BEST $\lambda$-APPROXIMATION FOR ENTIRE FUNCTIONS OF FINITE ORDER

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Communicated by Stevan Pilipović

Abstract. We investigate so-called BLAS problem for entire functions whose logarithm of maximum moduli is regularly varying in the sense of Karamata or de Haan. We also give an interesting application on Hadamard-type convolutions with regularly varying sequences of arbitrary index.

Introduction

For a given entire function $f(z) := \sum_{k=0}^{\infty} a_k z^k$ we define, as usual, its partial sums $S_n(z) := \sum_{k=0}^{n} a_k z^k$ and maximum moduli $M_f(r) := \max |f(z)|_{|z|=r} = |f(re^{i\theta})| = |f(z_0)|$. The order $\rho$ of $f(z)$ is $\rho := \limsup_{r \to \infty} \log \log M_f(r)/\log r$.

In [5], we gave a notion of best $\lambda$-approximating (BLAS) partial sums for functions analytic on the unit disc. This can be easily reformulated for entire functions (analytic on the whole complex plane) as:

If there is an integer-valued function $n := n(r, \lambda) \to \infty \ (r \to \infty)$ such that

$$\frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} o(1), & 0 < \lambda < 1; \\
1 + o(1), & \lambda > 1; 
\end{cases} \quad (r \to \infty) \quad (I)$$

we call $S_{n(r,\lambda)}(z_0)$ the best $\lambda$-approximating partial sum (BLAS).

In this way, we are going to find the “shortest” partial sum which is well approximating $f(z)$ at the point(s) of maximal growth, for $r$ sufficiently large.

Note that analogous to (I) is the relation between moduli of BLAS and $M_f(r)$.

An important role in measuring the growth of entire functions of order $\rho > 0$ have the class $R_\rho$ consisting of regularly varying functions in the sense of Karamata; i.e., $g(x) \in R_\rho$ can be represented in the form $g(x) = x^\rho l(x), x > 0, \rho \in R$, where $\rho$ is the index of regular variation and $l(x) \in R_0$ is a slowly varying function i.e., positive, measurable and satisfying $l(tx)/l(x) \sim 1, \forall t > 0 (x \to \infty)$.

1991 Mathematics Subject Classification. Primary 30D20, 30D30.
An immediate consequence which we are going to use in the sequel is
\[
g(x) \in R_\rho \iff g(tx)/g(x) \sim t^\rho, \quad \forall t > 0, \quad (x \to \infty) \tag{0.1}
\]

For further information on regular variation we recommend [1] and [4]. In order
to study entire functions of order zero we shall consider a subclass of \( R_0 \) i.e. de
Haan’s class \( \Pi_t \),
\[
h(x) \in \Pi_t \iff \frac{h(tx) - h(x)}{l(x)} \sim \log t, \quad \forall t > 0; \quad (x \to \infty) \tag{0.2}
\]
where \( l(x) \in R_0 \) is called the auxiliary function and we can take \( h(x) = l(x) + \int_{x}^{\infty} l(t)/t \, dt \) [1, pp. 160–165].

We are going to apply our BLAS results to entire functions with non-negative
coefficients i.e., to determine the asymptotic behavior of Hadamard-type con-
volutions \( T_f(r) := \sum n^\alpha l_n a_n r^n \), where \( (l_n) \) are slowly varying sequences; therefore
improving our results from [6].

Results

Let \( f(z), M_f(r), n(r, \lambda), \rho, z_0 \) be defined as above. Then we have the following.

