AN ASYMPTOTIC FORMULA
FOR A SUM INVOLVING ZEROS
OF THE RIEMANN ZETA-FUNCTION

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Abstract. E. Landau gave an interesting asymptotic formula for a sum involving zeros of the Riemann zeta-function. We give an asymptotic formula which can be regarded as a smoothed version of Landau’s formula.

1. Introduction

Let \( \zeta(s) \) be the Riemann zeta-function. It is important to study non-trivial zeros \( \rho = \beta + i\gamma \) of \( \zeta(s) \). Weil’s explicit formula is one of useful formulas for the study of \( \rho \). Roughly speaking, it connects certain sums involving \( \rho \) with sums involving prime numbers in terms of test functions and those Mellin transforms. We refer to Lang [6] or Patterson [7] for the details of Weil’s explicit formula.

In this paper, as an application of Weil’s explicit formula with a certain test function, we shall study the asymptotic behaviour of a quantity involving \( \rho \), that is,

\[
\sum_{\rho} e^{u\rho^2-v\rho},
\]

Some suitable choice of the test function enables us to get asymptotic formulas for (1.1).

Theorem 1.1. (i) For \( v = u \) or \( v = 0 \) we have

\[
\sum_{\rho} e^{u\rho^2-v\rho} = \frac{1}{\sqrt{16\pi u}} \log \frac{1}{u} - \frac{\log(16\pi^2) + \zeta}{\sqrt{16\pi u}} + O(1), \quad u \to +0,
\]

where \( \zeta \) is the Euler constant, and the sum \( \sum_{\rho} \) runs over all non-trivial zeros \( \rho \) counting with multiplicity.

(ii) For any integer \( m \geq 2 \) we have

\[
\sum_{\rho} e^{u\rho^2+(\log m)\rho} = \frac{\Lambda(m)}{\sqrt{4\pi u}} + O(1), \quad u \to +0,
\]

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where $\Lambda(m) = \log p$ if $m$ is a power of a prime $p$ and $\Lambda(m) = 0$ otherwise. The implied constant depends on $m$.

(ii') Let $K$ be a closed interval contained in $(-\infty, 0) - \bigcup_m \{-\log m\}$, where $m$ is a power of a prime. Then we have
\[ \sum_{\rho} e^{\pi \rho^2 - v \rho} = O(1), \quad u \to +0, \]
uniformly for $v$ in $K$.

(iii) For any integer $m \geq 2$ we have
\[ \sum_{\rho} e^{\pi \rho^2 - (\log m) \rho} = -\frac{\Lambda(m)}{m \sqrt{4\pi u}} + O(1), \quad u \to +0. \]
The implied constant depends on $m$.

(iii') Let $K$ be a closed interval contained in $(0, \infty) - \bigcup_m \{\log m\}$, where $m$ is a power of a prime. Then we have
\[ \sum_{\rho} e^{\pi \rho^2 - v \rho} = O(1), \quad u \to +0, \]
uniformly $v$ in $K$.

We can see asymptotic behaviours different from each other for the quantity (1.1) and the difference depends on the choice of $v$. The first and second terms on the right-hand side of the asymptotic formula in (i) come from the logarithmic derivative of the gamma factor appeared in the functional equation of $\zeta(s)$. On the other hand, the first terms on the right-hand sides of the asymptotic formulas in (ii) and (iii) come from the logarithmic derivative of $\zeta(s)$.

The asymptotic formula in (ii) is related to the results of Landau [5], Gonek [3] [4], and Fujii [2]. Landau [5] proved that, for fixed $x > 1$,
\[ \sum_{0 < x \leq T} x^\rho = -\frac{T}{2\pi} \Lambda(x) + O(\log T) \]
holds. Gonek [3] [4] gave uniform versions of Landau’s result, and Fujii [2] gave a refined formula for it under the Riemann Hypothesis. The asymptotic formula in (ii) may be regarded as a smoothed version of Landau’s with the measure given by the Gaussian function.

The asymptotic formula in (i) may be regarded as a smoothed version of the asymptotic formula for $N(T)$, number of non-trivial zeros $\rho$ with $0 < \gamma < T$. To see this, let us consider the case $v = u$ in (i) under the Riemann Hypothesis. Then the asymptotic formula in (i) is
\[ \sum_{\gamma} e^{-\pi (1/4 + \gamma^2)} = \frac{1}{\sqrt{16\pi u}} \log \frac{1}{u} - \frac{\log(16\pi^2) + C}{\sqrt{16\pi u}} + O(1). \]
The sum on the left-hand side is written as an integral form, and, by integration by parts, it follows that

\[- \int_0^\infty N(T)d(e^{-T^2}) = \frac{1}{2\sqrt{16\pi u}} \log \frac{1}{u} - \log(16\pi^2) + O(1).\]

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2. An explicit formula for a sum involving zeros of the Riemann zeta-function

In this section we give an explicit formula, which is a variant of Weil’s explicit formula.

