A LITTLEWOOD–PALEY THEOREM FOR
SUBHARMONIC FUNCTIONS

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Abstract. If \( u(z) > 0 \) (\(|z| < 1\)) is a subharmonic function of class \( C^2 \) such
that \( \Delta u \) is subharmonic and if \( \int u(r e^{i\theta}) \, dt \) (\( q > 1 \)) is bounded when \( 0 < r < 1 \),
then
\[
\int \int (1 - |z|)^{q-1} (\Delta u(z))^q \, dx \, dy < \infty.
\]
In the case \( u = h^2 \) and \( q = p/2 < 1 \), where \( h \) is harmonic, this reduces to the
Littlewood–Paley theorem. In the case \( 0 < q < 1 \) we prove a theorem in the
opposite direction.

1. Introduction

Let \( D \) denote the open unit disk in the complex plane. For a function \( u \) defined
on \( D \) we write
\[
I(r, u) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \, dt
\]
provided the integral is defined for all \( r < 1 \), and
\[
I(u) = \sup_{0 < r < 1} I(r, u),
\]
where the value \( \infty \) is permitted. In this paper we prove the following theorem.

Theorem 1.1. Let \( u \geq 0 \) be a subharmonic function of class \( C^2(D) \) such that
its Laplacian, \( \Delta u \), is subharmonic as well. If \( q \geq 1 \) and \( I(u^q) < \infty \), then
\[
(1.1) \quad \int_D (1 - |z|)^{2q-1} (\Delta u(z))^q \, dm(z) \leq C_q (I(u^q) - u(0)^q),
\]
where \( C_q \) is a constant depending only on \( q \).

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Here $dm$ denotes the area measure in the plane.

An important special case of (1.1) is the Littlewood-Paley inequality [3]; namely, if $p \geq 2$ and $I(|h|^{p}) < \infty$, where $h$ is a real-valued function harmonic in $D$, then

$$\int_{D} (1 - |z|)^{p-1} |\nabla u|^{p} dm < C_{p} I(|h|^{p} - |h(0)|^{p}).$$

To obtain (1.2) from (1.1) we take $u = h^{2}$ and $q = p/2$. The function $u$ satisfies the hypotheses of Theorem 1.1 because $\Delta u = 2|\nabla h|^{2}$.

Inequality (1.2) is usually stated in the weaker form

$$\int_{D} (1 - |z|)^{p-1} |\nabla h|^{p} dm \leq C_{p} I(|h|^{p}) \quad (p > 2).$$

The usual method of proving (1.3) is to use the Riesz-Thorin theorem. A quick elementary proof is given in [6]; it is based on the Hardy-Stein identity and the inequality $|\nabla h(z)| \leq 2h(z)/(1 - |z|)$ which holds when $h > 0$. An earlier proof based on the Hardy-Stein inequality and some local estimates is due to Luecking [5]. Our proof of Theorem 1.1 is similar to Luecking’s proof of (1.3) (see Lemma 2.2 and 3.1). However, some simplifications are made so that we can treat the case $q < 1$ as well (see Theorem 4.1). This provides, in particular, a new proof of the reverse Littlewood-Paley inequality which holds for harmonic functions when $1 < p < 2$ and for analytic functions when $0 < p < 2$. Moreover, a special case of Theorems 1.1 and 4.1 is the Littlewood-Paley inequality for vector valued functions. More precisely, inequality (1.3) remains true for $p \geq 2$ if we assume that $h$ is a harmonic function with values in $L^{p}$, $|h(z)|^{2} = \sum h_{n}(z)^{2}$ and $|\nabla h(z)|^{2} = \sum |\nabla h_{n}(z)|^{2}$. The reverse inequality holds for $1 < p < 2$.

## 2. Local estimates for Riesz’ measure

From now on we shall assume that $u$ is an arbitrary nonnegative subharmonic function defined on $D$. Then there exists a positive measure $d\mu$ on $D$, called the Riesz measure of $u$, such that $\Delta u = d\mu$ in the sense of distribution theory. (If $u$ is of class $C^{2}$, then $d\mu(z) = \Delta u(z) dm(z)$.) There holds the formula

$$I(r, u) - u(0) = \frac{1}{2\pi} \int_{rD} \log \frac{r}{|z|} d\mu(z) \quad (0 < r < 1),$$

which can be deduced, for example, from the Riesz representation formula (see [4], Theorem 3.3.6.)

