DRAZIN INVERSES OF OPERATORS
WITH RATIONAL RESOLVENT

Christoph Schmoeger

Communicated by Stevan Pilipović

Abstract. Let $A$ be a bounded linear operator on a Banach space such that
the resolvent of $A$ is rational. If $0$ is in the spectrum of $A$, then it is well
known that $A$ is Drazin invertible. We investigate spectral properties of the
Drazin inverse of $A$. For example we show that the Drazin inverse of $A$ is a
polynomial in $A$.

1. Introduction and terminology

In this paper $X$ is always a complex Banach space and $\mathcal{L}(X)$ the Banach algebra
of all bounded linear operators on $X$. For $A \in \mathcal{L}(X)$ we write $N(A)$ for its kernel
and $A(X)$ for its range. We write $\sigma(A)$, $\rho(A)$ and $R_\lambda(A)$ for the spectrum, the
resolvent set and the resolvent operator $(A - \lambda)^{-1}$ ($\lambda \notin \sigma(A)$) of $A$, respectively.
The ascent of $A$ is denoted by $\alpha(A)$ and the descent of $A$ is denoted by $\delta(A)$.
An operator $A \in \mathcal{L}(X)$ is Drazin invertible if there is $C \in \mathcal{L}(X)$ such that
(i) $CAC = C$, (ii) $AC = CA$ and (iii) $A^{\nu+1}C = A^\nu$ for nonnegative integer $\nu$.

In this case $C$ is uniquely determined (see [2]) and is called the Drazin inverse
of $A$. The smallest nonnegative integer $\nu$ such that (iii) holds is called the index
$i(A)$ of $A$. Observe that

$0 \in \rho(A) \Leftrightarrow A$ is Drazin invertible and $i(A) = 0$.

The following proposition tells us exactly which operators are Drazin invertible
with index $> 0$:

1.1. Proposition. Let $A \in \mathcal{L}(X)$ and let $\nu$ be a positive integer. Then the
following assertions are equivalent:

(1) $A$ is Drazin invertible and $i(A) = \nu$.
(2) $\alpha(A) = \delta(A) = \nu$.
(3) $R_\lambda(A)$ has a pole of order $\nu$ at $\lambda = 0$.

2000 Mathematics Subject Classification: Primary 47A10.
Key words and phrases: rational resolvent, Drazin inverse.
Proof. [2, §5.2] and [3, Satz 101.2]. \qed

The next result we will use frequently in our investigations.

1.2. Proposition. Suppose that \( A \in \mathcal{L}(X) \) is Drazin invertible, \( i(A) = \nu \geq 1 \), \( P \) is the spectral projection of \( A \) associated with the spectral set \( \{0\} \) and that \( C \) is the Drazin inverse of \( A \). Then

\[
P = I - AC, \quad N(C) = N(A^\nu) = P(X),
\]

\[
C(X) = N(P) = A^\nu(X),
\]

\( C \) is Drazin invertible, \( i(C) = 1 \), \( ACA \) is the Drazin inverse of \( C \),

\[
0 \in \sigma(C) \text{ and } \sigma(C) \setminus \{0\} = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(A) \setminus \{0\} \right\}.
\]

Proof. We have \( P = I - AC, N(A^\nu) = P(X) \) and \( \sigma(C) \setminus \{0\} = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(A) \setminus \{0\} \right\} \) by [2, §52]. It is clear that \( 0 \in \sigma(C) \). From Proposition 1.1 and [3, Satz 101.2] we get \( N(P) = A^\nu(X) \). If \( x \in X \) then \( Cx = 0 \iff Px = x \), hence \( N(C) = P(X) \). From \( P = I - AC = I - CA \) it is easily seen that \( N(P) = C(X) \).

Let \( B = ACA \). Then

\[
C^2B = CBC = CACAC = CAC = C,
\]

\[
CB = CACA = ACAC = BC
\]

\[
BCB = ACACACA = ACACA = ACA = B.
\]

This shows that \( C \) is Drazin invertible, \( B \) is the Drazin inverse of \( C \) and that \( i(C) = 1 \). \qed