Lemma 2.1. For any positive \(u\) and any real \(v\) we have

\[
\sum \rho e^{\rho^2-v\rho} = e^{u-v} - \frac{\log \pi}{\sqrt{4\pi u}} e^{-v^2/4u} + 1
\]

\[- \frac{1}{
\sqrt{4\pi u} \sum_{n=2}^{\infty} \Lambda(n) e^{-(v+\log n)^2/4u}} - \frac{1}{\sqrt{4\pi u} \sum_{n=2}^{\infty} \Lambda(n) e^{-(v-\log n)^2/4u}}\]

\[+ \frac{e^{u/4-v/2}}{2\pi} \int_{-\infty}^{\infty} \log \left| \frac{1}{4} + i \frac{t}{2} \right| e^{-u t^2 + i t (u-v)} dt = (E * G_u)(v),\]

where the functions \(E\) and \(G_u\) are defined by

\[E(x) = \left( \frac{1}{e^{2|x|} - 1} - \frac{1}{2|x|} + 1 \right) e^{-|x|^2 x/2}, \quad G_u(x) = \frac{1}{\sqrt{4\pi u}} e^{-x^2/4u},\]

and \(E \ast G_u\) means the convolution of \(E\) and \(G_u\), that is,

\[(E \ast G_u)(v) = \int_{-\infty}^{\infty} E(x) G_u(v-x) dx.\]

Proof. Since

\[\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi u}} e^{-(v+\log x)^2/4u} \frac{dx}{x} = e^{u x^2-v},\]

we have

\[\frac{1}{2\pi i} \int_{1+\delta-i\infty}^{1+\delta+i\infty} \frac{\zeta'(s)}{s} e^{u s^2-v s} ds = - \sum_{n=2}^{\infty} \Lambda(n) \frac{1}{\sqrt{4\pi u}} e^{-(v+\log n)^2/4u} \frac{1}{\sqrt{4\pi u}} e^{-(v-\log n)^2/4u}.\]

We also have

\[\frac{1}{2\pi i} \int_{1+\delta-i\infty}^{1+\delta+i\infty} \frac{\zeta'(s)}{s} e^{u s^2-v s} ds = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{\zeta'(s)}{s} e^{u s^2-v s} ds + \sum_{\rho} e^{\rho^2-v\rho} - e^{u-v}.\]
The first term on the right-hand side can be expressed in the following form by the functional equation of $\zeta(s)$:

$$
\begin{align*}
\frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \left( \frac{\zeta'(s)}{s} + \log \pi - \frac{1}{2} \Gamma' \left( \frac{s}{2} \right) - \frac{1}{2} \Gamma' \left( \frac{1}{2} - s \right) \right) e^{\rho s} ds \\
= \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} e^{-(v \log n)^2/4u} + \log \pi - e^{-v^2/4u} - 1 \\
- \frac{1}{4\pi} \int_{-\infty}^{\infty} \left( \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + i \frac{t}{2} \right) + \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} - i \frac{t}{2} \right) \right) e^{u(1/2+it)^2-v(1/2+it)} dt.
\end{align*}
$$

Hence we have

$$
\sum_{\rho} e^{\rho s} = -\log \pi - \frac{\log \pi}{\sqrt{4\pi u}} e^{-v^2/4u} + 1 \\
- \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} e^{-(v \log n)^2/4u} - \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} e^{-(v \log n)^2/4u} \\
+ \frac{e^{\rho s} e^{-v^2/2}}{4\pi} \int_{-\infty}^{\infty} \left( \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + i \frac{t}{2} \right) + \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} - i \frac{t}{2} \right) \right) e^{-ut^2+it(v-u)} dt.
$$

This formula is a special case of Weil’s explicit formula (with the test function $\frac{1}{\sqrt{4\pi u}} e^{-(v \log x)^2/4u}$), but we supply a proof to make the paper self-contained.

