SPECTRAL RADIUS AND SPECTRUM OF THE COMPRESSION OF A SLANT TOEPLITZ OPERATOR

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Communicated by Miroslav Jevtić

Abstract. A slant Toeplitz operator \( A_\varphi \) with symbol \( \varphi \) in \( L^\infty(T) \), where \( T \) is the unit circle on the complex plane, is an operator whose representing matrix \( M = (a_{kj}) \) is given by \( a_{kj} = \langle \varphi, z^{k-j} \rangle \), where \( \langle \cdot, \cdot \rangle \) is the usual inner product in \( L^2(T) \). The operator \( B_\varphi \) denotes the compression of \( A_\varphi \) to \( H^2(T) \) (Hardy space). In this paper, we prove that the spectral radius of \( B_\varphi \) is greater than the spectral radius of \( A_\varphi \), and if \( \varphi \) and \( \varphi^{-1} \) are in \( H^\infty \), then the spectrum of \( B_\varphi \) contains a closed disc and the interior of this disc consists of eigenvalues with infinite multiplicity.

1. Introduction

Let \( \varphi \in L^\infty(T) \). Then \( \varphi(z) \sim \sum_{i=-\infty}^{\infty} a_i z^i \), where \( a_i = \langle \varphi, z^i \rangle \) is the \( i \)-th Fourier coefficient of \( \varphi \) and \( \{z^i : i \in \mathbb{Z} \} \) is the usual basis, and \( \mathbb{Z} \) is the set of integers.

The slant Toeplitz operator \( A_\varphi \) is defined as follows: \( A_\varphi(z^k) = \sum_{i=-\infty}^{\infty} a_{2i-k} z^i \).

Furthermore, by [4, Proposition 1] \( A_\varphi = WM_\varphi \), where \( M_\varphi \) is a multiplication operator and \( Wz^{2n} = z^n, Wz^{2n-1} = 0 \), for \( n \in \mathbb{Z} \).

\( B_\varphi \), the compression of \( A_\varphi \) to \( H^2(T) \), is by definition \( B_\varphi = PA_\varphi|_{H^2} \). Equivalently, \( B_\varphi P = PA_\varphi P \), where \( P \) is the orthogonal projection from \( L^2 \) on to \( H^2 \). By [4, p. 846], \( B_\varphi = WT_\varphi \), where \( T_\varphi \) is the Toeplitz operator on \( H^2(T) \).

2. Spectral radius

Our aim is to prove that the spectral radius of \( B_\varphi \) is greater than the spectral radius of \( A_\varphi \). To do this we need the following lemmas.

\[ 2000 \text{ Mathematics Subject Classification. Primary 47B35; Secondary 47A10.} \]
\[ \text{Key words and phrases. Toeplitz Operator, Slant Toeplitz Operator, Compression, Spectrum.} \]
LEMMA 2.1. $(I - P)M_{z^n} \to 0$, as $n \to \infty$ in the strong operator topology, where $M_{z^n}$ is the multiplication by $z^n$ on $L^2(T)$.

PROOF. Let $f \in L^2(T)$ and $f(z) \sim \sum_{i=-\infty}^{\infty} a_i z^i$ be its Fourier expansion. Then

$$(I - P)M_{z^n} f = (I - P)\left( \sum_{i=-\infty}^{\infty} a_i z^{i+n} \right) = \sum_{i=-\infty}^{-n-1} a_i z^{i+n}.$$ Since, $\left| \sum_{i=-\infty}^{-n-1} a_i z^{i+n} \right|^2 = \sum_{i=-\infty}^{-n-1} |a_i|^2 \to 0$, as $n \to \infty$, the assertion follows. \hfill \square

The proof of the following lemma is similar to that of [1, Theorem 5].

LEMMA 2.2. $M_{z^n} B_\varphi P M_{z^n} \to A_\varphi$, as $n \to \infty$, in the strong operator topology.

PROOF. From Lemma 2.1, we know $(I - P)M_{z^n} \to 0$. This implies that $M_{z^n}(I - P)M_{z^n} \to 0$ which is equivalent to $M_{z^n}PM_{z^n} \to I$. Consider

$$M_{z^n} B_\varphi P M_{z^n} = M_{z^n} PA_\varphi P M_{z^n} = (M_{z^n}PA_\varphi)(M_{z^n}PM_{z^n}).$$

Since, for each $n = 1, 2, \ldots$, $M_{z^n}A_\varphi P M_{z^n} = A_\varphi$, [4, Proposition 3], and the first and the last factors converge to $I$, as $n \to \infty$, the assertion follows. \hfill \square

The following theorem is proved in [4, p. 85] but we give here a different proof.