Now we introduce the class of operators which we will consider in this paper. We say that \( A \in \mathcal{L}(X) \) has a rational resolvent if

\[
R_\lambda(A) = \frac{P(\lambda)}{q(\lambda)}
\]

where \( P(\lambda) \) is a polynomial with coefficients in \( \mathcal{L}(X) \), \( q(\lambda) \) is polynomial with coefficients in \( \mathbb{C} \) and where \( P \) and \( q \) have no common zeros. We use the symbol \( \mathcal{F}(X) \) to denote the subclass of \( \mathcal{L}(X) \) consisting of those operators whose resolvent is rational. For \( A \in \mathcal{L}(X) \) let \( \mathcal{H}(A) \) be the set of all functions \( f : \triangle(f) \to \mathbb{C} \) such that \( \triangle(f) \) is an open set in \( \mathbb{C} \), \( \sigma(A) \subseteq \triangle(f) \) and \( f \) is holomorphic on \( \triangle(f) \). For \( f \in \mathcal{H}(A) \) the operator \( f(A) \in \mathcal{L}(X) \) is defined by the usual operational calculus (see [3] or [4]).

The following proposition collects some properties of operators in \( \mathcal{F}(X) \). An operator \( A \in \mathcal{L}(X) \) is called algebraic if \( p(A) = 0 \) for some nonzero polynomial \( p \).

1.3. Proposition. Let \( A \in \mathcal{L}(X) \). Then

\begin{enumerate}
  \item \( A \in \mathcal{F}(X) \) if and only if \( \sigma(A) \) consists of a finite number of poles of \( R_\lambda(A) \).
  \item \( A \in \mathcal{F}(X) \) if and only if \( A \) is algebraic.
  \item If \( \dim A(X) < \infty \), then \( A \in \mathcal{F}(X) \).
\end{enumerate}
(4) If $A \in \mathcal{F}(X)$ and $f \in \mathcal{H}(A)$, then $f(A) = p(A)$ for some polynomial $p$.
(5) If $A \in \mathcal{F}(X)$, the $p(A) \in \mathcal{F}(X)$ for every polynomial $p$.

Proof. [4, Chapter V.11] \hfill \square

1.4. Corollary. Suppose that $A \in \mathcal{F}(X)$ and $0 \in \rho(A)$. Then $A^{-1} \in \mathcal{F}(X)$ and $A^{-1}$ is a polynomial in $A$.

Proof. Define the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $f(\lambda) = \lambda^{-1}$. Then $f \in \mathcal{H}(A)$ and $f(A) = A^{-1}$. Now apply Proposition 1.3 (4) and (5). \hfill \square

Remark. That $A \in \mathcal{F}(X)$ and $0 \in \rho(A)$ implies $A^{-1} \in \mathcal{F}(X)$ is also shown in [1, Theorem 2]. In Section 2 we will give a further proof of this fact.

2. Drazin inverses of operators in $\mathcal{F}(X)$

Throughout this section $A$ will be an operator in $\mathcal{F}(X)$ and $\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}$, where $\lambda_1, \ldots, \lambda_k$ are the distinct poles of $R_\lambda(A)$ of orders $m_1, \ldots, m_k$ (see Proposition 1.3 (1)).

Recall that $m_j = \alpha(A - \lambda_j) = \delta(A - \lambda_j)$ ($j = 1, \ldots, k$). Let

$$m_A(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}.$$ (2.1)

By [4, Theorem V.10.7],

$$m_A(A) = 0.$$

The polynomial $m_A$ is called the minimal polynomial of $A$. It follows from [4, Theorem V.10.7] that $m_A$ divides any other polynomial $p$ such that $p(A) = 0$. In what follows we always assume that $m_A$ has degree $n$, thus $n = m_1 + \cdots + m_k$ and that $m_A$ has the representations (2.1) and

$$m_A(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_{n-1} \lambda^{n-1} + \lambda^n.$$ (2.2)

Observe that

$$0 \in \rho(A) \iff a_0 \neq 0$$

and that

$$0$$

is a pole of order $\nu \geq 1$ of $R_\lambda(A) \iff a_0 = \cdots = a_{\nu-1} = 0$ and $a_\nu \neq 0$.

Now we are in a position to state our first result. Recall from Proposition 1.1 that if $\lambda_0 \in \sigma(A)$, then $A - \lambda_0$ is Drazin invertible.

2.1. Theorem. If $\lambda_0 \in \sigma(A)$ and if $C$ is the Drazin inverse of $A - \lambda_0$, then there is a scalar polynomial $p$ such that $C = p(A)$.