**Lemma 2.1.** We have

$$I(u) - u(0) = \frac{1}{2\pi} \int_{D} \log \frac{1}{|z|} d\mu(z).$$

**Proof.** Write (2.1) in the form

$$I(r, u) - u(0) = \frac{1}{2\pi} \int_{D} K_{r}(z) \log \frac{r}{|z|} d\mu(z),$$

where $K_{r}(z)$ is the Kelvin transform of the function $u$.
where \( K_r(z) \) is the characteristic function of the disk \( rD \). Since \( K_r(z) \log(r/|z|) \) increases with \( r \) we have

\[
\lim_{r \to 1} (r, u) - u(0) = \frac{1}{2\pi} \int_D \lim_{r \to 1} K_r(z) \log \frac{r}{|z|} d\mu(z).
\]

And since \( I(r, u) \) increases with \( r \) we have \( I(u) = \lim_{r \to 1} I(r, u) \). The result follows. \( \square \)

**Lemma 2.2.** Let \( q \geq 1 \) and let \( \mu \) and \( \mu_q \) be the Riesz measures of \( u \) and \( u^q \) respectively. Then

\[
(2.2) \quad \mu(E)^q \leq C_q \mu_q(5E)
\]

for any disk \( E \) such that \( 6E \subset D \). The constant \( C_q \) depends only on \( q \).

If \( E \) is a disk of radius \( R \), then \( rE \) denotes the concentric disk of radius \( rR \).

**Proof.** By translation the proof is reduced to the case where \( E \) is centered at 0. Then since \( \mu(E) = \nu((1/r)E) \), where \( \nu \) is the Riesz measure of the function \( u(rz) \), we can assume that the radius of \( E \) is fixed. e.g., \( E = \varepsilon D \) with \( \varepsilon = 1/6 \). Assuming this we use the simple inequalities

\[
(I(r, u) - u(0))^q \leq (I(r, u))^q - u(0)^q
\]

and \( (I(r, u))^q \leq I(r, u^q) \), which hold because \( q > 1 \), to deduce from (2.1) (applied to \( u \) and \( u^q \) that

\[
(2.3) \quad \left( \frac{1}{2\pi} \int_D \log \frac{r}{|z|} d\mu(z) \right)^q \leq \frac{1}{2\pi} \int_D \log \frac{r}{|z|} d\mu_q(z).
\]

Putting \( r = 4\varepsilon \) we get

\[
(2.4) \quad \mu(2\varepsilon D)^q \leq C \int_{2\varepsilon D} |z|^{-1} d\mu_q(z),
\]

where we have used the estimates \( \log(4\varepsilon/|z|) \geq \log 2 \) for \( |z| < 2\varepsilon \) and \( \log(4\varepsilon/|z|) \leq 1/|z| \). Thus to prove (2.2) we have to eliminate \( |z|^{-1} \) in the integral. To do this we change the ‘center’ of (2.4) and we get

\[
\mu(2\varepsilon D_a)^q \leq C \int_{4\varepsilon D_a} |z-a|^{-1} d\mu_q(z)
\]

for \( a \in \varepsilon D \), where \( D_a = \{ z : |z-a| < 1 \} \). Since \( \varepsilon D \subset 2\varepsilon D_a \) and \( 4\varepsilon D_a \subset 5\varepsilon D \) we have

\[
\mu(\varepsilon D)^q \leq C \int_{4\varepsilon D_a} |z-a|^{-1} d\mu_q(z).
\]

Now we integrate this inequality over \( \varepsilon D \) with respect to \( dm(a) \) and use Fubini’s theorem. This concludes the proof because

\[
\sup_{\varepsilon \in D} \int_{\varepsilon D} |z-a|^{-1} dm(a) < \infty.
\]

\( \square \)
3. Proof of Theorem 1.1

Theorem 1.1 is a consequence of the following.

**Theorem 3.1.** Let \( u \geq 0 \) be a subharmonic function in \( D \) and let \( \mu \) be the Riesz measure of \( u \). If \( q \geq 1 \) and \( I(u^q) < \infty \), then there holds the inequality

\[
\left(1 - |z|\right)^{-1} (\mu(E_\varepsilon(z)))^q \, dm \leq C_q (I(u^q) - u(0)^q),
\]

where \( \varepsilon = 1/6 \) and

\[
E_\varepsilon(z) = \{w : |w - z| < \varepsilon(1 - |z|)\}.
\]

If in addition \( u \) is \( C^2 \) and \( \Delta u \) is subharmonic, then

\[
\mu(E_\varepsilon(z)) = \int_{E_\varepsilon(z)} \Delta u \, dm \geq \pi \varepsilon^2 (1 - |z|)^2 \Delta u(z)
\]

because of the sub-mean-value property of \( \Delta u \), and this shows that (3.1) implies (1.2).

