ON REGULARLY VARYING MOMENTS FOR POWER SERIES DISTRIBUTIONS

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Abstract. For the power series distribution, generated by an entire function of finite order, we obtain the asymptotic behavior of its regularly varying moments. Namely, we prove that $E_w X^\alpha \ell(X) \sim (E_w X)^\alpha \ell(E_w X)$, $\alpha > 0$ ($w \to \infty$), where $\ell(\cdot)$ is an arbitrary slowly varying function.

0. Introduction

0.1. Denote by $A_\rho$ the class of transcendental entire functions with positive Taylor coefficients and of finite order $\rho$, $0 \leq \rho < \infty$.

Definition 1. Let $f(w) = \sum a_n w^n$, $f \in A_\rho$. A power series distribution with parameter $w > 0$, generated by $f$, is defined by (cf. [2])

$$P(X = n) := a_n w^n / f(w), \quad n = 0, 1, 2, \ldots$$

Our aim is to obtain the asymptotic behavior of the $k$-th moment $E_w X^k$ when $w \to \infty$, where

$$E_w X^k := \sum n^k P(X = n) = \sum n^k a_n w^n / f(w), \quad k = 1, 2, \ldots$$

Note that the expectation $E_w X$ is equal to

$$E_w X := \sum n a_n w^n / f(w) = w f'(w) / f(w). \quad (1)$$

For any $k$, consider the sequence of functions $f_k(w)$ defined recursively by

$$f_k(w) = w f_{k-1}(w), \quad k = 1, 2, \ldots; \quad f_0(w) = f(w) = \sum a_n w^n.$$

Then $f_k(w) = \sum n^k a_n w^n \in A_\rho$ and

$$E_w X^k = f_k(w) / f(w), \quad k = 1, 2, \ldots \quad (2)$$

We shall derive the asymptotic behavior of $E_w X^k$ for large $w$ by applying our following recent result:

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Theorem 1. [5] For an arbitrary \( f \in A_\rho \), we have
\[
\frac{f(w)f''(w)}{(f'(w))^2} \to 1 \quad (w \to \infty).
\]

independently of the order \( \rho \).

0.2. Further generalization leads to the concept of regularly varying moments \( E_wX^\alpha \ell(X) \) (cf. [1, p. 335]),
\[
E_wX^\alpha \ell(X) := \sum n^\alpha \ell(n)a_n w^n / f(w),
\]
where \( \alpha \) is a positive real number and \( \ell(\cdot) \) is a slowly varying function.

Definition 2. A positive continuous function \( \ell(\cdot) \), defined on \([x_0, \infty)\), is slowly varying if the asymptotic equivalence \( \ell(tx) \sim \ell(x) \) \((x \to \infty)\), holds for each \( t > 0 \).

For \( x \in [0, x_0) \) we can take \( f(x) := f(x_0) \). Some examples of slowly varying functions are
\[
\log x; \quad \log^b (\log x); \quad \exp (\log^c x); \quad \exp (\log x / \log \log x); \quad a, b \in \mathbb{R}, \quad 0 < c < 1.
\]

Functions \( g(\cdot) \) of the form \( g(x) = x^\mu \ell(x) \) are regularly varying with index \( \mu \in \mathbb{R} \) (cf. [1, p.18]). Each regularly varying function \( x^\mu \ell(x) \) generates a regularly varying sequence of the form \( \{n^\mu \ell(n)\}_{n=1}^\infty \).

The main tool for asymptotic estimation of regularly varying moments is the following theorem on matrix transforms with slowly varying sequences (cf. [4]).

Theorem 2. For a given complex-valued matrix \( (A_{nk})_{n,k=1}^\infty \) define \( t_n(\rho) := \sum k^\rho |A_{nk}| \). Suppose that for some positive constants \( a, A \), \( t_n(\rho) \) exists for \(-a \leq \rho \leq 0 \) and, for sufficiently large \( n \),
\[
(i) \quad \sum |A_{nk}| \geq A; \quad (ii) \quad t_n(0) \to 1; \quad (iii) \quad t_n(1) \to \infty; \quad (iv) \quad t_n(-a) = O((t_n(1))^{-a}).
\]

Then the asymptotic relation
\[
\sum A_{nk} \ell(k) = \ell(t_n(1)) \left( \sum A_{nk} \right) (1 + o(1)) \quad (n \to \infty),
\]
holds for all slowly varying sequences \( \{\ell(k)\}_{k=1}^\infty \).