Proof. Without loss of generality we can assume that $\lambda_0 = \lambda_1 = 0$. Let $\nu = m_1$. Then we have

$$m_A(\lambda) = a_\nu \lambda^\nu + a_\nu + 1 \lambda^{\nu+1} + \cdots + \lambda^{n-1} + \lambda^n$$

and that $a_\nu \neq 0$. Let

$$q_\ell(\lambda) = -\frac{1}{a_\nu}(a_{\nu+1} + a_{\nu+2} \lambda + \cdots + \lambda^{n-(\nu+1)}).$$
Then
\[ A^{\nu+1}q_1(A) = -\frac{1}{a_\nu}(a_{\nu+1}A^{\nu+1} + a_{\nu+2}A^{\nu+2} + \cdots + A^n) = -\frac{1}{a_\nu}(m_A(A) - a_\nu A^\nu) = A^\nu. \]

Let \( B = q_1(A) \). Then \( A^{\nu+1}B = A^\nu \) and \( BA = AB \). For the Drazin inverse \( C \) we have
\[ A^{\nu+1}C = A, \ CAC = C \quad \text{and} \quad CA = AC. \]

Thus
\[ A^{\nu+1}(B - C) = A^{\nu+1}B - A^{\nu+1}C = A^\nu - A^\nu = 0 \]

This shows that \((B - C)(X) \subseteq N(A^{\nu+1})\). By Proposition 1.1, \( \alpha(A) = \nu \), thus \((B - C)(X) \subseteq N(A^\nu)\), therefore \((B - C)(X) \subseteq P_1(X)\), where \( P_1 \) denotes the spectral projection of \( A \) associated with the spectral set \( \{0\} \) (see Proposition 1.2). Since \( P_1 = I - AC = I - CA \), it follows that
\[ B - C = P_1(B - C) = P_1B - P_1C = P_1B - (I - CA)C \]
\[ = P_1B - C + CAC = P_1B, \]

thus \( C = B - P_1B \). We have \( P_1 = f(A) \) for some \( f \in \mathcal{H}(A) \). By Proposition 1.3 (4), \( f(A) = q_2(A) \) for some polynomial \( q_2 \). Now define the polynomial \( p \) by \( p = q_1 - q_2q_1 \). It results that
\[ p(A) = q_1(A) - q_2(A)q_1(A) = B - P_1B = C. \]
\[ \square \]

### 2.2. Corollary
If \( \lambda_0 \in \sigma(A) \) and if \( C \) is the Drazin inverse of \( A - \lambda_0 \), then \( C \in \mathcal{F}(X) \).

**Proof.** Theorem 2.1 and Proposition 1.3 (5).
\[ \square \]

### 2.3. Corollary
Let \( A \) be a complex square matrix and \( \lambda_0 \) a characteristic value of \( A \). Then the Drazin inverse of \( A - \lambda_0 \) is a polynomial in \( A \).

**Proof.** Theorem 2.1 and Proposition 1.3 (3).
\[ \square \]

Let \( T \in \mathcal{L}(X) \). An operator \( S \in \mathcal{L}(X) \) is called a pseudo inverse of \( T \) provided that \( TST = T \). In general the set of all pseudo inverses of \( T \) is infinite and this set consists of all operators of the form \( STSU + U - STUT \), where \( U \in \mathcal{L}(X) \) is arbitrary (see [2, Theorem 2.3.2]). Observe that if \( T \) is Drazin invertible with \( i(T) = 1 \), then the Drazin inverse of \( T \) is a pseudo inverse of \( T \).

### 2.4. Corollary
If \( \lambda_0 \in \sigma(A) \), then the following assertions are equivalent:
1. \( \lambda_0 \) is a simple pole of \( R_\lambda(A) \);
2. there is a pseudo inverse \( B \) of \( A - \lambda_0 \) such that \( B(A - \lambda_0) = (A - \lambda_0)B \);
3. there is a polynomial \( p \) such that \( p(A) \) is a pseudo inverse of \( A - \lambda_0 \).
Proof. (1) $\iff$ (2): Proposition 1.1.

(1) $\implies$ (3): We can assume that $\lambda_0 = 0$. Let $q_1$ and $B$ as in the proof of Theorem 2.1. Then $A^2B = A$ and $AB = BA$, hence $ABA = A$.

(3) $\implies$ (1): Again we can assume that $\lambda_0 = 0$. With $B = p(A)$ we have $ABA = A$ and $AB = BA$. Set $C = BAB$; then $ACA = A$, $CAC = C$ and $AC = CA$. This shows that $C$ is the Drazin inverse of $A$ and that $i(A) = 1$. By Proposition 1.1, $\lambda_0 = 0$ is a simple pole of $R(A)$.

