) \) and \( |T_{yi}^0| \leq 2 \) \( (H_2^{10} \subset G) \);
3. \( |T_{yiy_j}^0| \leq 1 \) \( (H_2^{10} \subset G \cap G_4 \subset G) \);
4. The sets \( T_{yi}^0 \) and \( T_{yiy_j}^0 \) are not coexistent \( (H_{19} \subset G \cap G_4 \subset G) \). The sets \( T_{yi}^0 \) and \( T_{yiy_j}^0 \) are not coexistent, too \( (H_{19} \subset G \cap G_4 \subset G) \);
5. The graph which is induced by vertices of the set \( T_0^0 \) is the graph without edges \( (G_2 \subset G) \).

By properties (1)--(5) we conclude that the graph \( G \) is an induced subgraph of the graph \( F_{11} \) in Fig. 2.
Case B. In this case we have

\[ T = T_0 \cup T_{x_1} \cup T_{x_2} \cup T_{x_1x_2}. \]

The vertices of the set \( T \) have the following properties:

1. \( T_0 = \emptyset \) (\( G_{10} \subset G \cap H_{20} \subset G \));
2. The set \( T_{x_i} \) does not contain adjacent vertices (\( G_{13} \subset G \)) and \( |T_{x_i}| \leq 2 \) (\( H_{27} \subset G \));
3. \( |T_{x_1x_2}| \leq 1 \) (\( H_{23} \subset G \cap H_{30} \subset G \));
4. If \( T_{x_1} \neq \emptyset \) and \( T_{x_1x_2} \neq \emptyset \), then a vertex from the set \( T_{x_1} \) is adjacent to a vertex from the set \( T_{x_1x_2} \) (\( H_{31} \subset G \));
5. If \( T_{x_1} \neq \emptyset \) and \( T_{x_2} \neq \emptyset \) and if vertices \( x \in T_{x_1} \) and \( y \in T_{x_2} \) are adjacent, then \( |T_{x_1}| = |T_{x_2}| = 1 \) (\( H_{21} \subset G \cap H_{25} \subset G \)) and \( T_{x_1x_2} = \emptyset \) (\( G_{18} \subset G \)).

In view of properties (1)-(5) we have that the graph \( G \) is an induced subgraph of the graphs \( F_9 \) or \( F_{10} \) from Fig. 2.

Case C. The vertices of the set \( T \) have the following properties:

1. \( T_0 = \emptyset \) (\( G_{10} \subset G \cap H_{22} \subset G \));
2. The set \( T_{x_i} \) does not contain adjacent vertices (\( G_{13} \subset G \)) and \( |T_{x_i}| \leq 2 \) (\( H_{27} \subset G \));
3. \( |T_{x_1x_2}| \leq 1 \) (\( H_{23} \subset G \cap H_{30} \subset G \));
4. The sets \( T_{x_1} \) and \( T_{x_1x_2} \) are not coexistent (\( G_{10} \subset G \cap H_{31} \subset G \));
5. If \( |T_2| \leq 1 \), then the graph which is induced by vertices of the set \( T \) is the graph without edges (\( G_{10} \subset G \cap H_{24} \subset G \));
6. If \( |T_2| > 1 \), then the graph which is induced by vertices of the set \( T = T_2 \) is the complete graph (\( H_{31} \subset G \)), and \( |T_2| = 2 \) (\( G_{18} \subset G \)).

In view of properties (1)-(6) we have that the graph \( G \) is an induced subgraph of the graphs \( F_9 \) or \( F_{11} \) from Fig. 2.

Case D. The vertices of the set \( T \) have the properties (1)-(4) from Case C. The following properties also hold:

5. The sets \( T_{x_1x_j} \) and \( T_{x_2x_k} \) are not coexistent (\( H_{31} \subset G \cap G_{10} \subset G \));
6. The graph which is induced by vertices of the set \( T \) is the graph without edges (\( G_{10} \subset G \cap H_{22} \subset G \cap H_{24} \subset G \)).

Now using the mentioned properties we find that the graph \( G \) is an induced subgraph of the graph \( F_{11} \) displayed in Fig. 2. \( \Box \)

**Lemma 3.** If \( G = L(H; a_1, \ldots, a_n) \in \mathcal{L}_0 \), \( a_1 \geq \cdots \geq a_n \) and \( a_1 > 1 \), then \( G \) is an induced subgraph of the graph \( F_{11} \) displayed in Fig. 2.

