ON A THEOREM OF M. VUILLEUMIER

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Abstract. We give an improvement of a well-known theorem on matrix transforms of slowly varying sequences in the sense of Karamata.

1. Introduction

A sequence of positive numbers \( (\ell_n) \) is said to be slowly varying in the sense of Karamata if

\[
\lim_{n \to \infty} \left( \frac{\ell_{cn}}{\ell_n} \right) = 1, \quad \forall c > 0.
\]

The essential properties of these sequences were studied by Karamata [5], [6], Bojanic and Seneta [2] and many others.

Some examples of \( \ell_n \) are:

\[
1, \log^a 2n, \log^b (\log 3n), \exp(\log^c 2n); \quad a, b \in \mathbb{R}; \quad 0 < c < 1.
\]

The main tool in dealing with matrix transforms of slowly varying sequences is a theorem of Vuilleumier [4]. Her result specialized to triangular real-valued matrices \( (A_{nk}) \), \( 1 \leq k \leq n \) can be stated as follows:

Theorem A. In order that

\[
\sum_{k \leq n} A_{nk} \ell_k \sim \ell_n \quad (n \to \infty),
\]

for each slowly varying sequence \((\ell_n)\), it is necessary and sufficient that

\[
(1) \quad I. \sum_{k \leq n} A_{nk} \to 1 \quad (n \to \infty); \quad II. \sum_{k \leq n} |A_{nk}| k^{-\eta} = O(n^{-\eta}) \quad (n \to \infty),
\]

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for some \( n > 0 \).

This theorem plays a fundamental role in the theory of R-regular or R-mercerian matrices [3], [10]. But, although it is self-sufficient, there are some inner limitations as we are going to show.

Consider a real-valued sequence \((a_n)\), \(\forall M \in N : \sum_{n \leq M} a_n \neq 0\), and let
\[
A_{nk} := a_k / \sum_{i \leq n} a_i.
\]

Then, the condition I of the Theorem A is trivially satisfied and for II, using an inequality for convex means (Lemma 3, below), we obtain
\[
\sum_{k \leq n} |A_{nk}| k^{-\eta} = \frac{\sum_{k \leq n} |a_k| \sum_{k \leq n} |a_k| k^{-\eta}}{\sum_{k \leq n} |a_k|} \\
= \frac{\sum_{k \leq n} |a_k|}{\sum_{k \leq n} |a_k|} \left( \frac{\sum_{k \leq n} k |a_k|}{\sum_{k \leq n} |a_k|} \right)^{-\eta} \\
= n^{-\eta} \left( \frac{\sum_{k \leq n} a_k}{\sum_{k \leq n} |a_k|} \right)^{-1} \left( \frac{\sum_{k \leq n} k |a_k|}{\sum_{k \leq n} |a_k|} \right)^{-\eta}.
\]

Since both expressions in parenthesis are positive and not greater than one, we see from (2) that if
\[
\liminf_{n} \frac{\sum_{k \leq n} a_k}{\sum_{k \leq n} |a_k|} = 0 \quad \text{or} \quad \liminf_{n} \frac{\sum_{k \leq n} k |a_k|}{\sum_{k \leq n} |a_k|} = 0,
\]
the condition II is not satisfied so that Theorem A is not applicable.

We will remove such obstacles and thus extend the field of applications.

2. Results

In order to produce a proof of Theorem B below, we need some well-known properties of slowly varying sequences and some elementary inequalities.

Lemma 1. For each \( c > 0 \), a slowly varying sequence \( \ell_n \) satisfies (cf. [4, p. 52])
\[
\ell_{[x]} \sim \ell_{[c \cdot x]} \quad (x \to \infty).
\]

Lemma 2. For \( \eta > 0 \), the following relations hold
\[
\sup_{k \leq y} k^\eta \ell_k \sim y^\eta \ell_{[y]}; \quad \sup_{k \geq y} k^{-\eta} \ell_k \sim y^{-\eta} \ell_{[y]} \quad (y \to \infty).
\]

A variant of the convex means inequality (cf. [7, p. 76]) is

Lemma 3. For a sequence of non-negative numbers \( \alpha_k \) and \( \mu \leq 0 \) or \( \mu \geq 1 \),
\[
\frac{\sum k^\mu \alpha_k}{\sum \alpha_k} \geq \left( \frac{\sum k \alpha_k}{\sum \alpha_k} \right)^\mu,
\]
and the converse inequality holds for \( 0 < \mu < 1 \).