2.5. Corollary. Let $X$ be a complex Hilbert space and suppose that $N \in L(X)$ is normal and that $\sigma(N)$ is finite. We have:

(1) $N \in F(X)$,
(2) If $\lambda_0 \in \sigma(N)$, then there is a polynomial $p$ such that
$$
(N - \lambda_0)p(N - \lambda_0) = N - \lambda_0.
$$

Proof. By [3, Satz 111.2], each point in $\sigma(N)$ is a simple pole of $R_N$, thus $N \in F(X)$. Now apply Theorem 2.4.

Our results suggest the following.

Question. If $A \in F(X)$ and if $B$ is a pseudo inverse such that $AB = BA$, does there exist a polynomial $p$ with $B = p(A)$?

The answer is negative:

Example. Consider the square matrix
$$
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
$$

It is easy to see that the minimal polynomial of $A$ is given by $m_A(\lambda) = \lambda^2 - 3\lambda = \lambda(\lambda - 3)$, hence $\sigma(A) = \{0, 3\}$ and $A^2 = 3A$. Let
$$
B = \frac{1}{3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

Then $AB = BA = \frac{1}{3}A$, thus $ABA = \frac{1}{3}A^2 = A$, hence $B$ is a pseudo inverse of $A$. Since $A^2 = 3A$, any polynomial in $A$ has the form $\alpha I + \beta A$ with $\alpha, \beta \in \mathbb{C}$. But there are no $\alpha$ and $\beta$ such that $B = \alpha I + \beta A$. An easy computation shows that the Drazin inverse of $A$ is given by $\frac{1}{3}A$ and that $i(A) = 1$.

If $0$ is a simple pole of $R_N$, then we have seen in Theorem 2.4 that $A$ has a pseudo inverse. If $0$ is a pole of $R_N$ of order $\geq 2$, then, in general $A$ does not have a pseudo inverse, as the following example shows.

Example. Let $T \in L(X)$ be any operator with $T(X)$ not closed (of course $X$ must be infinite dimensional). Define the operator $A \in L(X \oplus X)$ by the matrix
$$
A = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.
$$
Then the range of \( A \) is not closed. By [2, Theorem 2.1], \( A \) has no pseudo inverse. From \( A^2 = 0 \) it follows that \( A \in \mathcal{F}(X \oplus X) \) and that 0 is a pole of order 2 of \( R_\lambda(A) \).

Now we return to the investigations of our operator \( A \in \mathcal{F}(X) \). To this end we need the following propositions.

2.6. PROPOSITION. Suppose that \( T \in \mathcal{L}(X) \), \( 0 \in \rho(T), \lambda \in \mathbb{C} \setminus \{0\} \) and that \( k \) is a nonnegative integer. Then:

1. \( N(T - \lambda)^k = N(T^{-1} - \frac{1}{\lambda})^k; \)
2. \( \alpha(T - \lambda) = \alpha(T^{-1} - \frac{1}{\lambda}). \)

PROOF. We only have to show that \( N((T - \lambda)^k) \subseteq N((T^{-1} - \frac{1}{\lambda})^k) \). Take \( x \in N((T - \lambda)^k) \). Then \( 0 = (T - \lambda)^k x \), thus \( 0 = (T^{-1} - \frac{1}{\lambda})^k x = (1 - \lambda T^{-1})^k x \), hence \( x \in N((T^{-1} - \frac{1}{\lambda})^k). \)

2.7. PROPOSITION. Suppose that \( T \in \mathcal{L}(X) \), \( 0 \in \sigma(T), \lambda \in \mathbb{C} \setminus \{0\} \) and \( k \) is a nonnegative integer. Furthermore suppose that \( T \) is Drazin invertible and that \( C \) is the Drazin inverse of \( T \). Then:

1. \( N((T - \lambda)^k) = N((C - \frac{1}{\lambda})^k); \)
2. \( \alpha(T - \lambda) = \alpha(C - \frac{1}{\lambda}); \)

PROOF. (2) follows from (1).

