ON THREE CONJECTURES INVOLVING
THE SIGNLESS LAPLACIAN
SPECTRAL RADIUS OF GRAPHS

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1. Introduction

In this paper, we consider only simple connected graphs and follow the notation of [1]. Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The adjacency matrix of $G$ is $A(G) = (a_{ij})$, where $a_{ij} = 1$ if two vertices $i$ and $j$ are adjacent in $G$ and $a_{ij} = 0$ otherwise. The characteristic polynomial of $G$ is just $P_G(x) = \det(xI - A(G))$. Let $D(G)$ be the diagonal degree matrix of $G$. We call the matrix $L(G) = D(G) - A(G)$ the Laplacian matrix of $G$, and the matrix $Q(G) = D(G) + A(G)$ the signless Laplacian matrix or $Q$-matrix of $G$. We denote the largest eigenvalues of $A(G)$, $L(G)$, $Q(G)$ by $\rho(G)$, $\lambda(G)$, $\mu(G)$, respectively, and call them the adjacency spectral radius, the Laplacian spectral radius, the signless Laplacian spectral radius (or the $Q$-spectral radius) of $G$, respectively.

The study of the signless Laplacian spectral radius has recently attracted researchers’ attention. In [10], Fan et al. studied the signless Laplacian spectral radius of bicyclic graphs with fixed order. In [9], the authors discussed the smallest eigenvalue of $Q(G)$ as a parameter reflecting the nonbipartiteness of the graph $G$. In [7], the authors studied the smallest signless Laplacian eigenvalue of non-bipartite graphs. In [11], the extremal graphs with maximal signless Laplacian spectral radius and fixed diameter were studied. More information about the signless Laplacian can be found in [2], [3], [5], [6]. For more information about the spectral radius of graphs, the reader can refer to [4].

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In [5], the authors proposed the following conjectures and in this paper we confirm that they are true.

**Theorem 1.1.** [5] Conjecture 6] Let $G$ be a connected graph of order $n \geq 4$. Then
\[
\mu(G) - \frac{4m}{n} \leq n - 4 + \frac{4}{n}.
\]
Equality holds if and only if $G = K_{1,n-1}$.

**Theorem 1.2.** [5] Conjecture 7] Let $G$ be a connected graph of order $n \geq 5$. Then
\[
\mu(G) - \frac{2m}{n} \leq n - 1.
\]
Equality holds if and only if $G = K_n$.

**Theorem 1.3.** [5] Conjecture 10] Let $G$ be a connected graph of order $n \geq 4$. Then
\[
\mu(G) - \lambda(G) \leq n - 2.
\]
Equality holds if and only if $G = K_n$.

2. Lemmas and results

Let $G$ be a connected graph. The degree of $u$ in $G$ is denoted by $d_u$, the average degree of $u$, denoted by $m_u$, satisfies $d_u m_u = \sum_{uv \in E} d_v$, where $E = E(G)$.

**Lemma 2.1.** [8] Let $G$ be a graph with $n$ vertices, and $m$ edges. Then
\[
\max\{d_u + m_v \mid v \in V(G)\} \leq \frac{2m}{n-1} + n - 2,
\]
with equality if and only if $K_{1,n-1} \subseteq G$ or $G = K_{n-1} \cup K_1$.

**Lemma 2.2.** [12] Let $M = (m_{ij})$ be an $n \times n$ irreducible nonnegative matrix with spectral radius $\rho(M)$, and let $R_i(M)$ be the $i$th row sum of $M$, i.e., $R_i(M) = \sum_{j=1}^n m_{ij}$. Then
\[
\min\{R_i(M) \mid 1 \leq i \leq n\} \leq \rho(M) \leq \max\{R_i(M) \mid 1 \leq i \leq n\}.
\]
Moreover, if the row sums of $M$ are not all equal, then both above inequalities are strict.

**Lemma 2.3.** Let $G$ be a connected graph. Then $\mu(G) \leq \max\{d_u + m_v \mid v \in V(G)\}$, with equality holding if and only if $G$ is either semiregular bipartite or regular.

**Proof.** We consider the matrix $K = D^{-1}QD$, where the row sum corresponding to the vertex $u$ is $d_u + m_u$. From Lemma 2.2 we obtain the required upper bound for $\mu(G)$.

If equality holds, then by Lemma 2.2 for a neighbor $v$ of $u$, $d_u + m_u = d_v + m_v$, thus $\sum_{uv \in E} (d_u + m_u) = \sum_{uv \in E} (d_v + m_v)$, that is $d_u^2 + d_u m_u = d_v m_v + \sum_{uv \in E} m_v$. So we get
\[
d_u^2 = \sum_{uv \in E} m_v.
Suppose $d_u$ is the maximum degree. Then $d_v m_v = \sum_{w \in E} d_w \leq d_u d_v$, whence $m_v \leq d_u$ for all $v \in V(G)$. Since $d_u^2 = \sum_{w \in E} m_v \leq d_u^2$, we have for any edge $uv$, $d_u = m_v$, and $d_u d_v = d_v m_v$, that is, $\sum_{w \in E} (d_u - d_w) = 0$. Since $d_u$ is the maximum degree, we have $d_u = d_w$ whenever there exists a vertex $v$ such that $uv, vw \in E$.

If $G$ does not contain odd cycles, then $G$ is bipartite. Suppose $V = S \cup T$ is a bipartition and $u \in T$; then $v \in S, w \in T$. This implies that the vertices in $T$ have the same degree. Similarly, the vertices in $S$ also have the same degree. So $G$ is semiregular bipartite.

If $G$ contains odd cycles, then $G$ must be regular. The converse is easy to check. \hfill \Box

**Lemma 2.4.** Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$\mu(G) \leq \frac{2m}{n-1} + n - 2,$$

with equality if and only if $G$ is $K_{1,n-1}$ or $K_n$.

**Proof.** By Lemmas 2.1 and 2.3 we can get the result. Note that $K_{1,n-1}$ is the only semiregular bipartite graph and $K_n$ is the only regular graph that arises in the case of equality. \hfill \Box

Now we can present the proof of the main results of this paper.

**Proof of Theorem 1.1.** By Lemma 2.4 we have

$$\mu(G) - \frac{4m}{n} \leq \frac{2m}{n-1} + n - 2 - \frac{4m}{n} = (n-2) \left(1 - \frac{2m}{n(n-1)}\right),$$

$$\leq (n-2) \left(1 - \frac{2(n-1)}{n(n-1)}\right) = n - 4 + \frac{4}{n}.$$  

The last inequality holds since $G$ is connected and so has at least $n-1$ edges. The equality case is easy to see from Lemma 2.4. \hfill \Box

**Proof of Theorem 1.2.** By Lemma 2.4 we have

$$\mu(G) - \frac{2m}{n} \leq \frac{2m}{n-1} + n - 2 - \frac{2m}{n} = \frac{2m}{n(n-1)} + n - 2 \leq 1 + n - 2 = n - 1.$$  

The last inequality holds since $G$ has at most $\frac{1}{2} n(n-1)$ edges. When this bound is attained, $G = K_n$. \hfill \Box

**Proof of Theorem 1.3.** Note that the sum of all the Laplacian eigenvalues is $2m$, so we have $(n-1)\lambda(G) \geq 2m$ and hence $\lambda(G) \geq \frac{2m}{n-1}$. By Lemma 2.4 we have

$$\mu(G) - \lambda(G) \leq \frac{2m}{n-1} + n - 2 - \frac{2m}{n-1} = n - 2.$$  

The equality case is easy to see from Lemma 2.4. \hfill \Box

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