Let us denote the last term on the right-hand side by $H$. By the expression (see, for example, [1, p. 28, 1.16])

$$
\frac{\Gamma'}{\Gamma} (z) = \log z - \int_{0}^{\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x + 1} \right) e^{-x|z|} dx, \quad \text{Re } z > 0,
$$

we have

$$
H = \frac{e^{\rho s} e^{-v^2/2}}{4\pi} \int_{-\infty}^{\infty} \left( \log \left( \frac{1}{4} + i \frac{t}{2} \right) + \log \left( \frac{1}{4} - i \frac{t}{2} \right) \right) e^{-ut^2+it(v-u)} dt \\
- \frac{e^{\rho s} e^{-v^2/2}}{4\pi} \int_{0}^{\infty} \left( \frac{1}{e^x - 1} - \frac{1}{x + 1} \right) e^{-x/4} \sqrt{\frac{z}{u}} \left( e^{-\frac{x^2}{4u} - 2x} + e^{-\frac{x^2}{4u} + 2x} \right) dx \\
= \frac{e^{\rho s} e^{-v^2/2}}{2\pi} \int_{-\infty}^{\infty} \log \left( \frac{1}{4} + i \frac{t}{2} \right) e^{-ut^2+it(v-u)} dt \\
- \frac{1}{\sqrt{4\pi u}} \int_{-\infty}^{\infty} \left( \frac{1}{2|x| - 1} - \frac{1}{2|x| + 1} \right) e^{-|x|/2} e^{-v^2/2} dx.
$$

Hence we obtain the lemma. 

\[ \square \]

3. Proof of Theorem

To obtain the estimates in the theorem we consider separately each term on the right-hand side of Lemma 2.1.

**Lemma 3.1.** We have $0 \leq (E * G_{v})(v) \leq 1$. 

Proof. It is easy to verify that $0 \leq E(x) \leq 1$. Hence

$$0 \leq (E * G_u)(v) \leq \int_{-\infty}^{\infty} G_u(x)dx = 1. \quad \Box$$

Here, we remark on the convolution $(E * G_u)(v)$. The assertion of Lemma 2.1 is enough for the proof of the theorem, but we can obtain a more precise behaviour of the convolution. It is not hard to verify that the function $E$ has the property $|E(v - x) - E(v)| \leq C|x|$, where $C$ is a positive absolute constant. Hence we have

$$|E * G_u(v) - E(v)| \leq \int_{-\infty}^{\infty} |E(v - x) - E(v)|G_u(x)dx \leq C \int_{-\infty}^{\infty} |x|G_u(x)dx = C \sqrt{\frac{4u}{\pi}},$$

that is, $(E * G_u)(v) = E(v) + O(\sqrt{u})$.

The next lemma is the key for the proof of the theorem.

Lemma 3.2. For $0 < u < 1$ we have

$$\int_{-\infty}^{\infty} \log \left| \frac{1}{4} + \frac{t}{2} + e^{-at^2 + it(u-v)} \right| dt = \begin{cases} O \left( \frac{1}{\sqrt{u}} \right) & \text{if } v \neq u \text{ and } v \neq 0, \\ \sqrt{\frac{4u}{\pi}} \log \frac{1}{u} - \sqrt{\frac{4u}{\pi}} (4 \log 2 + C) + O(1), & \text{if } v = u \text{ or } v = 0, \end{cases}$$

where the implied constants are absolute.

Proof. Firstly, we consider the case $v \neq u$ and $v \neq 0$. We have

$$\int_{-\infty}^{\infty} \log \left| \frac{1}{4} + \frac{t}{2} + e^{-at^2 + it(u-v)} \right| dt$$

$$= \int_{0}^{\infty} \log \left( \frac{1}{16} + \frac{t^2}{4u} \right) \cdot e^{-\frac{1}{2}t^2} - it(u-v) \cdot \frac{dt}{\sqrt{u}}$$

$$= -\frac{1}{u-v} \int_{0}^{\infty} \frac{dt}{u-v} \left( \log \left( \frac{1}{16} + \frac{t^2}{4u} \right) \cdot e^{-it^2} \cos \left( \frac{t(u-v)}{\sqrt{u}} \right) \right) dt$$

$$= -\frac{\sqrt{u}}{(u-v)^2} \int_{0}^{\infty} \frac{dt}{(u-v)^2} \left( \log \left( \frac{1}{16} + \frac{t^2}{4u} \right) \cdot e^{-it^2} \cos \left( \frac{t(u-v)}{\sqrt{u}} \right) \right) dt$$

$$= -\frac{\sqrt{u}}{(u-v)^2} \left\{ \int_{0}^{\infty} \frac{2(-t^2 + u/4)}{(t^2 + u/4)^2} e^{-t^2} \cos \left( \frac{t(u-v)}{\sqrt{u}} \right) dt \right. \right.$$

$$\left. - \int_{0}^{\infty} \frac{8t^2}{t^2 + u/4} e^{-it^2} \cos \left( \frac{t(u-v)}{\sqrt{u}} \right) dt \right. \right.$$