Let \( \nu = i(T) \). We use induction. First we show that \( N(T - \lambda) = N(C - \frac{1}{\lambda}). \) Let \( x \in N(T - \lambda) \), then \( Tx = \lambda x \) and \( T^\nu x = \lambda^\nu x \). We have

\[
\lambda C^2 x = C^2 Tx = CT Cx = Cx,
\]

hence \( C(1 - \lambda C)x = 0 \), thus \( (1 - \lambda C)x \leq N(C) \). By Proposition 1.2, \( N(C) = N(T^\nu) \), therefore

\[
0 = T^\nu(1 - \lambda C)x = (1 - \lambda C)T^\nu x = (1 - \lambda C)\lambda^\nu x,
\]

therefore \( x \in N(C - \frac{1}{\lambda}). \) Now let \( x \in N(C - \frac{1}{\lambda}). \) From \( Cx = \frac{1}{\lambda}x \) we see that \( x \in C(X) = N(P) \), where \( P \) is as in Proposition 1.2. From \( P = I - TC \) we get \( x = TCx = T(\frac{1}{\lambda}x) \), thus \( Tx = \lambda x \), hence \( x \in N(T - \lambda) \). Now suppose that \( n \) is a positive integer and that

\[
N((T - \lambda)^n) = N((C - \frac{1}{\lambda})^n).
\]

Take \( x \in N((T - \lambda)^{n+1}) \). Then \( (T - \lambda)x \in N((T - \lambda)^n) = N((C - \frac{1}{\lambda})^n) \), thus

\[
0 = (C - \frac{1}{\lambda})^n(T - \lambda)x = (T - \lambda)(C - \frac{1}{\lambda})^n x.
\]

This gives

\[
(C - \frac{1}{\lambda})^n x \in N(T - \lambda) = N(C - \frac{1}{\lambda}),
\]

therefore \( x \in N((C - \frac{1}{\lambda})^{n+1}) \). Similar arguments show that \( N((C - \frac{1}{\lambda})^{n+1}) \subseteq N((T - \lambda)^{n+1}) \).

In what follows we use the notation of the beginning of this section. Recall that we have \( \sigma(A) = \{\lambda_1, \ldots, \lambda_k\} \). If \( 0 \in \sigma(A) \), then we always assume that \( \lambda_1 = 0 \), hence \( \sigma(A) \setminus \{0\} = \{\lambda_2, \ldots, \lambda_k\}. \)
2.8. Proposition.
(1) If $0 \notin \rho(A)$, then $\sigma(A^{-1}) = \{ \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k} \}$.
(2) If $0 \notin \sigma(A)$ and if $C$ is the Drazin inverse of $A$, then $0 \notin \sigma(C)$ and $\sigma(C) \setminus \{0\} = \{ \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_k} \}$.

Proof. (1) follows from the spectral mapping theorem. (2) is a consequence of Proposition 1.2.

For our next result recall from Corollary 1.4 that if $0 \notin \rho(A)$, then $A^{-1} \in \mathcal{F}(X)$.

2.9. Theorem. Suppose that $0 \notin \rho(A)$. Then

(1) If the minimal polynomial $m_A$ has the representation (2.1), then the minimal polynomial $m_{A^{-1}}$ of $A^{-1}$ is given by

$$m_{A^{-1}}(\lambda) = \left( \lambda - \frac{1}{\lambda_1} \right)^{m_1} \cdots \left( \lambda - \frac{1}{\lambda_k} \right)^{m_k}.$$ 

(2) If the minimal polynomial $m_A$ has the representation (2.2), then $m_{A^{-1}}$ is given by

$$m_{A^{-1}}(\lambda) = \frac{1}{a_0} + \frac{a_{n-1}}{a_0} \lambda + \cdots + \frac{a_1}{a_0} \lambda^{n-1} + \lambda^n.$$ 

Proof. Proposition 2.6 shows that

$$\alpha(A^{-1} - \frac{1}{\lambda_1}) = \alpha(A - \lambda_j) = m_j \quad (j = 1, \ldots, k),$$

thus (1) is shown. Furthermore $m_{A^{-1}}$ has degree $m_1 + \cdots + m_k = n$. Now define the polynomial $q$ by

$$q(\lambda) = \frac{1}{a_0} + \frac{a_{n-1}}{a_0} \lambda + \cdots + \frac{a_1}{a_0} \lambda^{n-1} + \lambda^n.$$ 

Then

$$a_0 A^n q(A^{-1}) = A^n \left( a_0 (A^{-1})^n + a_1 (A^{-1})^{n-1} + \cdots + a_{n-1} A^{-1} + 1 \right) = m_A(A) = 0.$$ 

Since $a_0 \neq 0$ and $0 \notin \rho(A)$, it results that $q(A^{-1}) = 0$. Because of degree of $q = n =$ degree of $m_{A^{-1}}$, we get $q = m_{A^{-1}}$. \qed

Remark. The proof just given shows that there is a polynomial $q$ such that $q(A^{-1}) = 0$. Therefore we have a simple proof for the fact that $A^{-1} \in \mathcal{F}(X)$.

