ON THE SOLID HULL
OF THE HARDY–LORENTZ SPACE

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Abstract. The solid hulls of the Hardy–Lorentz spaces $H^{p,q}$, $0 < p < 1$, $0 < q \leq \infty$ and $H^{p,\infty}$, $0 < p < 1$, as well as of the mixed norm space $H^{p,\infty,\alpha}$, $0 < p \leq 1$, $0 < \alpha < \infty$, are determined.

Introduction

In [JP1] the solid hull of the Hardy space $H^p$, $0 < p < 1$, is determined. In this article we determine the solid hulls of the Hardy–Lorentz spaces $H^{p,q}$, $0 < p < 1$, $0 < q \leq \infty$ and $H^{p,\infty}$, $0 < p < 1$, as well as of the mixed norm space $H^{p,\infty,\alpha}$, $0 < p \leq 1$, $0 < \alpha < \infty$. Since $H^{p,p} = H^p$ our results generalize [JP1] Theorem 1.

Recall, the Hardy space $H^p$, $0 < p \leq \infty$, is the space of all functions $f$ holomorphic in the unit disk $U$, $(f \in H(U))$, for which $\|f\|_p = \lim_{r \to 1} M^p(r,f) < \infty$, where, as usual,

$$M^p(r,f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p}, \quad 0 < p < \infty,$$

$$M^\infty(r,f) = \sup_{0 \leq t < 2\pi} |f(re^{it})|.$$

Now we introduce a generalization and refinement of the spaces $H^p$; the Hardy–Lorentz spaces $H^{p,q}$, $0 < p < \infty$, $0 < q \leq \infty$.

Let $\sigma$ denotes normalized Lebesgue measure on $T = \partial U$ and let $L^0(\sigma)$ be the space of complex-valued Lebesgue measurable functions on $T$. For $f \in L^0(\sigma)$ and $s \geq 0$ we write

$$\lambda_f(s) = \sigma(\{\xi \in T : |f(\xi)| > s\})$$

for the distribution function and

$$f^*(s) = \inf(\{t \geq 0 : \lambda_f(t) \leq s\})$$

for the decreasing rearrangement of $|f|$ each taken with respect to $\sigma$.

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The Lorentz functional \( \| \cdot \|_{p,q} \) is defined at \( f \in L^0(\sigma) \) by
\[
\| f \|_{p,q} = \left( \int_0^1 (f^*(s)s^{1/p}) q \frac{ds}{s} \right)^{1/q} \quad \text{for} \quad 0 < q < \infty,
\]
\[
\| f \|_{p,\infty} = \sup \{ f^*(s)s^{1/p} : s \geq 0 \}.
\]
The corresponding Lorentz space is \( L^{p,q}(\sigma) = \{ f \in L^0(\sigma) : \| f \|_{p,q} < \infty \} \). The space \( L^{p,0}(\sigma) \) is separable if and only if \( q \not= \infty \). The class of functions \( f \in L^{p,0}(\sigma) \) satisfying \( \lim_{r \to 0} (f^*(s)s^{1/p}) = 0 \) is a separable closed subspace of \( L^{p,\infty}(\sigma) \), which is denoted by \( L_0^{p,\infty}(\sigma) \).

The Nevanlinna class \( N \) is the subclass of functions \( f \in H(U) \) for which
\[
\sup_{0 < r < 1} \int_T \log^+ |f(r\xi)| \, d\sigma(\xi) < \infty.
\]
Functions in \( N \) are known to have non-tangential limits \( \sigma \)-a.e. on \( T \). Consequently every \( f \in N \) determines a boundary value function which we also denote by \( f \).

Thus
\[
f(\xi) = \lim_{r \to 1} f(r\xi) \quad \sigma \text{-a.e.} \quad \xi \in T.
\]
The Smirnov class \( N^+ \) is the subclass of \( N \) consisting of those functions \( f \) for which
\[
\lim_{r \to 1} \int_T \log^+ |f(r\xi)| \, d\sigma(\xi) = \int_T \log^+ |f(\xi)| \, d\sigma(\xi).
\]

We define the Hardy–Lorentz space \( H^{p,q} \), \( 0 < p < \infty \), \( 0 < q \leq \infty \), to be the space of functions \( f \in N^+ \) with boundary value function in \( L^{p,q}(\sigma) \) and we put \( \| f \|_{H^{p,q}} = \| f \|_{p,q} \). The functions in \( H^{p,\infty} \) with a boundary value function in \( L_0^{p,\infty}(\sigma) \) form a closed subspace of \( H^{p,\infty} \), which is denoted by \( H_0^{p,\infty} \). The cases of major interest are of course \( p = q \) and \( q = \infty \); indeed \( H^{p,p} \) is nothing but \( H^p \), and \( H^{p,\infty} \) is the weak-\( H^p \).

