UNIVALENT HARMONIC MAPPINGS BETWEEN JORDAN DOMAINS

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ABSTRACT. We give a classification of univalent harmonic functions of a Jordan domain onto a convex Jordan domain with boundary which does not contain linear segments. It is interesting that the boundary function must be continuous but not necessarily a univalent function, contrary to the case of conformal mappings.

1. Introduction and notation

The complex twice-differentiable function \( w = f(z) = u + iv \) is called harmonic if \( u \) and \( v \) are real harmonic functions. Let be \( f \) a harmonic diffeomorphism. If there is \( k < 1 \) such that \( |f_u| \leq k|f_z| \), then we say that \( f \) is a quasiconformal function (q.c.). We denote by QCH the class of harmonic quasiconformal functions.

The following formula is the Poisson integration formula and it plays a very important role in harmonic function theory. For any bounded harmonic function \( f \) defined on the unit disc \( U \) there is a bounded \( L^1 \) function \( g \) defined on the unit circle \( S^1 \) such that:

\[
f(z) = P[g](z) = \int_0^{2\pi} P(z, \theta) g(e^{i\theta}) \, d\theta
\]

where

\[
P(z, \theta) = \frac{1 - |z|^2}{2\pi |z - e^{i\theta}|^2}
\]

is the Poisson kernel.

**Lemma 1.1.** [1] If \( g \) is a continuous function then \( f \) has a continuous extension on \( U \).

**Lemma 1.2.** [3] Let \( \{\varphi_n, n \in \mathbb{N}\} \) be a sequence of non-decreasing functions from \([0, 2\pi]\) to \([-2\pi, 4\pi]\). Then there is a subsequence \( (\varphi_{n_k}) \subset (\varphi_n) \) and a function \( \varphi \) such that \( \varphi_{n_k}(x) \to \varphi(x) \) for \( x \in [0, 2\pi] \) and \( \varphi \) is a non-decreasing function.

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Lemma 1.3. Let $f : U \to V$ be a harmonic mapping of the unit disk $U$ into the Jordan domain $V$. If $f = P[e^{i\theta}]$ and if there exists a $\theta_0 \in [0, 2\pi]$ such that
\[ \lim_{r \to \theta_0} f(e^{i\theta}) = A_0 \quad \text{and} \quad \lim_{\theta \to \theta_0} f(e^{i\theta}) = B_0, \]
then for $\lambda \in [-1, 1]$ we have:
\[ \lim_{r \to \infty} f \left( e^{i\theta_0} \left( \frac{Re^{i\lambda\pi/2} - 1}{Re^{i\lambda\pi/2} + 1} \right) \right) = \frac{1}{2} (1 - \lambda) A_0 + \frac{1}{2} (1 + \lambda) B_0. \]

The proof of this lemma follows from definition, but it has some technical difficulties. For details see [9].

Proposition 1.1 (Choquet). Let $f(z) = P[e^{i\varphi(z)}](z)$, such that $\varphi$ is a non-decreasing 1-1 function and $\varphi(0) = 0$ and $\varphi(2\pi) = 2\pi$. Then $f$ is a univalent harmonic function of the unit disk onto itself.

Note that, this proposition is valid in a more general form. Indeed, only the convexity of the co-domain is important.

Proposition 1.2. Let $f_n : \Omega \to U$ be a sequence of k-q.c. mappings of a Jordan domain $\Omega$ into the unit disc $U$. Also let $f = \lim_{n \to \infty} f_n$; then either:

1. $f$ is constant or
2. $f$ is function with two points value or
3. $f$ is k-q.c.

This proposition is due to Lehto and Virtanen [5].

Proposition 1.3. Let $f_n : U \to V$, be a sequence of harmonic diffeomorphisms of the unit disc $U$ onto a Jordan domain $V$ such that $f_n \to f$. Then:

1. $f$ is univalent of $U$ into $V$ or
2. $f(z) = c + e^{i\varphi} \cdot R(z)$ where $R$ is real harmonic function $\neq 0$ or
3. $f \equiv \text{const}.