2.10. Theorem. Suppose that $0 \notin \sigma(A)$ and that $0$ is a pole of $R_{\lambda}(A)$ of order $\nu \geq 1$. Let $C$ denote the Drazin inverse of $A$ (recall from Corollary 2.2 that $C \in \mathcal{F}(X)$).

(1) If $m_A$ has the representation (2.1), then

$$m_C(\lambda) = \lambda (\lambda - \frac{1}{\lambda_2})^{m_2} \cdots (\lambda - \frac{1}{\lambda_k})^{m_k}.$$ 

(2) If $m_A$ has the representation (2.2), then

$$m_C(\lambda) = \frac{1}{a_\nu} \lambda + \frac{a_{\nu-1}}{a_\nu} \lambda^2 + \cdots + \frac{a_{\nu+1}}{a_\nu} \lambda^{n+1-(\nu+1)} + \lambda^{n+1-\nu}.$$
Proof. Proposition 2.7 gives
\[ \alpha(C - \frac{1}{\lambda}) = \alpha(A - \lambda_j) = m_j \quad (j = 2, \ldots, k). \]
By Proposition 1.1 and Proposition 1.2, \( \alpha(C) = 1. \) Thus (1) is valid. We have
\[ m_A(\lambda) = a_\nu \lambda^\nu + a_{\nu+1} \lambda^{\nu+1} + \cdots + a_{n-1} \lambda^{n-1} + \lambda^n, \]
hence
\[ (2.3) \quad 0 = m_A(A) = a_\nu A^\nu + a_{\nu+1} A^{\nu+1} + \cdots + a_{n-1} A^{n-1} + A^n. \]
If \( \nu \leq l \leq n, \) then
\[ C^{n+1} A^l = C^{n+1} C A^l = C^{n+1-l} (CA)^l \]
\[ = C^{n+1-l} C A = C^{n-1} C A C = C^{n+1-l}. \]
Then multiplying (2.3) by \( C^{n+1} \), it follows that
\[ 0 = a_\nu C^{n+1-\nu} + a_{\nu+1} C^{n+1-(\nu+1)} + \cdots + a_{n-1} C^2 + C. \]
Now define the polynomial \( q \) by
\[ q(\lambda) = \frac{1}{a_\nu} \lambda + \frac{a_{n-1}}{a_\nu} \lambda^2 + \cdots + \frac{a_{\nu+1}}{a_\nu} \lambda^{n+1-\nu} + \lambda^{n+1}. \]
Then \( q(C) = 0. \) Since degree of \( q = n+1 - \nu = 1 + m_2 + \cdots + m_k = \text{degree of } m_C, \)
we get \( q = m_C. \)

2.11. Corollary. With the notation in Theorem 2.10 we have
\[ C(A - \lambda_j)^{m_2} \cdots (A - \lambda_k)^{m_k} = 0. \]
Proof. Let \( D = (A - \lambda_j)^{m_2} \cdots (A - \lambda_k)^{m_k}. \) From \( A^\nu D = m_A(A) = 0 \) we see
that \( D(X) \subseteq N(A^\nu). \) Since \( N(A^\nu) = N(C) \) (Proposition 1.2), \( CD = 0. \)

Notation. \( X^* \) denotes the dual space of \( X \) and we write \( T^* \) for the adjoint
of an operator \( T \in \mathcal{L}(X). \) Recall from [4, Theorem IV. 8.4] that
\[ \overline{\text{tr}(X)} = N(T^*) \quad (T \in \mathcal{L}(X)). \]

2.12. Proposition. Suppose that \( T \in \mathcal{L}(X), \lambda \in \mathbb{C} \setminus \{0\} \) and that \( j \) is a
nonnegative integer. Then
(1) If \( 0 \in \rho(T), \) then \( (T - \lambda)^j(X) = (T^{-1} - \frac{1}{\lambda})^j(X). \)
(2) If \( 0 \in \sigma(T), \) if \( T \) is Drazin invertible and if \( C \) denotes the Drazin inverse
of \( T, \) then \( (T - \lambda)^j(X) = (C - \frac{1}{\lambda})^j(X). \)