$$\left. + \int_{0}^{\infty} \log \left( \frac{1}{16} + \frac{t^2}{4u} \right) \cdot e^{-it^2} \cos \left( \frac{t(u-v)}{\sqrt{u}} \right) dt \right\}$$

$$= -\frac{\sqrt{u}}{(u-v)^2} \left\{ I_1 + I_2 + I_3 \right\},$$

say. As for $I_1$ and $I_2$ we easily have
(3.2) \[ |I_1| \leq 2 \int_0^\infty \frac{1}{t^2 + u/4} e^{-t^2} dt \leq \frac{8}{u} \int_0^{\sqrt{\pi}} \frac{1}{t^2} dt = \frac{10}{\sqrt{u}}. \]

(3.3) \[ |I_2| \leq 8 \int_0^\infty e^{-t^2} dt = 4\sqrt{\pi}. \]

As for \( I_3 \) we have

(3.4) \[ |I_3| \leq \int_0^\infty \left| \log\left( \frac{1}{16} + \frac{t^2}{4u} \right) \right| e^{-t^2} (4t^2 + 2) dt \]

\[ \leq \log \frac{1}{u} \cdot \int_0^\infty e^{-t^2} (4t^2 + 2) dt + \int_0^2 \log\left( \frac{u}{16} + \frac{t^2}{4} \right) e^{-t^2} (4t^2 + 2) dt \]

\[ + \int_2^\infty \log\left( \frac{u}{16} + \frac{t^2}{4} \right) e^{-t^2} (4t^2 + 2) dt \]

\[ \ll \log \frac{1}{u} + \int_2^\infty \log t \cdot e^{-t^2} (4t^2 + 2) dt \ll \log \frac{1}{u}. \]

Substituting (3.2), (3.3), and (3.4) into (3.1), we obtain the first estimate of this lemma.

Next, we consider the case \( v = u \). We have

(3.5) \[ \int_{-\infty}^{\infty} \log\left( \frac{1}{4} + \frac{t^2}{2} \right) \cdot e^{-u^2} dt \]

\[ = \int_0^{\infty} \log\left( \frac{1}{16} + \frac{t^2}{4u} \right) \cdot e^{-t^2} \frac{dt}{\sqrt{u}} \]

\[ = \frac{1}{\sqrt{u}} \int_0^{\infty} e^{-t^2} dt + \frac{1}{\sqrt{u}} \int_0^{\infty} \log\left( \frac{u}{16} + \frac{t^2}{4} \right) \cdot e^{-t^2} dt \]

\[ = \frac{\sqrt{\pi}}{2 \sqrt{u}} + \frac{1}{\sqrt{u}} \int_0^{\infty} \log \frac{1}{4} \cdot e^{-t^2} dt + \frac{1}{\sqrt{u}} \int_0^{\infty} \log\left( \frac{1}{16} + \frac{u}{4t^2} \right) \cdot e^{-t^2} dt \]

\[ = \frac{\sqrt{\pi}}{2} + \frac{1}{\sqrt{u}} J_1 + \frac{1}{\sqrt{u}} J_2, \]

say. As for \( J_2 \) we have

(3.6) \[ J_2 = \int_0^{\sqrt{\pi}} \log\left( 1 + \frac{u}{4t^2} \right) \cdot e^{-t^2} dt + \int_{\sqrt{\pi}}^{\infty} \log\left( 1 + \frac{u}{4t^2} \right) \cdot e^{-t^2} dt \]

\[ \leq \int_0^{\sqrt{\pi}} \log\left( 1 + \frac{u}{4t^2} \right) dt + \frac{u}{4} \int_{\sqrt{\pi}}^{\infty} \frac{1}{t^2} e^{-t^2} dt \]

\[ = \sqrt{u} \log \frac{5}{4} + \int_0^{\sqrt{\pi}} \frac{2u}{4t^2} dt + \frac{u}{4} \int_{\sqrt{\pi}}^{\infty} \frac{1}{t^2} e^{-t^2} dt \]

\[ \leq \sqrt{u} \log \frac{5}{4} + 2\sqrt{u} + \frac{u}{4} \int_{\sqrt{\pi}}^{\infty} \frac{1}{t^2} dt \ll \sqrt{u}. \]
For \( J_1 \) we have

\[
J_1 = \int_0^\infty \log t \cdot e^{-t} \frac{dt}{2\sqrt{t}} - \log 4 \cdot \int_0^\infty e^{-t^2} dt
\]

\[
= \frac{1}{2} \Gamma'(\frac{1}{2}) - \sqrt{\pi} \log 2 = -\frac{\sqrt{\pi}}{2} (4 \log 2 + O). 
\]

Substituting (3.6) and (3.7) into (3.5), we obtain the second asymptotic formula in this lemma in the case \( v = u \).