**Proof.** It follows from (2.2) that

\[
\int_D \left(1 - |z|\right)^{-1} (\mu(E_\varepsilon(z)))^q \, dm \int_D \left(1 - |z|\right)^{-1} \mu_q(E_{5\varepsilon}(z)) \, dm(z).
\]

Next we write

\[
\mu_q(E_{5\varepsilon}(z)) = \int_{E_{5\varepsilon}(z)} \, dm_q(w)
\]

and use Fubini’s theorem to conclude that the right hand side of (3.2) is equal to

\[
\int_D dm_q(w) \int_{G(w)} \left(1 - |z|\right)^{-1} \, dm(z),
\]

where \( G(w) = \{z : |z - w| < 5\varepsilon(1 - |z|)\} \). Since \( z \in G(w) \) implies \( |z| - |w| < 5\varepsilon (1 - |z|) \), whence \( |z| < (1 + 5\varepsilon)(1 - |z|) \), we have

\[
\int_{G(w)} \left(1 - |z|\right)^{-1} \, dm(z) \leq (1 + 5\varepsilon) m(G(w)) \left(1 - |w|\right)^{-1}.
\]

And since \( (1 + 5\varepsilon)(1 - |z|) < 1 - |w| \) for \( z \in G(w) \), we have \( m(G(w)) \leq C' (1 - |w|)^2 \), where \( C' = \pi (5\varepsilon/(1 - 5\varepsilon))^2 \). Combining the previous results we see that

\[
\int_D \left(1 - |z|\right)^{-1} (\mu(E_\varepsilon(z)))^q \, dm \leq C_q \int_D (1 - |w|) \, dm_q(w).
\]

This finishes the proof of (3.1) because of Lemma 2.1 and the inequality \( 1 - |w| \leq \log(1/|w|) \).
\( \square \)
4. The case \( q < 1 \)

**Theorem 4.1.** Let \( 0 < q < 1 \) and let \( u \geq 0 \) be a \( C^2 \)-function such that \( u^q \) and \( \Delta u \) are subharmonic. If \( \int_D (1 - |z|)^{2q-1} (\Delta u)^q \, dm < \infty \), then \( I(u^q) < \infty \) and there holds the inequality

\[
I(u^q) - u(0)^q \leq C_q \int_D (1 - |z|)^{2q-1} (\Delta u)^q \, dm.
\]

Observe that, in contrast to the case \( q > 1 \), the function \( u^q \) need not be smooth.

**Proof.** Fix \( \varepsilon < 1/6 \). Applying Lemma 2.2 to the pair \( u^q, (u^q)^{1/q} \) we get, because \( 1/q > 1 \),

\[
\mu_q(E_{\varepsilon}(z)) \leq C_q (\mu(E_{\varepsilon}(z)))^{q},
\]

where \( \mu_q \) and \( \mu \) are the Riesz measure of \( u^q \) and \( u \). On the other hand

\[
(\mu(E_{\varepsilon}(z)))^{q} = \left( \int_{E_{\varepsilon}(z)} \Delta u \, dm \right)^q
\leq C' (1 - |z|)^{2q} \sup \{(\Delta u(w))^q : w \in E_{\varepsilon}(z)\}.
\]

The function \((\Delta u)^q\) need not be subharmonic. Nevertheless, by a result of Hardy and Littlewood [2] and Fefferman and Stein [1], it possesses a weak form of the sub-mean-value property, namely

\[
(\Delta u(z))^q \leq \frac{C}{m(E)} \int_E (\Delta u)^q \, dm,
\]

where \( E \subseteq D \) is any disk centered at \( z \), and \( C \) depends only on \( q \). Using (4.3) one shows that

\[
\sup_{E_{\varepsilon}(z)} (\Delta u)^q \leq C'' (1 - |z|)^{-2} \int_{E_{\varepsilon}(z)} (\Delta u)^q \, dm.
\]

It follows that

\[
\int_D (1 - |z|)^{-1} \mu_q(E_{\varepsilon}(z)) \, dm(z) \leq C \int_D (1 - |z|)^{2q-3} \, dm(z) \int_{E_{\varepsilon}(z)} (\Delta u)^q \, dm,
\]

where \( C \) depends only on \( q \). Hence, as in the proof of Theorem 3.1,

\[
\int_D (1 - |z|) \, d\mu_q(z) \leq C_q \int_D (1 - |z|)^{2q-1} (\Delta u)^q \, dm.
\]

This implies that \( I(u^q) < \infty \) because of Lemma 2.1 applied to \( u^q \).

In order to prove (4.1) additional work is needed. We rewrite (2.3) as

\[
\left( \frac{1}{2\pi} \int_{rD} \log \frac{r}{|z|} \, d\mu_q(z) \right)^q \leq \frac{1}{2\pi} \int_{rD} \log \frac{r}{|z|} \, d\mu(z).
\]

Hence

\[
\int_{\varepsilon D} \log \varepsilon |z| \, d\mu_q(z) \leq C \sup_{\varepsilon D} (\Delta u)^q \leq C' \int_{2\varepsilon D} (\Delta u)^q \, dm,
\]
where we have used (4.3). Now it is easy to show that (4.4) remains true if we replace the left integral by
\[
\frac{1}{2\pi} \int_{\mathbb{D}} \log \frac{1}{|z|} \, d\mu_q(z) = I(u^q) - u(0)^q.
\]

\[\square\]

References


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