The mixed norm space \( H^{p,q,\alpha} \), \( 0 < p \leq \infty \), \( 0 < q, \alpha < \infty \), consists of all \( f \in H(U) \) for which
\[
\| f \|_{H^{p,q,\alpha}} = \| f \|_{p,q,\alpha} = \left( \int_0^1 (1-r)^{\alpha-1} M_p(r, f) \, dr \right)^{1/q} < \infty.
\]
\( H^{p,q,\alpha} \) can also be defined when \( q = \infty \), in which case it is sometimes known as the weighted Hardy space \( H^{p,\infty,\alpha} \), and consists of all \( f \in H(U) \) for which
\[
\| f \|_{p,\infty,\alpha} = \sup_{0 < r < 1} (1-r)^\alpha M_p(r, f) < \infty.
\]
The functions in \( H^{p,\infty,\alpha} \), \( \alpha > 0 \) for which \( \lim_{r \to 1} (1-r)^\alpha M_p(r, f) = 0 \) form a closed subspace which is denoted by \( H_0^{p,\infty,\alpha} \).

Throughout this paper, we identify the holomorphic function \( f(z) = \sum_{k=0}^\infty \hat{f}(k)z^k \) with its sequence of Taylor coefficients \( \{ \hat{f}(k) \}_{k=0}^\infty \).

If \( f(z) = \sum_{k=0}^\infty \hat{f}(k)z^k \) belongs to \( H^{p,q} \), then
\[
\hat{f}(k) = O((k+1)^{(1/p)-1}) \quad \text{if} \quad 0 < p < 1 \quad \text{and} \quad 0 < q \leq \infty.
\]
(See [Al] and [Co] for this result.)
In this paper we find the strongest condition that the moduli of an $H^{p,q}$, $0 < p < 1$, $0 < q \leq \infty$, satisfy. Our result shows that the estimate (1) is optimal only if $q = \infty$.

To state our results in a form of theorems we need to introduce some more notations.

A sequence space $X$ is solid if $\{b_n\} \in X$ whenever $\{a_n\} \in X$ and $|b_n| \leq |a_n|$. More generally, we define $S(X)$, the solid hull of $X$. Explicitly,

$$S(X) = \{\{\lambda_n\} : \exists \{a_n\} \in X \text{ such that } |\lambda_n| \leq |a_n|\}.$$ 

A complex sequence $\{a_n\}$ is of class $l(p,q)$, $0 < p, q \leq \infty$, if

$$\|\{a_n\}\|_{l(p,q)} = \|\{a_n\}\|_{l(p,q)} = \sum_{n=0}^{\infty} \left( \sum_{k \in I_n} |a_k|^p \right)^{q/p} < \infty,$$

where $I_0 = \{0\}$, $I_n = \{k \in N : 2^{n-1} \leq k < 2^n\}$, $n = 1, 2, \ldots$ In the case where $p$ or $q$ is infinite, replace the corresponding sum by a supremum. Note that $l(p,p) = l^p$.

For $t \in R$ we write $D^t$ for the sequence $\{(n+1)^t\}$, for all $n \geq 0$. If $\lambda = \{\lambda_n\}$ is a sequence and $X$ a sequence space, we write $\lambda X = \{\{\lambda_n x_n\} : \{x_n\} \in X\}$; thus, for example, $\{a_n\} \in D^t l^\infty$ if and only if $|a_n| = O(n^t)$.

We are now ready to state our first result.

**Theorem 1.** If $0 < p < 1$ and $0 < q \leq \infty$, then $S(H^{p,q}) = D^{(1/p) - 1} l(\infty, q)$.

In particular, $S(H^p) = D^{(1/p) - 1} l(\infty, p)$, $0 < p < 1$. This was proved in [JP1].