Proof. Without loss of generality we may assume that $f_n$ are sense preserving mappings. Let $a_n = \frac{f_n}{f_n*z}$. Then $a_n : U \to U$ is an analytic function. Since $f_n \to f$ one gets $f_n \to f$ and $\frac{f_n}{f_n*z} \to \frac{f}{f*z}$. Because $|f_n| \leq |f_n*z|$ it follows that $|f_n| < |f_n*z|$. If $f_n \equiv 0$, then $f \equiv \text{const}$ which gives (1).

Otherwise $a = \frac{f}{f*z}$ is an analytic function except for some points. Since $|a| \leq 1$, these points are admissible singularities of $a$. Hence $a$ is analytic on $U$. If $|a(z)| = 1$ at some point, then there exists a $\varphi$ such that $a(z) \equiv e^{i\varphi}$. Hence
\[ f(z) = c + e^{i\varphi} \sum_{n=1}^{\infty} (a_n e^{-i\varphi} z^n + \overline{a_n} e^{i\varphi} \bar{z}^n) = c + e^{i\varphi} \cdot R(z). \]

In this case (2) is true.

Let us now suppose $|a(z)| < 1$. We are going to prove that (1) holds. Let $\alpha, \beta$ be distinct points in $U$ such that $|\alpha| < r$, $|\beta| < r$, and $r < 1$. Let $D_r = \{ z : |z| < r \}$. Then the functions $F_n = f_n|_{D_r}$ are q.c. Since $F_n \to f = f|_{D_r}$, from Proposition 1.2 it follows that $F : D_r \to F(D_r)$ is k’ q.c. Consequently $f(\alpha) = F(\alpha) \neq F(\beta) = f(\beta)$. It follows that $f$ is 1-1. This completes the proof. □
2. The main results

**Lemma 2.1.** Let \( f : U \to V \) be a harmonic sense preserving diffeomorphism of the unit disk \( U \) into a Jordan domain \( V \). Then there exists a function \( \varphi : S \to \partial V \) with at most countably many points of discontinuity, all of them of the first type, such that: \( f = P[\varphi] \).

**Proof.** Let \( g : V \to U \) be a biholomorphism, which exists by Riemann mapping theorem. Then the function \( F = g \circ f : U \to U \) is a sense preserving diffeomorphism. Let \( U_n = \{z : |z| < \frac{n}{n-1}\} \), \( \Delta_n = F^{-1}(U_n) \) and let \( g_n \) be a biholomorphism of the Jordan domain \( U \) onto the domain \( \Delta_n \) such that \( g_n(0) = 0 \), and \( g'_n(0) > 0 \). Without loss of generality we can suppose \( 0 \in \Delta_n \) because the last relation holds for \( n \) large enough. Then the function:

\[
F_n = \frac{n}{n-1} F \circ g_n = \frac{n}{n-1} g \circ f \circ g_n : U \to \overline{U}
\]

is a sense preserving homeomorphism. Let \( \varphi_n = F_n|_S \) and let \( (\varphi_n) \) be a convergent subsequence of \( (\varphi_n) \) which exists because of Lemma 1.1. Then \( \varphi_n(e^{i\theta}) = e^{\varphi_n(\theta)} \) where \( \varphi_n(\theta) \) is a monotone non-decreasing function. Let \( \varphi_0 = \lim \varphi_n \). Then \( \varphi_0 \) is a monotone non-decreasing function. Hence

\[
\lim_{k \to \infty} \frac{n_k}{n_k - 1} g \circ f \circ g_{n_k}|_S \to \varphi_0 \text{ if } k \to \infty.
\]

And consequently

\[
\lim_{k \to \infty} f \circ g_{n_k}(e^{i\theta}) = g^{-1}(\varphi_0(e^{i\theta})) \text{ for all } \theta
\]