For an application we need the following
Lemma 4. For each $a, \eta \in R$, we have
\[
\sum_{1 \leq k \leq n} \left( \frac{n + a}{n - k} \right)^k \sim n^{\eta/2} L_n^{(a)}(-1) \quad (n \to \infty),
\]
where $L_n^{(a)}(z)$ is the Laguerre polynomial \([8]\).

Now, we can prove our main result. For a complex-valued triangular matrix $(a_{nk})$, $1 \leq k \leq n$, define
\[
\sigma_n := \frac{\sum_{k=1}^{n} |a_{nk}|}{\sum_{k=1}^{n} |a_{nk}|}; \quad t_n := \frac{\sum_{k=1}^{n} k|a_{nk}|}{\sum_{k=1}^{n} |a_{nk}|}.
\]
We can prove the following

Theorem B. If the matrix $(a_{nk})$ satisfies for $n \to \infty$

(i) $t_n \to \infty$, $t_n = o(n)$; \quad (ii) $\liminf_{n} \sigma_n > 0$; \quad (iii) $\sum_{k=1}^{n} k^{-\eta}|a_{nk}| \sim O(t_n^{-\eta})$

for some $\eta > 0$, then
\[
\sum_{k=1}^{n} \ell_k a_{nk} \to 1 \quad (n \to \infty),
\]
for all slowly varying sequences $(\ell_n)$.

Proof. The condition (ii) guarantees that, for sufficiently large $n$, we have $\sum_{k=1}^{n} a_{nk} \neq 0$ and $1/\sigma_n = O(1)$. Therefore, for such $n$ and all fixed $c$, $0 < c < 1$, we obtain
\[
\sigma_n \left| \sum_{k=1}^{n} \frac{\ell_k a_{nk}}{\sum_{k=1}^{n} a_{nk}} - 1 \right| \leq \left| \frac{\sum_{k=1}^{n} a_{nk}(\ell_k / \ell_{[\ell]} - 1)}{\sum_{k=1}^{n} |a_{nk}|} \right| \leq S_1 + S_2 + S_3. \quad (B_1)
\]
Applying Lemmas 1 and 2, by (iii), we get
\[
S_1 \leq \sum_{k=1}^{n} \frac{|a_{nk}| |\ell_k / \ell_{[\ell]} - 1|}{|a_{nk}|} = \sum_{k=1}^{n} \frac{k^{-\eta}|a_{nk}| |k^\eta \ell_k / \ell_{[\ell]} - k^\eta|}{|a_{nk}|} \leq \sup_{k \leq ct_n} (k^\eta |\ell_k / \ell_{[\ell]}| + k^\eta) \sum_{k=1}^{n} \frac{k^{-\eta}|a_{nk}|}{|a_{nk}|} \sim 2(ct_n)^{\eta} \cdot O(t_n^{-\eta}) \ll c^\eta. \quad (B_2)
\]
Similarly, using Lemmas 1, 2 and 3 with $0 < \mu < 1$, we obtain
\[
S_3 \leq \sup_{k \geq \ell_n} (k^{-\mu} |\ell_k / \ell_{[\ell]}| + k^{-\mu}) \sum_{k=1}^{n} \frac{k^\mu |a_{nk}|}{|a_{nk}|} \sim 2(1/c t_n)^{-\mu} \cdot O(t_n)^{\mu} \ll c^\mu. \quad (B_3)
\]
Finally, arguing as before and using (i), one has
\[
S_2 \leq \frac{\sum_{c_n < k < 1/c_n} |a_{nk}| |\ell_k/\ell_{[t_n]} - 1|}{\sum_{k \leq n} |a_{nk}|} \leq \sup_{c_n < k < 1/c_n} |\ell_k/\ell_{[t_n]} - 1| = o(1) \quad (t_n \to \infty),
\]
by the uniform convergence (see [1, pp. 6–11]).