Theorem 1. If \( \log M_f(r) \in R_\rho, \rho > 0, \) and
\[
n(r, \lambda) \sim \lambda \rho \log M_f(r), \quad (r \to \infty) \tag{1}
\]

Then
\[
\frac{S_n(r, \lambda)(z_0)}{(z_0)} = \left\{ \begin{array}{ll}
\epsilon_1(r, \lambda), & 0 < \lambda < 1; \\
1 + \epsilon_2(r, \lambda), & \lambda > 1,
\end{array} \right.
\]

with
\[
|\epsilon_i(r, \lambda)| \leq \frac{1}{|\lambda^{1/\rho - 1}|} M_f(r)^{-\lambda \log \lambda - \lambda + 1 + o(1)}, \quad i = 1, 2 \quad (r \to \infty).
\]

Proof. An implementation of Cauchy’s Integral formula gives
\[
\frac{1}{2\pi i} \int_C f(w) \frac{z_0/w^{n+1}}{w - z_0} \, dw = \left\{ \begin{array}{ll}
-S_n(z_0), & z_0 \notin \text{int } C, \\
f(z_0) - S_n(z_0), & z_0 \in \text{int } C.
\end{array} \right. \tag{2}
\]

Let the contour \( C \) be a circle \( w = r\lambda^{1/\rho}e^{i\phi} \). Since
\[
|z_0| = r \left\{ \begin{array}{ll}
> |w|, & 0 < \lambda < 1; \\
< |w|, & \lambda > 1;
\end{array} \right.
\]

from (2) we get
\[
I := \frac{1}{2\pi} \int_{0}^{2\pi} f(r\lambda^{1/\rho}e^{i\phi}) \frac{\lambda - n/r\rho e^{i(\phi_0 - \phi)}}{\lambda^{1/\rho}e^{i(\phi_0 - \phi)} - 1} \, d\phi = \left\{ \begin{array}{ll}
-S_n(z_0)/f(z_0), & 0 < \lambda < 1; \\
1 - S_n(z_0)/f(z_0), & \lambda > 1.
\end{array} \right. \tag{3}
\]
Taking into account that \( |f(z_0)| = M_f(r) \), a simple estimation of \( I \) gives

\[
|I| \leq \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(r\lambda^{1/\rho}e^{i\phi})|}{|f(z_0)|} e^{-\frac{\pi \log \lambda}{\lambda^{1/\rho} - 1}} \, d\phi \right|
\]

i.e.,

\[
|I| \leq \frac{M_f(r\lambda^{1/\rho})}{M_f(r)} e^{-\frac{\pi \log \lambda}{\lambda^{1/\rho} - 1}}
\]

Since, for sufficiently large \( r \) (cf. (0,1)),

\[
\log M_f(r) \in R_\rho \Rightarrow \frac{\log M_f(r\lambda^{1/\rho})}{\log M_f(r)} = \lambda(1 + o(1)),
\]

putting in (4), \( n = n(r, \lambda) = \lambda \rho \log M_f(r)(1 + o(1)) \), we finally obtain

\[
|I| \leq \frac{1}{\lambda^{1/\rho} - 1} \exp(-\log M_f(r)(\lambda \log \lambda - \lambda + 1 + o(1))) \quad (r \to \infty)
\] (5)

For \( \lambda > 0, \lambda \neq 1, \lambda \to \lambda \log \lambda - \lambda + 1 \) is strictly positive, hence the assertion of Theorem 1 follows.

In the case of entire functions of order zero, we shall treat the subclass whose logarithm of the maximum modulus belongs to de Haan’s class \( \Pi_l \) with unbounded auxiliary function \( l \in R_0 \).

Taking in (2) the contour \( C : |w| = \lambda r; \lambda > 0, \lambda \neq 1 \), the estimation (4) can be rewritten as

\[
|I| \leq \frac{1}{\lambda^{1/\rho} - 1} \exp(I(r) \frac{\log M_f(\lambda r) - \log M_f(r)}{l(r)} - n \log \lambda)
\] (6)

Putting there \( n = n(r, \lambda) = \lambda l(1 + o(1)) \) (\( r \to \infty \)) and taking into account the definition (0.2), (6) yields

\[
|I| \leq \frac{1}{\lambda^{1/\rho} - 1} \exp(-l(r)((\lambda - 1) \log \lambda + o(1))) \quad (r \to \infty).
\]

Since \( \lambda \to (\lambda - 1) \log \lambda \) is strictly positive for \( \lambda > 0, \lambda \neq 1 \), we obtain