Proof. (1) Let \( y = (T - \lambda)^j x \in (T - \lambda)^j(X) \quad (x \in X). \) Then
\[ (T^{-1} - \frac{1}{\lambda})^j T^j x = ((T^{-1} - \frac{1}{\lambda}) T)^j x = (1 - \frac{\lambda}{\lambda})^j x \]
\[ = \frac{(-1)^j}{\lambda^j} (T - \lambda)^j x = \frac{(-1)^j}{\lambda^j} y, \]
therefore \( y \in (T^{-1} - \frac{1}{\lambda})^j(X). \)
(2) Let \( \nu = i(T). \) Then \( T^{\nu+1} C = T^\nu, \) \( TC = CT \) and \( C T C = C. \) Hence
\[ (T^*)^{\nu+1} C^* = (T^*)^\nu, \] \( T^* C^* = C^* T^* \) and \( C^* T^* C^* = C^*. \)
Thus $T^*$ is Drazin invertible and $C^*$ is the Drazin inverse of $T^*$. By Proposition 2.7, 
\[ N((T^* - \lambda)^2) = N((C^* - \frac{1}{\lambda})^2), \]
therefore the result follows in view of (2.4). \hfill \Box

2.13. **Corollary.**

(1) If $0 \not\in \rho(A)$, then $(A - \lambda_j)^{m_j}(X) = (A^{-1} - \frac{1}{\lambda_j})^{m_j}(X)$ \ $(j = 1, \ldots, k)$.

(2) If $0 \not\in \sigma(A)$ is a pole of order $\nu \geq 1$ of $R_\lambda(A)$ and if $C$ is the Drazin inverse of $A$, then $A^\nu(X) = C(X)$ and
\[ (A - \lambda_j)^{m_j}(X) = (C - \frac{1}{\lambda_j})^{m_j}(X) \quad (j = 2, \ldots, k). \]

**Proof.** (1) is a consequence of Proposition 2.12.

(2) That $A^\nu(X) = C(X)$ is a consequence of Proposition 1.2. Now let $j \leq \{2, \ldots, k\}$. Because of Proposition 1.1 and Theorem 2.10 we see that
\[ \alpha(C - \frac{1}{\lambda_j}) = \delta(C - \frac{1}{\lambda_j}) = m_j = \alpha(A - \lambda_j) = \delta(A - \lambda_j). \]
By [3, Satz 101.2], the subspaces $(A - \lambda_j)^{m_j}(X)$ and $(C - \frac{1}{\lambda_j})^{m_j}(X)$ are closed. Now apply Proposition 2.12. \hfill \Box

For $j = 1, \ldots, k$ let $P_j$ denote the spectral projection of $A$ associated with the spectral set $\{\lambda_j\}$. Observe that
\[ P_jP_k = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad P_1 + \cdots + P_k = 1. \]

If $0 \in \rho(A)$, then denote by $Q_j$ the spectral projection of $A^{-1}$ associated with the spectral set $\{\frac{1}{\lambda_j}\}$ \ $(j = 1, \ldots, k)$. If $0 \in \sigma(A)$ and if $C$ is the Drazin inverse, then denote by $Q_1$ the spectral projection of $C$ associated with the spectral set $\{0\}$ and by $Q_j$ the spectral projection of $C$ associated with the spectral set $\{\frac{1}{\lambda_j}\}$ \ $(j = 2, \ldots, k)$.

2.14. **Corollary.** $P_j = Q_j$ \ $(j = 1, \ldots, k)$.

**Proof.** By [3, Satz 101.2], we have
\[ P_j(X) = N((A - \lambda_j)^{m_j}) \quad \text{and} \quad N(P_j) = (A - \lambda_j)^{m_j}(X) \]
$(j = 1, \ldots, k)$. If $0 \in \rho(A)$, then
\[ Q_j(X) = N((A^{-1} - \frac{1}{\lambda_j})^{m_j}) \quad \text{and} \quad N(Q_j) = (A^{-1} - \frac{1}{\lambda_j})^{m_j}(X) \]
$(j = 1, \ldots, k)$. Now apply Proposition 2.6 and Corollary 2.13 (1) to get
\[ P_j(X) = Q_j(X) \quad \text{and} \quad N(P_j) = N(Q_j), \]

hence $P_j = Q_j$ \ $(j = 1, \ldots, k)$.

Now let $0 \in \sigma(A)$. By Proposition 1.2, Proposition 2.7, Corollary 2.13 (2) and [3, Satz 101.2], we derive
\[ P_1(X) = N(C) = Q_1(X), \quad N(P_1) = C(X) = N(Q_1), \]
\[ P_j(X) = N((C - \frac{1}{\lambda_j})^{m_j}) = Q_j(X) \]
\[ N(P_j) = (C - \frac{1}{\lambda_j})^{m_j}(X) = N(Q_j) \]
(j = 2, ..., k). Hence \( P_j = Q_j \) (j = 1, ..., k).