Since $c$ can be taken arbitrarily small, from $(B_1), (B_2), (B_3)$ we deduce
\[
\left| \frac{\sum_{k \leq n} \ell_k a_{nk}}{\ell_{[t_n]} \sum_{k \leq n} a_{nk}} - 1 \right| = \frac{1}{\sigma_n} (S_1 + S_2 + S_3) = O(1)o(1) = o(1) \quad (n \to \infty),
\]
i.e., Theorem B is proved.

**Remark.** Comparing theorems A and B, two advantages of the second one become clear. Firstly, Theorem A is not applicable when $t_n = o(n)$ (see Introduction), while in Theorem B it is enough that $t_n \to \infty \quad (n \to \infty)$. Secondly, Theorem B is valid for complex-valued matrices. Closer connection between theorems A and B can be established if we replace the condition (i) by: $t_n \to \infty$ and $\lim_n t_n/n$ exist. The proof is carried out as before.

To illustrate the power of the assertion from Theorem B, we give a nontrivial example.

Consider the class of Laguerre polynomials $L_n^{(a)}(z)$ defined by
\[
L_n^{(a)}(z) := \sum_{k \leq n} \frac{(n+a)}{n-k} \frac{(-z)^k}{k!}
\]
and take $a_{nk} := \frac{(n+a)}{n-k} \frac{\exp(|b_k n^{-1/4}|)}{k!}$, $b \in R$. Then
\[
\sum_{k \leq n} a_{nk} = L_n^{(a)}(-1), \quad \sum_{k \leq n} |a_{nk}| = L_n^{(a)}(-1), \quad t_n = \frac{L_n^{(a)}(-1)}{L_n^{(a)}(-1)}.
\]
Perron’s formula for the asymptotic behavior of Laguerre polynomials in the complex plane cut along the positive part of the real axis says that (cf. [9, p. 197])
\[
L_n^{(a)}(z) \sim 1/2 \pi^{-1/2} e^{z/2} (-z)^{-a/2-1/4} n^{a/2-1/4} \exp(2(-n z)^{1/2})(1 + O(n^{-1/2}))
\]
when $n \to \infty$. Using this formula and the properties of Laguerre polynomials we get $t_n \sim \sqrt{n}$, $(n \to \infty)$ i.e., Theorem A is not applicable in this case.

On the other hand, taking into account Lemma 4, we see that the condition (iii) is satisfied and
\[
\sigma_n \sim \frac{|\exp(2\sqrt{n} e^{b/2n^{-1/4}})|}{\exp(2\sqrt{n})} = \exp(2\sqrt{n}(\cos(b/2n^{-1/4}) - 1)) \to e^{-b^2/4} \quad (n \to \infty),
\]

i.e., (ii) is also valid; hence, applying Theorem B, we obtain for $n \to \infty$

$$\sum_{k \leq n} \frac{(n + a) \exp(ibn^{-1/4})}{k!} \ell_k = L_n^{(a)}(-\exp(ibn^{-1/4}))\ell_{\sqrt{n}}(1 + o(1)).$$

In addition, by separating the real and imaginary parts on the left and applying Perron’s formula on the right side of the last expression, we obtain the following two asymptotic relations for $n \to \infty$

$$\sum_{k \leq n} \frac{(n + a) \cos(bkn^{-1/4})}{k!} \ell_k \sim \frac{1}{2\sqrt{\pi}e} n^{\alpha/2-1/4} \ell_{\sqrt{n}} e^{2\sqrt{\pi}k^2/4} \cos(bn^{1/4});$$

$$\sum_{k \leq n} \frac{(n + a) \sin(bkn^{-1/4})}{k!} \ell_k \sim \frac{1}{2\sqrt{\pi}e} n^{\alpha/2-1/4} \ell_{\sqrt{n}} e^{2\sqrt{\pi}k^2/4} \sin(bn^{1/4}).$$

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