\textbf{Theorem 2.} If \( \log M_f(r) \in \Pi_l \) with auxiliary function \( R_0 \ni l(r) \to \infty (r \to \infty) \), and \( n(r, \lambda) \sim \lambda l(r) \) (\( r \to \infty \)), then

\[
S_{n(r, \lambda)}(z_0) \leq \begin{cases} \mu_1(r, \lambda), & 0 < \lambda < 1 \\ 1 + \mu_2(r, \lambda), & \lambda > 1 \end{cases}
\]

with

\[
|\mu_i| \leq \frac{1}{|\lambda - 1|} e^{-l(r)((\lambda - 1) \log \lambda + o(1))}, \quad i = 1, 2; \quad (r \to \infty).
\]

It is easy now to derive, from the Theorems above, various estimation formulae for the moduli of BLAS. We need the following in the sequel:
PROPOSITION 1. Under the conditions of the Theorem 1, for any \( \sigma > 1, \rho > 0 \) and \( n_1(r, \sigma) \sim e^{\sigma \rho \log M_f(r)} (r \to \infty) \),

\[
\left| \sum_{n > n_1(r, \sigma)} a_n \zeta_0^n \right| < C M_f(r)^{-\sigma \log \sigma + \sigma(1)}.
\]

Proof. Applying Theorem 1 with \( \lambda > 1 \), we get

\[
\sum_{n > n(r, \lambda)} a_n \zeta_0^n = f(z_0) - S_{n(r, \lambda)}(z_0) = -f(z_0) e_2(r, \lambda),
\]
i.e., since \( |f(z_0)| = M_f(r) \),

\[
\left| \sum_{n > n(r, \lambda)} a_n \zeta_0^n \right| = M_f(r) |e_2(r, \lambda)| \leq \frac{1}{\lambda^{1/\rho} - 1} M_f(r)^{\lambda(\log \lambda - 1) + \sigma(1)}.
\]

Putting there \( \lambda = e\sigma, n(r, \lambda) = n_1(r, \sigma) \) we obtain the proof with \( C = C(\rho, \sigma) = 1/(e\sigma)^{1/\rho - 1} \).

Now, we give some applications of our BLAS results. For a given entire function \( f(r) := \sum_n a_n r^n \) with non-negative coefficients, there is a classical problem of estimating asymptotic behavior of Hadamard-type convolutions \( T_f(r) := \sum_n c_n a_n r^n \) \( (r \to \infty) \).

In the well-known book [3, pp. 20, 197, 198] this is solved in the case

\[
c_n := n^\alpha, \quad \alpha \in R; \quad \log f(r) \sim ar^\rho, \quad a, \rho > 0 \quad (r \to \infty).
\]

In [6] we obtain a result for regularly varying \( c_n := n^\alpha l_n, \alpha \in R, c_0 := 1 \) and \( \log f(r) \in SR_{\rho}, \rho > 0 \).

Here \( l_n \) are slowly varying sequences [2], for example:

\[
\log^a 2n, \quad \log^b (\log 3n), \quad \exp \left( \frac{\log n}{\log \log 3n} \right), \quad \exp(\log^c 2n); \quad a, b \in R; \quad 0 < c < 1;
\]

and \( SR_{\rho} \subset R_{\rho} \) is the class of smoothly varying functions [1, pp. 44–47].

Using Theorem 1 and Lemmas 1 and 2 below, we are going to prove the next:

THEOREM 3. Let an entire function \( f(r) := \sum_n a_n r^n, a_n \geq 0, \) of order \( \rho > 0 \), satisfy \( \log f(r) \in R_{\rho} \). Then

\[
T_f(r) := \sum_n c_n a_n r^n \sim \rho^\alpha c_{[\log f(r)]} f(r) \quad (r \to \infty),
\]

for any regularly varying sequence \( (c_n) \) of arbitrary index \( \alpha \in R \).

For the proof we need two lemmas.
Lemma 1. Define

\[ S(\lambda, r) := \sum_{n \leq \lambda n(r)} a_n r^n, \quad a_n \geq 0, \quad n \in N, \]

where \( n(r) \) increases to infinity with \( r \), and an operator \( T \) acting on \( S \):

\[ TS(\lambda, r) := \sum_{n \leq \lambda n(r)} c_n a_n r^n, \quad n \in N, \]

where \( (c_n)_{n \in N} \) is a regularly varying sequence of index \( \alpha \in R \).