0.3. Here we quote some well-known assertions we shall need in the sequel.

Lemma 1. If \( a(x) \sim b(x) \to \infty \), then \( \ell(a(x)) \sim \ell(b(x)) \) \((x \to \infty)\).

Lemma 2. [3, Vol. I, p. 36]. Let \( g(x) = \sum a_n x^n \), \( h(x) = \sum b_n x^n \), \( g, h \in A_\rho \). If \( a_n \sim b_n \) \((n \to \infty)\), then \( g(x) \sim h(x) \) \((x \to \infty)\).

Jensen’s inequality: \( EX^t \geq (EX)^t \), \( t > 1 \), and vice versa for \( 0 < t < 1 \).

Lemma 4. Lyapunov moments inequality asserts that, for \( r > s > t > 0 \),
\[
(EX^r)^{r-t} \leq (EX)^{s-t}(EX^t)^{r-s}.
\]
1. Results

1.1. The above Theorem 2 has many applications in real or complex analysis (cf. [4]). We shall apply it here to derive the following theorem on regularly varying moments for discrete laws.

**Theorem 3.** Let a discrete law $G$ be given by $P(X_n = k) = p_{nk} \geq 0$, $\sum_k p_{nk} = 1$. If $EX_n \to \infty$ and $EX_n^\beta \sim C_\beta(EX_n)^\beta$ ($n \to \infty$) for $\beta \in (0, B]$, $B > 1$, $C_\beta > 0$ then, for an arbitrary slowly varying function $\ell(\cdot)$, the asymptotic relation

$$EX_n^\beta \ell(X_n) \sim C_\beta(EX_n)^\beta \ell(EX_n) \quad (n \to \infty),$$

holds.

a. for each $\beta \in (0, B - 1]$;

b. for each $\beta \in (B - 1, B]$, if $EX_n^{B+1}$ exists and $EX_n^{B+1} = O((EX_n)^{B+1})$ ($n \to \infty$).

**Proof.** Putting $A_{nk} := p_{nk}k^\beta/C_\beta(EX_n)^\beta$, we find out that conditions (i) and (ii) of Theorem 2 are satisfied. For $\beta \in (0, B - 1]$, we obtain

$$t_n(1) = EX_n^{\beta+1}/C_\beta(EX_n)^\beta \sim (C_{\beta+1}/C_\beta)EX_n \to \infty \quad (n \to \infty).$$

Also,

$$t_n(-\beta/2) = EX_n^{\beta/2}/C_\beta(EX_n)^\beta \sim (C_{\beta/2}/C_\beta)(EX_n)^{-\beta/2} = O(t_n(1))^{-\beta/2} \quad (n \to \infty).$$

Therefore, the conditions of Theorem 2 are satisfied with $A = 1$, $a = \beta/2$ and the result for $\beta \in (0, B - 1]$ follows.

For the case $\beta \in (B - 1, B]$, we need the following

**Lemma 5.** Under the condition b of Theorem 3, we have

$$EX_n^{\beta+1} = O((EX_n)^{\beta+1}) \quad (n \to \infty),$$

for each $\beta \in (B - 1, B]$.

**Proof.** Indeed, applying Lyapunov moments inequality (Lemma 4) with $r = B + 1$, $s = \beta + 1$, $t = B$, we get

$$EX_n^{\beta+1} \leq (EX_n^B)^{B-\beta}(EX_n^{B+1})^{\beta+1-B}$$

$$= O((EX_n)^{B(B-\beta)}(EX_n)^{(B+1)(\beta+1-B)}) = O((EX_n)^{\beta+1}).$$

Now, by Jensen’s inequality $EX_n^{\beta+1} \geq (EX_n)^{\beta+1}$ i.e., $t_n(1) \geq EX_n/C_\beta \to \infty$ ($n \to \infty$).