THEOREM 2.3. $\|A_\varphi\| = \|B_\varphi\|$.

PROOF. For each $n = 1, 2, \ldots$, we have $\|M_{z^n} B_\varphi P M_{z^n}\| \leq \|B_\varphi\|$. So from Lemma 2.2, we get $\|A_\varphi\| \leq \|B_\varphi\|$. Since $B_\varphi$ is the compression of $A_\varphi$, we get $\|A_\varphi\| \geq \|B_\varphi\|$. The proof is complete. \hfill \square

We are now ready to prove our main result.

THEOREM 2.4. The spectral radius of $B_\varphi$ is greater than the spectral radius of $A_\varphi$.

PROOF. First, we prove the following claim by induction.

Claim: For $k = 1, 2, 3, \ldots$, $M_{z^n} B_\varphi^k P M_{z^n} \to A_\varphi^k$, as $n \to \infty$ in the strong operator topology.

For $k = 1$, the claim is true by Lemma 2.2. Let $m$ be any positive integer and assume that it is true for $k \leq m$, and consider

$$\|M_{z^n} B_\varphi^{m+1} P M_{z^n} - A_\varphi^{m+1}\| = \|M_{z^n}(B_\varphi^{m+1} P M_{z^n} - M_{z^n} A_\varphi^{m+1})\| = \|B_\varphi^{m+1} P M_{z^n} A_\varphi^{m+1} - M_{z^n} A_\varphi^{m+1}\| \\
\leq \|B_\varphi^{m+1} P M_{z^n} A_\varphi^{m+1} - PM_{z^n} A_\varphi^{m+1}\| + \|PM_{z^n} A_\varphi^{m+1} - M_{z^n} A_\varphi^{m+1}\|$$

By Lemma 2.1, the second term tends to 0 as $n$ approaches infinity. As to the first term, we use $M_{z^n} A_\varphi = A_{\varphi^n} M_{z^n}$ [4, Proposition 3] and get the following
approximation
\[
\|B^{m+1}_\varphi PM_{2m+1} - PM_{2m} A^{m+1}_\varphi\| = \|PA_\varphi B^m_\varphi PM_{2m+1} - PA_\varphi M_{2m} A^m_\varphi\| \\
\leq \|A_\varphi\| \|B^m_\varphi PM_{2m+1} - M_{2m} A^m_\varphi\| \\
= \|A_\varphi\| \|(B^m_\varphi PM_{2m} - M_{2m} A^m_\varphi)M_{2m_n}\| \\
\leq \|A_\varphi\| \|M_{2m} B^m_\varphi PM_{2m} - A^m_\varphi\|
\]

By the induction assumption, the last expression tends to 0 as \(n\) approaches infinity. Therefore, the claim is proved. The fact that \(\|M_{2m} B^k_\varphi PM_{2m} A^m_\varphi\| \leq \|B^k_\varphi\|\) for \(k = 1, 2, \ldots\), and the above claim imply that \(\|B^k_\varphi\| \geq \|A^m_\varphi\|\). This in turn implies that \(r(B_\varphi) \geq r(A_\varphi)\), where \(r\) represents spectral radius. \(\square\)

**Definition 2.5.** \(H^\infty = \{\varphi \in L^\infty(T) : \langle \varphi, z^n \rangle = 0 \text{ for } n < 0\}\). The elements of \(H^\infty\) are called analytic and their conjugates are called coanalytic.

**Corollary 2.6.** \(r(A_\varphi) = r(B_\varphi)\), if \(\varphi\) is analytic or coanalytic.

**Proof.** If \(\varphi\) is analytic, then \(B_\varphi = A_\varphi|_{H^2}\). Therefore, for each \(k = 1, 2, 3, \ldots\), \(\|B^k_\varphi\| \leq \|A^k_\varphi\|\). This implies \(r(B_\varphi) \leq r(A_\varphi)\). This together with Theorem 2.4, gives the required assertion. If \(\varphi\) is coanalytic, then \(B^*_\varphi = A^*_\varphi|_{H^2}\). By the same argument, we get \(r(A^*_\varphi) = r(B^*_\varphi)\). Consequently \(r(A_\varphi) = r(B_\varphi)\). \(\square\)

The following fact is indicated in [4, p. 856], but we give a different proof below.