Also, $S(H^{p,\infty}) = D^{(1/p) - 1} l^\infty$ means that the estimate (1) valid for the Taylor coefficients of an $H^{p,\infty}$, $0 < p < 1$, function is sharp.

Our second result is as follows:

**Theorem 2.** If $0 < p < 1$, then $S(H_0^{p,\infty}) = D^{(1/p) - 1} c_0$, where $c_0$ is the space of all null sequences.

Our method of proving Theorem 1 and Theorem 2 depend upon nested embedding [Le Theorem 4.1] for Hardy–Lorentz spaces. Thus, the strategy is to trap $H^{p,q}$ between a pair of mixed norm spaces and then deduce the results for $H^{p,q}$ from the corresponding results for the mixed norm spaces. Our Theorem 1 will follow from the following two theorems:

**Theorem L.** [Le] Let $0 < p_0 < p < s \leq \infty$, $0 < q \leq t \leq \infty$ and $\beta > (1/p_0) - (1/p)$. Then

$$D^{-\beta} H_0^{p_0,q,\beta+(1/p_0) - (1/p)} \subset H^{p,q} \subset H_0^{s,q,(1/p)-(1/s)},$$

$$D^{-\beta} H_0^{p_0,\infty,\beta+(1/p_0) - (1/p)} \subset H_0^{p,\infty,(1/p)-(1/s)}.$$ 

**Theorem JP 1.** [JP1] If $0 < p \leq 1$, $0 < q \leq \infty$ and $0 < \alpha < \infty$, then $S(H_0^{p,q,\alpha}) = D^{(1/p) - 1} l(\infty, q)$.

To prove Theorem 2 we first determine the solid hull of the space $H_0^{p,\infty,\alpha}$, $0 < p \leq 1$, $0 < \alpha < \infty$. More precisely, we prove

**Theorem 3.** If $0 < p \leq 1$ and $0 < \alpha < \infty$, then $S(H_0^{p,\infty,\alpha}) = D^{(1/p) - 1} c_0$. 


Given two vector spaces $X, Y$ of sequences we denote by $(X, Y)$ the space of multipliers from $X$ to $Y$. More precisely, 

$$(X, Y) = \{ \lambda = \{ \lambda_n \} : \{ \lambda_n a_n \} \in Y, \text{ for every } \{a_n\} \in X \}.$$ 

As an application of our results we calculate multipliers $(H^{p,q}, l(u,v))$, $0 < p < 1$, $0 < q \leq \infty$, $(H_0^{p,\infty}, l(u,v))$, $0 < p < 1$, and $(H^{p,\infty}, X)$, $0 < p < 1$, where $X$ is a solid space. These results extend some of the results obtained by Lengfield [LE] Section 5. 

1. The solid hull of the Hardy–Lorentz space $H^{p,q}$, $0 < p < 1$, $0 < q \leq \infty$

Proof of Theorem 1. Let $0 < p < 1$. Choose $p_0$ and $s$ so that $p_0 < p < s \leq 1$ and a real number $\beta$ so that $\beta + (1/p) - (1/p_0) > 0$. As an easy consequence of Theorem JP we have

$$S(D^{-\beta}H^{p_0,q;\beta+(1/p)-(1/p_0)}) = D^{(1/p)-1}(\infty, q).$$

Also, by Theorem JP,

$$S(H^{p,q}(1/p)-(1/s)) = D^{(1/p)-1}(\infty, q),$$

and consequently $S(H^{p,q}) = D^{(1/p)-1}(\infty, q)$, by Theorem L.

2. The solid hull of mixed norm space $H_0^{p,\infty;\alpha}$, $0 < p \leq 1$, $0 < \alpha < \infty$

If $f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k$ and $g(z) = \sum_{k=0}^{\infty} \hat{g}(k)z^k$ are holomorphic functions in $U$, then the function $f \ast g$ is defined by $(f \ast g)(z) = \sum_{k=0}^{\infty} \hat{f}(k)\hat{g}(k)z^k$.

The main tool for proving Theorem 3 are polynomials $W_n$, $n \geq 0$, constructed in [JPT] and [JPS]. Recall the construction and some of their properties.