Finally, we consider the case \( v = 0 \). We have

\[
\int_{-\infty}^{\infty} \log \left| \frac{1}{4} + i \frac{t^2}{2} \right| e^{-ut^2+iv} dt 
\]

\[
= \int_0^\infty \log \left( 1 + \frac{t^2}{4} \right) \cdot e^{-it^2} \cos(\sqrt{u}t) dt + \frac{1}{\sqrt{u}} \int_0^\infty \log \left( \frac{1}{4} + \frac{u}{4t^2} \right) \cdot e^{-it^2} \cos(\sqrt{u}t) dt 
\]

\[
= \log \frac{1}{\sqrt{u}} K_1 + \frac{1}{\sqrt{u}} K_2 + \frac{1}{\sqrt{u}} K_3, 
\]

say. As for \( K_3 \) we have

\[
|K_3| \ll J_2 \ll \sqrt{u}, 
\]

For \( K_1 \) and \( K_2 \) we use

\[
\cos(\sqrt{u}t) = 1 + O(u^2). 
\]

From (3.10) it follows that

\[
K_1 = \int_0^\infty e^{-t^2} dt + O\left( u \int_0^\infty e^{-t^2} dt \right) = \frac{\sqrt{\pi}}{2} + O(u), 
\]

\[
K_2 = \int_0^\infty \log \frac{t^2}{4} \cdot e^{-t^2} dt + O\left( u \int_0^\infty \log \frac{t^2}{4} \cdot e^{-t^2} dt \right) 
\]

\[
= -\frac{\sqrt{\pi}}{2} (4 \log 2 + O) + O(u). 
\]

Substituting (3.9), (3.11), and (3.12) into (3.8), we obtain the second asymptotic formula in this lemma in the case \( v = 0 \).

To obtain the theorem we now consider the asymptotic behaviour of the quantity

\[
- \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{\infty} \frac{A(n)e^{-(v+\log n)^2/4u}}{n} = \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^{\infty} \frac{A(n)}{n} e^{-(v-\log n)^2/4u} 
\]
in Lemma 2.1. The behaviour of this quantity depends on the choice of \( v \). For the case \( v = 0 \) and \( 0 < u < 1 \)

\[
e^{-\left(\log n\right)^2/4u} = e^{-\left(\log n\right)^2/8\sigma} e^{-\left(\log n\right)^2/8\sigma} \leq e^{-\left(\log n\right)^2/8} e^{-\left(\log 2\right)^2/8},
\]

and hence (3.13) is of exponential decay as \( u \to +0 \). For the case \( v = -\log m \), \( m \geq 2 \) is an integer, and \( 0 < u < 1 \) we have

\[
e^{-\frac{1}{4u}\left(-\log m + \log n\right)^2} \leq e^{-\frac{1}{4u}\left(-\log m + \log n\right)^2} e^{-\frac{1}{4u}\left(-\log m + \log (m+1)\right)^2}
\]

\[
\leq e^{-\frac{1}{4u}\left(\log n\right)^2\left(1 - \frac{\log m}{\log n}\right)^2} e^{-\frac{1}{4u}\left(-\log m + \log (m+1)\right)^2}, \quad n \neq m,
\]

and

\[
e^{-\frac{1}{4u}\left(-\log m - \log n\right)^2} \leq e^{-\frac{1}{4u}\left(-\log m + \log n\right)^2} \leq e^{-\frac{1}{8}\left(\log n\right)^2} e^{-\frac{1}{8}\left(\log 2\right)^2},
\]

and hence (3.13) is

\[
= -\frac{\Lambda(m)}{\sqrt{4\pi u}} + O\left(\frac{e^{-\frac{1}{4u}\left(\log 2\right)^2}}{\sqrt{u}} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} e^{-\frac{1}{8}\left(\log n\right)^2}\right)
\]

\[
+ \frac{e^{-\frac{1}{4u}\left(-\log m + \log (m+1)\right)^2}}{\sqrt{u}} \left(\sum_{m \neq n=2}^{m^2} \Lambda(n) + \sum_{n=m^2}^{n} \Lambda(n) e^{-\frac{1}{8}\left(\log n\right)^2}\right).
\]

For other \( v \) we can similarly consider the asymptotic behaviour of (3.13).

Combining the above arguments and Lemmas 2.1, 3.1, and 3.2, we obtain the assertion of the theorem.

References