**Theorem 2.7.** If \(\varphi\) is invertible in \(L^\infty(T)\), then \(r(A_\varphi) \geq r(A_{\varphi^{-1}})^{-1}\).

**Proof.** First, we show that \(\varphi(z)\) is invertible if and only if \(\varphi(z^2)\) is invertible. Suppose \(\varphi\) is invertible. Then \(\varphi \varphi^{-1} = 1\). Therefore, by [4, p. 846] \(W^* \varphi W \varphi^{-1} = 1\). Equivalently \(\varphi(z^2) \varphi^{-1}(z^2) = 1\). Hence \(\varphi(z^2)\) is invertible. Conversely, if \(\varphi(z^2)\) is invertible, then \(\varphi(z^2) \varphi^{-1}(z^2) = 1\). This and [4, p. 847] implies \(W \varphi(z^2) W \varphi^{-1}(z^2) = 1\), which is equivalent to \(\varphi \varphi^{-1} = 1\). Therefore \(\varphi\) is invertible.

Let \(h(z) = \varphi(z^2)\) be invertible. Then \(hh^{-1} = 1\). This and [4, p. 847] implies that \((Wh)(Wh)^{-1} = 1\). Therefore \((Wh)^{-1} = W(h^{-1})\). This in turn implies that, for each \(n = 1, 2, 3, \ldots\)

\[(\varphi^{-1})_n = (\varphi_n)^{-1},\]

where

\[\varphi_n = \underbrace{W(W(\ldots (W(h^2)))h^2 \ldots)h^2)}_{n \text{ times}}\]

From this and [4, p. 851], we get

\[r(A_{h^{-1}}) = \lim_{n \to \infty} \|((\varphi^{-1})_n)^{1/2n} = \lim_{n \to \infty} \|((\varphi_n)^{-1})^{1/2n}}\]

\[\geq \left[ \lim_{n \to \infty} \|\varphi_n^{1/2n}\right]^{-1} = \left[ r(A_h)\right]^{-1}\]

Therefore, \(r(A_h) \geq [r(A_{h^{-1}})]^{-1}\).
Since $\sigma(A_{\varphi}) = \sigma(A_{\varphi^{-1}})$, where $\sigma$ denotes the spectrum [4, Lemma 9], it follows that $r(A_{\varphi}) \geq [r(A_{\varphi^{-1}})]^{-1}$.

**Corollary 2.8.** $r(B_{\varphi}) \geq r(B_{\varphi^{-1}})^{-1}$, if $\varphi$ and $\varphi^{-1}$ are analytic or $\varphi$ and $\varphi^{-1}$ are coanalytic.

**Proof.** This is an immediate consequence of Corollary 2.6 and Theorem 2.7. \hfill $\Box$

### 3. Spectrum

Ho [4] showed that, for invertible $\varphi$ in $L^\infty$, the spectrum of $A_{\varphi}$ contains a closed disk consisting of eigenvalues of $A_{\varphi}$. We also show, for $\varphi$ and $\varphi^{-1}$ in $H^\infty$, that the spectrum of $B_{\varphi}$ contains a closed disc and the interior of this disc consists of eigenvalues with infinite multiplicity, by using the idea of the proof of Proposition 10 in [4].

**Theorem 3.1.** Let $\varphi$ and $\varphi^{-1}$ be in $H^\infty$. Then the spectrum of $B_{\varphi}$ contains a closed disc and the interior of this disc consists of eigenvalues with infinite multiplicity.