For \( A \) we have the representation \( \text{\textit{A}} = \sum_{j=1}^{k} \lambda_j P_j + N \), where \( N \in \mathcal{L}(X) \) is nilpotent and \( N = \sum_{j=1}^{k} (A - \lambda_j) P_j \) (see [4, Chapter V. 11]). If \( p = \max\{m_1, \ldots, m_k\} \), then it is easily seen that \( N^p = 0 \). If \( A \) has only simple poles, then \( N = 0 \).

2.15. Corollary.

(1) If \( 0 \in \rho(A) \), then there is a nilpotent operator \( N_1 \in \mathcal{L}(X) \) with

\[
A^{-1} = \sum_{j=1}^{k} \frac{1}{\lambda_j} P_j + N_1
\]

(2) If \( 0 \in \sigma(A) \) and if \( C \) is the Drazin inverse of \( A \), then

\[
C = \sum_{j=2}^{k} \frac{1}{\lambda_j} P_j + N_1, \quad \text{where} \ N_1 \in \mathcal{L}(X) \ \text{is nilpotent}.
\]


With the notation of Corollary 2.15 (2) we have \( AC = 1 - P_1, CP_1 = 0 \) (see Proposition 1.2) and

\[
ACA = (1 - P_1) \left( \sum_{j=2}^{k} k\lambda_j P_j + N \right) = A - P_1 \left( \sum_{j=2}^{k} \lambda_j P_k + N \right) = A - P_1 N;
\]

hence \( A = ACA + P_1 N, P_1 N \) is nilpotent and

\[
(ACA)P_1 N = ACP_1 AN = 0 = NACP_1 A = P_1 N (ACA).
\]

Recall that \( ACA \) is the Drazin inverse of \( C \) and that \( i(ACA) = 1 \). The following more general result holds:

2.16. Theorem. Suppose that \( T \in \mathcal{L}(X) \) is Drazin invertible, \( i(T) = \nu \geq 1 \) and that \( C \) is the Drazin inverse of \( T \). Then there is a nilpotent \( N \in \mathcal{L}(X) \) such that \( T = TCT + N, N(TCT) = (TCT)N = 0 \) and \( N^\nu = 0 \).

This decomposition is unique in the following sense: if \( S, N_1 \in \mathcal{L}(X), S \) is Drazin invertible, \( i(S) = 1 \), \( N_1 \) is nilpotent, \( N_1 S = SN_1 = 0 \) and if \( T = S + N_1 \), then \( S = TCT \) and \( N = N_1 \).

Proof. Let \( N = T - TCT \); then

\[
N^\nu = (T(1 - CT))^\nu = T^\nu (1 - CT)^\nu = T^\nu (1 - CT)
\]

\[
= T^\nu - T^\nu CT = T^\nu - T^\nu + 1 C = T^\nu - T^\nu = 0.
\]

For the uniqueness of the decomposition we only have to show that \( S = TCT \).

There is \( R \in \mathcal{L}(X) \) such that \( SRS = S, RSR = R \) and \( SR = RS \). Consequently,

\[
N_1 R = N_1 RSR = N_1 R^2 = 0 = R^s S N_1 = RN_1,
\]

hence

\[
TR = (S + N_1) R = SR = RS = R(S + N_1) = RT.
\]
Now let $n$ be a nonnegative integer such that $N_1^n = 0$. Since $SN_1 = 0 = N_1S$, it follows that
\[ T^n = (S + N_1)^n = S^n + N_1^n = S^n. \]
We can assume that $n \geq \nu$. Thus
\[ T^{n+1}R = S^{n+1}R = S^n SRS = S^n = T^n. \]
Furthermore we have $TR = RT$ and
\[ RTR = R(S + N_1)R = RSR = R, \]
hence $R = C$. With $S_1 = TCT$ we get
\[ S_1RS_1 = TCTCTCT = TCT = S_1, \]
\[ RS_1R = CTCTC = CTC = RTR = R \]
\[ S_1R = TCTC = CTCT = RS_1. \]
This shows that $S = S_1 = TCT$. \hfill \Box

References


Institut für Analysis
Universität Karlsruhe (TH)
Kaiserstrasse 89
76133 Karlsruhe
Germany
christoph.schoege@math.uni-karlsruhe.de

(Received 17 04 2006)