Let $\omega: R \rightarrow R$ be a nonincreasing function of class $C^\infty$ such that $\omega(t) = 1$, for $t \leq 1$, and $\omega(t) = 0$, for $t \geq 2$. We define polynomials $W_n = W_\omega^n$, $n \geq 0$, in the following way:

$$W_0(z) = \sum_{k=0}^{\infty} \omega(k)z^k \quad \text{and} \quad W_n(z) = \sum_{k=2^{n-1}}^{2^{n+1}-1} \varphi\left(\frac{k}{2^{n-1}}\right)z^k, \quad \text{for } n \geq 1,$$

where $\varphi(t) = \omega(t/2) - \omega(t)$, $t \in R$.

The coefficients $W_n(k)$ of these polynomials have the following properties:

(4) \quad \text{supp}(W_n) \subset [2^{n-1}, 2^{n+1}]; \\
(5) \quad 0 \leq W_n(k) \leq 1, \quad \text{for all } k, \\
(6) \quad \sum_{n=0}^{\infty} W_n(k) = 1, \quad \text{for all } k, \\
(7) \quad W_n(k) + W_{n+1}(k) = 1, \quad \text{for } 2^n \leq k \leq 2^{n+1}, \ n \geq 0.
Property (5) implies that
\[ f(z) = \sum_{n=0}^{\infty} (W_n \ast f)(z), \quad f \in H(U), \]
the series being uniformly convergent on compact subsets of \( U \).

If \( 0 < p < 1 \), then there exists a constant \( C > 0 \) depending only on \( p \) such that
\[ \|W_n\|_p^p \leq C_p 2^{-n(1-p)}, \quad n \geq 0. \]

**Proof of Theorem 3.** Let \( f \in H_0^{p,\infty,\alpha} \), \( 0 < p < 1 \), \( 0 < \alpha < \infty \). By using the familiar inequality
\[ M_p(r, f) \geq C(1 - r)^{(1/p) - 1} M_1(r^2, f), \quad 0 < p \leq 1, \]
(see [Du Theorem 5.9]), we obtain
\[ \sup_{k \in \mathcal{I}_n} |\hat{f}(k)| r^{2k} \leq M_1(r^2, f) \leq CM_p(r, f)(1 - r)^{1/(1/p)}, \quad 0 < r < 1. \]

Now we take \( r_n = 1 - 2^{-n} \) and let \( n \to \infty \), to get \( \{\hat{f}(k)\} \in D^{(1/p) - 1}c_0 \). Thus \( H_0^{p,\infty,\alpha} \subset D^{(1/p) - 1}c_0 \).

To show that \( D^{(1/p) - 1}c_0 \) is the solid hull of \( H_0^{p,\infty,\alpha} \), it is enough to prove that if \( \{a_n\} \in D^{(1/p) - 1}c_0 \), then there exists \( \{b_n\} \in H_0^{p,\infty,\alpha} \) such that \( |b_n| \geq |a_n| \), for all \( n \).

Let \( \{a_n\} \in D^{(1/p) - 1}c_0 \). Define
\[ g(z) = \sum_{j=0}^{\infty} B_j(W_j(z) + W_{j+1}(z)) = \sum_{k=0}^{\infty} c_k z^k, \]
where \( B_j = \sup_{2^j \leq k < 2^{j+1}} |a_k| \). Using (4) and (8) we find that
\[ M_p(r, g) \leq \sum_{j=0}^{\infty} B_j^p (M_p^p(r, W_j) + M_p^p(r, W_{j+1})) \leq C \left( B_0^p + \sum_{j=1}^{\infty} B_j^p 2^{2j-1 - j(1/p)} \right) \]
Set \( B_j^p 2^{-j(\alpha p + 1 - p)} = \lambda_j \). Then
\[ M_p^p(r, g) \leq C \left( \lambda_0 + \sum_{j=1}^{\infty} \lambda_j r^{2j-1} 2^{j(\alpha p)} \right), \]
where \( \lambda_j \to 0 \), as \( j \to \infty \). From this it easily follows that \( (1 - r)^{\alpha p} M_p^p(r, g) \to 0 \), as \( r \to 1 \). Thus \( g \in H_0^{p,\infty,\alpha} \).