If there exist \( g, g_1, g_2 : R^+ \to R^+; b_1 : (0,1) \to R^+; b_2 : (1,\infty) \to R^+ \), and

\[ \lim_{r \to \infty} \frac{\log n(r)}{g_i(r)} = 0, \quad i = 1, 2; \]

such that

\[ \frac{S(\lambda, r)}{g(r)} = \begin{cases} O(e^{-b_1(\lambda)g_1(r)}), & 0 < \lambda < 1, \\ A + O(e^{-b_2(\lambda)g_2(r)}), & \lambda > 1, \end{cases} \quad A \in R^+, \quad (r \to \infty), \]

then

\[ \frac{TS(\lambda, r)}{g(r)} = \begin{cases} o(c_{[n]}), & 0 < \lambda < 1, \\ c_{[n]}(A + o(1)), & \lambda > 1; \end{cases} \quad (r \to \infty). \]

In this form Lemma 1 is proved in [7] as the Theorem A.

Lemma 2. For any regularly varying sequence \((c_n)\) of index \(\alpha \in R\),

\[ c_{[nx]} \sim c_{[nx]} \sim \lambda^\alpha c_{[x]} \quad (x \to \infty). \]

This is a well-known fact [1, pp. 49–53].

Now, we are able to prove cited Theorem 3.

First of all, note that the condition \( a_n \geq 0 \) implies that on the circle \( |z| = r \) we have

\[ |f(z)| = \left| \sum_{n} a_n z^n \right| \leq \sum_{n} a_n |z|^n = \sum_{n} a_n r^n = f(r). \]

Hence, \( M_f(r) = f(r), \) \( z_0 = r \) and, comparing the assertions from Theorem 1 and Lemma 1, we see that the conditions of Lemma 1 are satisfied with

\[ g(r) := f(r); \quad n(r) := \rho \log f(r); \quad g_1(r) = g_2(r) := \log f(r); \quad A := 1. \]

Write, in terms of Theorem 1,

\[ T_f(r) := \sum_{n} c_n a_n r^n = \sum_{n \leq n \leq 2r} c_n a_n r^n + \sum_{n > n \leq 2r} c_n a_n r^n = S_1 + S_2. \]
Applying Lemma 1 with $\lambda := 2e2^a > 1$ and Lemma 2, we obtain

$$S_1 \sim c_{[\log f(r)],[\log f(r)]} f(r), \quad \alpha \in R \quad (r \to \infty). \quad (3.1)$$

For estimating $S_2$, note that (0.1) implies $2^a \log f(r) \sim \log f(2r), \ (r \to \infty)$, i.e., using Proposition 1 with $n(r,2e2^a) = n_1(2r,2), \ we get$

$$S_2 \leq \sup_{n} (2^{-n}c_n) \sum_{n > n_1(2r,2)} a_n(2r)^n = O(1) \sum_{n > n_1(2r,2)} a_n(2r)^n = O(1) e^{-2e2^a \log 2 + o(1)} \log f(2r) = o(c_{[\log f(r)]} f(r)) \ (r \to \infty). \quad (3.1)$$

This, along with (3.1) yields the proof of Theorem 3.

In the same manner, using Theorem 2 and Lemmas 1 and 2 we can prove

**Theorem 4.** Under the conditions of Theorem 2, for a given entire function $f$ of order zero,

$$f(r) := \sum_{n} a_n r^n, \quad a_n \geq 0, \quad n \in N,$$

we have

$$T_f(r) := \sum_{n} c_n a_n r^n \sim c_{[r]} f(r) \quad (r \to \infty),$$

for any regularly varying sequence $(c_n)$ of arbitrary index.