**Proof.** The vertices of the set \( T \) have the following properties:

1. \( T_0 = \emptyset \) (\( G_{10} \subset G \));
2. The set \( T_{x_i} \) does not contain adjacent vertices (\( G_{13} \subset G \)) and \( |T_{x_i}| \leq 2 \) (\( H_{27} \subset G \));
3. \( |T_{x_1x_2}| \leq 1 \) (\( H_{23} \subset G \cap H_{30} \subset G \));
4. The sets \( T_{x_1} \) and \( T_{x_1x_2} \) are not coexistent (\( H_{31} \subset G \cap G_{10} \subset G \)). The sets \( T_{x_1x_j} \) and \( T_{x_2x_k} \) are not coexistent, too (\( H_{31} \subset G \cap G_{10} \subset G \)).
(5) The graph which is induced by vertices of the set $T$ is the graph without edges $(G_{10} \subset G \lor G_{13} \subset G)$.

By properties (1)–(5) we conclude that the graph $G$ is an induced subgraph of the graph $F_{11}$ in Fig. 2. □

Thus, collecting the former conclusions from Lemmas 1–3, we arrive to the following theorem. We note that a generalized cocktail party graph is an induced subgraph of the graph $F_{11}$.

**Theorem 2.** If a connected generalized line graph contains as an induced subgraph neither of the graphs $G_{11}$–$G_{21}$ in Fig. 3, then $G$ is an induced subgraph of some of the graphs $F_{1}$–$F_{11}$ in Fig. 2.

**Theorem 3.** A connected generalized line graph $G$ has the property $\lambda_2(G) \leq 1$ if and only if $G$ is an induced subgraph of some of the graphs $F_{1}$–$F_{11}$ in Fig. 2.

**Proof.** Assume that $G$ is a connected generalized line graph with the property $\lambda_2(G) \leq 1$. Then by the Interlacing theorem (cf. [3, p. 19]) we conclude that $G$ does not contain any of graphs $G_{11}$–$G_{21}$ in Fig. 3 as an induced subgraph. In view of Theorem 2, $G$ must be an induced subgraph of some of the graphs $F_{1}$–$F_{11}$ in Fig. 2.

Conversely, let a connected generalized line graph $G$ is an induced subgraph of any of the graphs $F_{1}$–$F_{11}$ in Fig. 2. Because the property (2) is hereditary and the Theorem 1 holds, we have that $\lambda_2(G) \leq 1$. □

In the sequel, we shall determine all minimal generalized line graphs with the property $\lambda_2(G) > 1$.

**Theorem 4.** There are exactly 21 minimal generalized line graphs with the property $\lambda_2(G) > 1$. These are the graphs $G_{11}$–$G_{21}$ in Fig. 3.

**Proof.** By a straightforward verification one can easily prove that graphs $G_{11}$–$G_{21}$ in Fig. 3 are minimal with respect to the property $\lambda_2(G) > 1$. We shall prove that they are only generalized line graphs which are minimal with respect to this property.

Let $G$ be an arbitrary connected generalized line graph which is minimal with respect to the property $\lambda_2(G) > 1$ and which is distinct from the graphs $G_{11}$–$G_{18}$. Then $G$ does not contain any of graphs $G_{11}$–$G_{21}$ as an induced subgraph. By Theorem 2 we get that $G$ is an induced subgraph of some of the graphs $F_{1}$–$F_{11}$ in Fig. 2. But Theorem 1 and the Interlacing theorem also give $\lambda_2(G) \leq 1$, which is a contradiction. Thus, $G_{11}$–$G_{18}$ are the only minimal connected generalized line graphs with the property $\lambda_2(G) > 1$.

Now assume that $G$ is an arbitrary disconnected generalized line graph which is minimal with respect to the property $\lambda_2(G) > 1$. Then $G$ has no isolated vertices and it has exactly two connected components $E_1$ and $E_2$, where $\lambda_1(E_1) > 1$ and $\lambda_1(E_2) > 1$. Hence, we have that graphs $E_1$ and $E_2$ contain the graph $P_3$ or the graph $K_3$ as an induced subgraph and $P_3 \cup P_3 \subset G$ or $P_3 \cup K_3 \subset G$ or $K_3 \cup K_3 \subset G$. So we get that $G_{19}$–$G_{21}$ are the only minimal disconnected generalized line graphs with the property $\lambda_2(G) > 1$. □
References


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