**Proof.** Assume first that $\lambda \neq 0$. Suppose that $B_{\varphi^{-1}}^* - \lambda$ is onto. Since $B_{\varphi^{-1}}^* = T_{\varphi} W$, we have, for $f$ in $H^2$,

\[(B_{\varphi^{-1}}^* - \lambda)f = (W^* T_{\varphi} - \lambda)f = (W^* T_{\varphi} - \lambda P_0)f \oplus (-\lambda P_0 f)\]

where $P_0$ is the projection on the closed span of $\{z^{2n} : n = 0, 1, 2 \ldots\}$ in $H^2(T)$ and $P_0 = I - P_{\varphi}$. Now let $0 \neq g_0$ be in $P_0(H^2)$. Since $B_{\varphi^{-1}}^* - \lambda$ is onto, there exists a nonzero vector $g$ in $H^2(T)$ such that $(B_{\varphi^{-1}}^* - \lambda)g = g_0$. But then from the computations above, we have $(W^* T_{\varphi} - \lambda P_0)f = 0$, because $g_0 \neq 0$. Since $\lambda \neq 0$ and $T_{\varphi}$ is invertible [2, Theorem 7.1], it follows that $\lambda W^* T_{\varphi}(\lambda^{-1} - T_{\varphi^{-1}})W = 0$, and the fact that $W^*$ is an isometry implies that $(\lambda^{-1} - T_{\varphi^{-1}})W = 0$. This in turn implies $\lambda^{-1} \in \sigma_p(B_{\varphi^{-1}}^*)$, where $\sigma_p$ denotes the point spectrum. Since $\dim P_0(H^2) = \infty$, it follows that $\lambda^{-1}$ is of infinite multiplicity. Now, for $\lambda \in \rho(B_{\varphi^{-1}}^*)$, the resolvent of $B_{\varphi^{-1}}^*$, the operator $B_{\varphi^{-1}}^* - \lambda$ is invertible (hence onto), so we have

\[D = \{\lambda^{-1} : \lambda \in \rho(B_{\varphi^{-1}}^*)\} \subseteq \sigma_p(B_{\varphi^{-1}}^*)\]

Since $B_{\varphi^{-1}}^* = T_{\varphi} W$, $B_{\varphi} = W T_{\varphi}$ and $T_{\varphi}$ is invertible, we have $\sigma_p(B_{\varphi^{-1}}^*) = \sigma_p(B_{\varphi})$ [3, Problem 61]. Therefore $D \subseteq \sigma_p(B_{\varphi}^{-1})$. So by replacing $\varphi^{-1}$ with $\varphi$, we have shown that for any invertible $\varphi$ in $H^\infty$, the spectrum of $B_{\varphi}$ contains a disc consisting of eigenvalues with infinite multiplicity. Therefore, by the fact that the spectrum of any operator is compact, $\sigma(B_{\varphi})$ contains a closed disc and the interior of this disc consists of eigenvalues with infinite multiplicity. \hfill $\Box$

**Remark 3.2.** If $\varphi$ and $\varphi^{-1}$ are coanalytic, then $T_{\varphi} T_{\varphi^{-1}} = T_{\varphi^{-1}} T_{\varphi} = I$ [2, Theorem 7.1]. Therefore, one can repeat the proof above and arrive at the same conclusion as Theorem 3.1, that is, if $\varphi$ and $\varphi^{-1}$ are coanalytic, then the spectrum
of $B_{\varphi}$ contains a closed disc and the interior of this disc consists of eigenvalues of infinite multiplicity.

Remark 3.3. The radius of the closed disc contained in $\sigma_p(B_{\varphi})$ is equal to $(r(B_{\varphi^{-1}}))^{-1}$, because if $D_0 = \{0\} \cup \{\lambda^{-1} : |\lambda| > r(B_{\varphi^{-1}})\}$, then $D_0 \subseteq \{\lambda^{-1} : \lambda \in \rho(B_{\varphi^{-1}}) \cup \{0\} \subseteq \sigma_p(B_{\varphi})$ and the radius of the disc $D_0$ is equal to $(r(B_{\varphi^{-1}}))^{-1}$. Hence $r(B_{\varphi}) \geq (r(B_{\varphi^{-1}}))^{-1}$. This relation is also proved in Theorem 2.7.

Remark 3.4. If $\varphi(z) = 1$, then $r(B_{\varphi^{-1}}) = r(B_{\varphi}) = 1$ by the spectral radius formula for $A_{\varphi}$ [4, p. 851] and Corollary 2.6. Hence, the spectrum of $B_{\varphi}$ is the closed unit disc by Theorem 3.1 and Remark 3.3. Since the eigenvalues are of infinite multiplicity, it follows that the essential spectrum of $B_{\varphi}$ is the same as the spectrum of $B_{\varphi}$.

References