because \( g \) is a homeomorphism from \( \overline{V} \) onto \( \overline{U} \). Since \( \varphi_k = f \circ g_{n_k}|_S \) is continuous and \( f \circ g_{n_k} \) is a harmonic function then from Lebesgue’s Dominated Convergence Theorem, (because the function \( g^{-1} \circ \varphi_0 \) is bounded), we obtain: \( f \circ g_{n_k} = P[\varphi_k] \to P[g^{-1} \circ \varphi_0] \), as \( k \to \infty \). It follows that the sequence \( g_{n_k} \) is convergent. Let \( \varphi_0(z) = \lim k \to \infty g_{n_k}(z) \). Since \( \varphi_0 \) is a conformal mapping from the unit disk onto itself which satisfies \( \varphi_0(0) = 0 \), and \( \varphi_0'(0) > 0 \) it follows that \( \varphi_0 \) is id. Hence \( f = P[g^{-1} \circ \varphi_0] = P[\varphi] \), where \( g^{-1} \) is continuous and \( \varphi_0 \) is a monotone non-decreasing function. Hence, it has no more than countably many points of discontinuity, which are of the first type. The lemma is proved.

**Theorem 2.1.** Let \( f : U \to \Omega \) be a harmonic diffeomorphism of the unit disk \( U \) onto a Jordan domain \( \Omega \) with boundary which contains no linear segments. Then the function \( f \) has a continuous extension from \( \overline{U} \) onto \( \overline{\Omega} \).

**Proof.** Follows from Lemma 1.2, Lemma 1.3 and homeomorphic properties of diffeomorphisms.

**Remark 2.1.** If a homeomorphism \( f \) reverses sense, then the homeomorphism \( \overline{f} \) preserves sense.

**Corollary 2.1.** Let \( f : \Omega \to V \) be a harmonic diffeomorphism of a Jordan domain \( \Omega \) onto a strict convex bounded domain \( V \). Then \( f \) has a continuous extension of \( \overline{\Omega} \) onto \( \overline{V} \).
Proof. Let \( \varphi : U \to \Omega \) be a conformal diffeomorphism of the unit disk \( U \) onto the Jordan domain \( \Omega \). Then \( F = f \circ \varphi : U \to \Omega \) is a harmonic diffeomorphism. Hence, the corollary follows from Carathéodory’s theorem and Theorem 2.1.

Remark 2.2. It is a natural question, whether the extension of this harmonic diffeomorphism is a homeomorphism. The answer to this question, in the general case, is negative.

Indeed, the next theorem holds.

Theorem 2.2. Let \( g_n \) be a convergent sequence of homeomorphisms between the unit circle and the convex Jordan domain \( \gamma = \text{int} \Omega \). Let \( g = \lim_{n \to \infty} g_n \) be a non constant and a non two valued function and let \( \text{conv}(g(S^1)) = \Omega \). Then \( f(z) = P[g](z) \) is a harmonic diffeomorphism of the unit disk onto \( \Omega \).

Proof. By Choquet’s theorem it follows that the functions \( f_n = P[g_n] \) is a univalent function from the unit disk onto \( \Omega \). On the other hand, because the family \( f_n \) is normal it has a convergent subsequence \( f_{n_k} \). Let \( f = \lim_{n \to \infty} f_{n_k} = P[g] \). Because \( g \) is not constant and it is a two-valued function, according to Proposition 1.3 it follows that it is univalent. (The function \( \theta \to e^{\ell}(\theta) \) has at last three value points.) On the other hand, because \( \text{conv}(f(S^1)) = \Omega \), it follows that \( f(U) = \Omega \).

Corollary 2.2. The harmonic function \( f : U \to U \) is a sense preserving diffeomorphism of the unit disk \( U \) onto itself if \( f = P[e^{\ell}(\theta)] \) where \( \ell \) is a continuous non-decreasing function such that \( \ell(0) = a \) and \( \ell(2\pi) = 2\pi + a \), \( a \in (-2\pi, 2\pi) \).

The proof follows from Theorem 2.1 and Theorem 2.2.

Example 2.1. Let \( \varphi(\theta) = \theta + k \sin \theta, \theta \in [0, 2\pi], 0 < k < 1 \). Then the function \( f = P[e^{\ell}(\theta)] \) is a harmonic diffeomorphism of the unit disk onto itself such that if \( 0 < k < 1 \) it is quasiconformal.

For the proof of the last assertion in the example, see [8].

Remark 2.3. In a private conversation I learned that A. Lyzzaik and W. Hengartner have similar unpublished results.

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References

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