To prove that \( |c_k| \geq |a_k| \), \( k = 1, 2, \ldots \), choose \( n \) so that \( 2^n \leq k < 2^{n+1} \). It follows from (7)
\[ c_k = \sum_{j=0}^{\infty} B_j(W_j(k) + W_{j+1}(k)) \geq B_n(W_n(k) + W_{n+1}(k)) \]
\[ = B_n = \sup_{2^n \leq j < 2^{n+1}} |a_j| \geq |a_k|. \]
Now the function \( h(z) = \sum_{n=0}^{\infty} b_n z^n \), where \( b_0 = a_0 \) and \( b_n = c_n \), for \( n \geq 1 \), belongs to \( H_0^{p,\infty,\alpha} \) and \( |b_n| \geq |a_n| \) for all \( n \geq 0 \). This finishes the proof of Theorem 3. \( \square \)
3. The solid hull of the space 
\( H_0^{p,\infty}, \ 0 < p < 1 \)

**Proof of Theorem 2.** Let \( 0 < p < 1 \). Choose \( p_0 \) and \( s \) so that \( p_0 < p < s \leq 1 \) and \( \beta \in R \) so that \( \beta + (1/p) - (1/p_0) > 0 \). Then
\[
S\left(D^{-\beta} H_0^{p_0,\infty,\beta+(1/p)−(1/p_0)}\right) = D^{(1/p)−1} c_0,
\]
\[
S\left(H_0^{s,\infty,(1/p)−(1/s)}\right) = D^{(1/p)−1} c_0,
\]
by Theorem 3. By Theorem L we have \( S(H_0^{p,\infty}) = D^{(1/p)−1} c_0 \). \( \square \)

4. Applications to multipliers

As it was noticed in the introduction, another objective of this paper is to extend some of the results given in [Le Section 5].

The next lemma due to Kellog (see [K]) (who states it for exponents no smaller than 1, but it then follows for all exponents, since \( \{\lambda_n\} \in (l(a,b), (l(c,d))) \) if and only if \( \{\lambda_n^{(1/\ell)}\} \in (l(at, bt), (l( ct, dt))) \).

**Lemma 1.** If \( 0 < a, b, c, d \leq \infty \), then \( (l(a,b), l(c,d)) = (l(a \circ c, b \circ d)) \), where \( a \circ c = \infty \) if \( a \leq c \), \( b \circ d = \infty \), if \( b \leq d \), and
\[
\frac{1}{a \circ c} = \frac{1}{c} - \frac{1}{a}, \text{ for } 0 < c < a,
\]
\[
\frac{1}{b \circ d} = \frac{1}{d} - \frac{1}{b}, \text{ for } 0 < d < b.
\]

In particular, \( (l, l(u,v)) = l(u,v) \). Also, it is known that \( (c_0, l(u,v)) = l(u,v) \).

In [AS] it is proved that if \( X \) is any solid space and \( A \) any vector space of sequences, then \( (A, X) = (S(A), X) \).

Since \( l(u,v) \) are solid spaces, we have \( (H_0^{p,q}, l(u,v)) = (S(H_0^{p,q}), l(u,v)) \) and \( (H_0^{p,\infty}, l(u,v)) = (S(H_0^{p,\infty}), l(u,v)) \). Using this, Lemma 1, Theorem 1 and Theorem 2 we get

**Theorem 4.** Let \( 0 < p < 1 \) and \( 0 < q \leq \infty \). Then
\[
(H_0^{p,q}, l(u,v)) = D^{1−(1/p)} l(u, q \circ v).
\]

**Theorem 5.** Let \( 0 < p < 1 \). Then
\[
(H_0^{p,\infty}, l(u,v)) = D^{1−(1/p)} l(u, v).
\]

In particular, \( (H^{p,\infty}, l(u,v)) = D^{1−(1/p)} l(u, v) \). In fact more is true.

**Theorem 6.** Let \( 0 < p < 1 \) and let \( X \) be a solid space. Then
\[
(H^{p,\infty}, X) = D^{1−(1/p)} X.
\]

**Proof.** Since \( X \) is a solid space, we have \( (l, X) = X \). Hence, using Theorem 1 we get
\[
(H^{p,\infty}, X) = (S(H^{p,\infty}), X) = (D^{(1/p)−1} l, X)
= D^{1−(1/p)} (l, X) = D^{1−(1/p)} X. \quad \square
\]
References


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