Also, by (4),

$$t_n(-\beta/2) \sim (C_{\beta/2}/C_\beta)(EX_n)^{-\beta/2} = O((EX_n^{\beta+1}/(EX_n)^{\beta})^{-\beta/2}) = O((t_n(1))^{-\beta/2}).$$

Therefore, the conditions of Theorem 2 are satisfied and we get

$$EX_n^\beta \ell(X_n) \sim C_\beta(EX_n)^\beta \ell(EX_n^{\beta+1}/C_\beta(EX_n)^\beta) \quad (n \to \infty),$$

for each $\beta \in (B - 1, B]$. 

But, since
\[ \frac{E X_n}{C_\beta} \leq \frac{E X_{n+1}}{C_\beta (E X_n)^\beta} = O(E X_n) \quad (n \to \infty), \]
it follows by the uniform convergence theorem for slowly varying functions (cf. [1, p.6]), that
\[ \ell\left( \frac{E X_{n+1}}{C_\beta (E X_n)^\beta} \right) \sim \ell(E X_n) \quad (n \to \infty). \]

\section*{1.2.} We turn back now to the asymptotic evaluation of regularly varying moments for power series distributions. Using Theorem 3 above, it will be shown that this evaluation is equivalent to the following theorem on moments of power series distributions.

\textbf{Theorem 4.} For each \( \alpha > 0 \), we have \( E w X^\alpha \sim (E w X)^\alpha \quad (w \to \infty) \).

For the generating entire function \( f(w) = \sum a_k w^k \in A_\rho \), recall (1) and (2):
\[
E_w X = \sum k a_k w^k / f(w) = w f'(w) / f(w);
\]
\[
E_w X^m = \sum k^m a_k w^k / f(w) = f_m(w) / f(w).
\]

The proof of Theorem 4 requires some preliminary lemmas.

\textbf{Lemma 6.} The expectation \( E_w X \) is a monotone increasing and unbounded function in \( w \).

\textbf{Proof.} Since
\[
\frac{d}{dw} \left( E_w X \right) = E_w X^2 - (E_w X)^2 > 0,
\]
we conclude that \( E_w X \) is a monotone increasing function in \( w \). If it is bounded, then there exists a \( d > 0 \) such that \( E_w X < d \) for each \( w > 0 \). By (1) we get \( f'(w) / f(w) < d / w \), and integrating we find \( f(w) = O(w^d) \). Hence in this case \( f \) is a polynomial, which contradicts our assumption that \( f \) is a transcendental entire function. \( \square \)

\textbf{Lemma 7.} For \( m \in \mathbb{N} \), \( f_m(w) \sim w^m f^{(m)}(w) \quad (w \to \infty) \).

\textbf{Proof.} Note that \( f \in A_\rho \) implies \( f^{(m)} \in A_\rho \), \( m = 1, 2, \ldots \). Since, for fixed \( m \in \mathbb{N} \),
\[
f_m(w) = \sum k^m a_k w^k, \quad w^m f^{(m)}(w) = \sum_{k \geq m} k(k-1) \cdots (k-m+1) a_k w^k;
\]
\[
k(k-1) \cdots (k-m+1) \sim k^m \quad (k \to \infty),
\]
the result follows by Lemma 2. \( \square \)

\textbf{Lemma 8.} For each \( m \in \mathbb{N} \) we have \( E_w X^{m+1} / E_w X^m \sim E_w X \quad (w \to \infty) \).
Proof. Applying Theorem 1, we obtain
\[ \frac{E_w X^2}{(E_w X)^2} \to 1 \quad (w \to \infty), \]  
because
\[ \frac{E_w X^2}{(E_w X)^2} - \frac{1}{E_w X} \frac{f(w)f''(w)}{(f'(w))^2} \to 1 \quad (w \to \infty), \]
and, by Lemma 6, \( 1/E_w X \to 0 \).