Finally, we shall give two examples. To illustrate the results from Theorems 1 and 3, we shall consider the Mittag-Leffler function $E_s(z)$,

$$E_s(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+ns)}, \quad s > 0.$$

Then, for $z = re^{i\phi},$

$$z_0 = r, \quad M_{E_s}(r) = E_s(r) = \sum_{n=0}^{\infty} \frac{r^n}{\Gamma(1+ns)}, \quad s > 0,$$

and (cf. [1, p. 329]),

$$E_s(r) \sim (1/s) e^{1/s}; \quad \log E_s(r) \sim r^{1/s} \quad (r \to \infty).$$

Hence $E_s(z)$ is an entire function of order $1/s$ and Theorem 1 gives:

**Proposition 2.** For the Mittag-Leffler function $E_s(z)$,

$$n(r, \lambda) \sim (\lambda/s)r^{1/s}, \quad s > 0, \quad (r \to \infty)$$

and

$$S_{n(r, \lambda)}(r) := \sum_{n \leq n(r, \lambda)} \frac{r^n}{\Gamma(1+ns)} = o(E_s(r)) \quad \text{for } 0 < \lambda < 1;$$

$$S_{n(r, \lambda)}(r) \sim E_s(r) \quad \text{for } \lambda > 1 \quad (r \to \infty).$$

Similarly, applying Theorem 3 and the properties of $E_s(r)$ mentioned above, we obtain
Proposition 3. For any slowly varying sequence \((\ell_n)\) and arbitrary \(\alpha \in R\),
\[
T_E(r) := \sum_{n=1}^{\infty} \frac{n^\alpha \ell_n}{1 + ns} r^n \sim (1/s)^{\alpha+1} r^{\alpha/s} \ell(r^{1/s}) \exp(r^{1/s}) \quad (r \to \infty).
\]

For the next example we take the function \(Q(z)\) of zero order,
\[
Q(z) := \prod_{n=1}^{\infty} \left(1 + \frac{z}{q^n}\right) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{(q-1)(q^2-1)\cdots(q^n-1)}; \quad q > 1.
\]
(Euler, cf. [8, p. 32])

For \(z = re^{\theta}\), we have
\[
z_0 = r; \quad M_Q(r) = Q(r) = \prod_{n=1}^{\infty} (1 + r/q^n).
\]

That \(\log Q(r)\) belongs to de Haan’s class \(\Pi_1\) follows from Hardy’s result (cf. [9, p. 171]),
\[
\log Q(r) = \frac{1}{2 \log q} \left(\log r - \frac{1}{2} \log q\right)^2 + O(1) \quad (r \to \infty).
\]
Therefore, for \(t > 0\),
\[
\log Q(tr) - \log Q(r) \sim \frac{\log t}{\log q} \log r,
\]
i.e.,
\[
\frac{\log Q(tr) - \log Q(r)}{\log r / \log q} \to \log t, \quad \forall t > 0 \quad (r \to \infty)
\]

According to (0.2), \(\log Q(r) \in \Pi_1\) and we can take for the auxiliary function \(l(r) = \frac{\log r}{\log q} \in R_0\).

Applying Theorem 2 we obtain

Proposition 4. For the function \(Q(r)\) defined above,
\[
n(r, \lambda) \sim \left(\frac{\lambda}{\log q}\right) \sim \log r \quad (r \to \infty),
\]
and
\[
S_{n(r, \lambda)}(r) := 1 + \sum_{n \leq n(r, \lambda)} \frac{r^n}{(q-1)(q^2-1)\cdots(q^n-1)} = o(Q(r)) \quad \text{for } 0 < \lambda < 1;
\]
\[
S_{n(r, \lambda)} \sim Q(r) \quad \text{for } \lambda > 1 \quad (r \to \infty).
\]

Theorem 4 also gives

Proposition 5. For any slowly varying sequence \((\ell_n)\), \(n \in N\) and arbitrary real \(\alpha\) we have (when \(r \to \infty\)),
\[
T_Q(r) := 1 + \sum_{n=1}^{\infty} \frac{n^\alpha \ell_n}{(q-1)(q^2-1)\cdots(q^n-1)} r^n \sim \frac{1}{\log^\alpha q} \log^\alpha r \ell(\log r) Q(r)
\]
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(Received 17 01 2001)