Since Theorem 1 is valid for each \( f \in A_p \) and \( f^{(m)} \in A_p, \ m = 1, 2, \ldots \), replacing \( f \) by \( f^{(m)} \), we get
\[ \frac{f^{(m+1)}(w)f^{(m-1)}(w)}{(f^{(m)}(w))^2} \to 1 \quad \text{i.e.} \quad \frac{f^{(m+1)}(w)}{f^{(m)}(w)} \sim \frac{f^{(m)}(w)}{f^{(m-1)}(w)} \quad (w \to \infty). \]

Hence by Lemma 7 and (6),
\[ \frac{E_w X^{m+1}}{E_w X^m} = \frac{f_{m+1}(w)}{f_m(w)} \sim \frac{w^{m+1}f^{(m+1)}(w)}{w^m f^{(m)}(w)} \sim \frac{w^m f^{(m)}(w)}{w^{m-1} f^{(m-1)}(w)} \]
\[ \sim \frac{f_m(w)}{f_{m-1}(w)} = \frac{E_w X^m}{E_w X_{m-1}}, \ n \in N. \]

Therefore,
\[ \frac{E_w X^{m+1}}{E_w X^m} \sim \frac{E_w X^m}{E_w X^m} \sim \cdots \sim \frac{E_w X^2}{E_w X} \sim E_w X \quad (w \to \infty). \]

A simple consequence of the previous lemma is the following:

Lemma 9. For each \( m \in N \), we have \( E_w X^m \sim (E_w X)^m \quad (w \to \infty). \)

Proof. Indeed,
\[ E_w X^m = (E_w X) \prod_{k=1}^{m-1} (E_w X^{k+1}/E_w X^k) \sim (E_w X)^m \quad (w \to \infty). \]

For the rest of the proof of Theorem 4 we apply Lemma 4.

Let \( m > \alpha > m - 1, \ m \in N \). Then Lyapunov’s inequality and Lemma 9 give
\[ E_w X^\alpha \leq (E_w X^m)^{\alpha - m + 1} (E_w X^{m - 1})^{n - \alpha} \sim (E_w X)^m (E_w X)^{(m - 1)(m - \alpha)} \]
\[ = (E_w X)^\alpha. \]

Hence
\[ \limsup_{w \to \infty} \frac{E_w X^\alpha}{(E_w X)^\alpha} \leq 1. \]

Now, let \( r = m + 1, s = m, t = \alpha \). We get \( (E_w X^m)^{m+1-\alpha} \leq (E_w X^\alpha)(E_w X^{m+1})^{n-\alpha} \),
i.e.,
\[ E_w X^\alpha \geq (E_w X^m)^{m+1-\alpha} (E_w X^{m+1})^{\alpha - m} \sim (E_w X)^m (E_w X)^{(m+1)(\alpha - m)} \]
\[ = (E_w X)^\alpha. \]
Therefore, \[ \liminf_{w \to \infty} \frac{E_w X^\alpha}{(E_w X)^\alpha} \geq 1, \]
and this concludes the proof of Theorem 4. \qed

1.3. Combining the last two theorems, we finally obtain a theorem on regularly varying moments for power series distributions.

**Theorem 5.** For a power series distribution generated by an entire function \( f(w) = \sum a_k w^k \in A_\rho \), we have
\[ E_w X^\alpha \ell(X) \sim (E_w X)^\alpha \ell(E_w x), \quad \alpha > 0 \quad (w \to \infty), \]
i.e.,
\[ \sum k^\alpha \ell(k)a_k w^k / f(w) \sim (w f'(w)/f(w))^\alpha \ell(w f'(w)/f(w)) \quad (w \to \infty), \]
where \( \ell(\cdot) \) is an arbitrary slowly varying function.

As an example we take the well-known Poisson distribution. Applying Theorem 5, we obtain

**Theorem 6.** For the Poisson law defined by \( P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \lambda > 0, \ k = 0, 1, 2, \ldots, \) we have
\[ E X^\alpha \ell(X) := \sum k^\alpha \ell(k) \frac{\lambda^k}{k!} e^{-\lambda} \sim \lambda^\alpha \ell(\lambda), \quad \alpha > 0 \quad (\lambda \to \infty). \]

### References