A NOTE ON DIFFERENCES OF POWER MEANS

Slavko Simić

Communicated by Stevan Pilipović

Abstract. We give some new inequalities concerning the differences of power means.

1. Introduction

Let \( \tilde{x}_n = \{x_i\}_1^n \) and \( \tilde{p}_n = \{p_i\}_1^n \) denote two sequences of positive real numbers with \( \sum_{i=1}^{n} p_i = 1 \). From the Theory of Convex Means (cf. \cite{1}, \cite{2}, \cite{3}), it is well known that for \( t > 1 \),

\[
\sum_{i=1}^{n} p_i x_i^t \geq \left( \sum_{i=1}^{n} p_i x_i \right)^t,
\]

and \textit{vice versa} for \( 0 < t < 1 \). The equality sign in (1) occurs if and only if all members of \( \tilde{x}_n \) are equal (cf. \cite{1}).

In this article we shall consider the difference

\[
d_t = d_t(n) = d_t([\tilde{x}_n, \tilde{p}_n]) := \sum_{i=1}^{n} p_i x_i^t - \left( \sum_{i=1}^{n} p_i x_i \right)^t, \quad t > 1,
\]

and thus generated sequence \( d = \{d_m\}_{m \geq 2} \) of non-negative real numbers.

By the above, if all members of the sequence \( \tilde{x}_n \) are equal, then all members of \( d \) are zero; hence this trivial case will be excluded in the sequel.

An interesting fact is that there exists an explicit constant \( c_m \), independent of the sequences \( \tilde{x}_n \) and \( \tilde{p}_n \), such that \( d_{m-1} d_{m+1} \geq c_m (d_m)^2 \), \( m \geq 3 \).

On the contrary, we show that there is no constant \( C_m \), depending only on \( m \), such that \( d_{m-1} d_{m+1} \leq C_m (d_m)^2 \).

Nontrivial lower bound for \( d_m \) and corresponding integral inequalities will also be given.

Finally we posed an open problem concerning the above matter.

2000 Mathematics Subject Classification: Primary 26D15.

Key words and phrases: power means; logarithmic convexity; best possible bounds.
2. Results

Denote by \(S_+\) the space of all positive sequences. Our main result is

**Theorem 1.** Let \(\tilde{p}_n, \tilde{x}_n \in S_+\) and \(d_m = d_m^{(n)} := \sum_1^n p_i x_i^m - (\sum_1^n p_i x_i)^m; m \in \mathbb{N}\). Then

\[
d_{m-1}d_{m+1} \geq c_m(d_m)^2, \ m \geq 3,
\]

with the best possible constant \(c_m = 1 - \frac{2}{m(m-1)}\).

This inequality is very precise. For example

\[
d_2(2) - \frac{2}{3}(d_3(2))^2 = \frac{1}{3}(p_1 p_2)^2(1 + p_1 p_2)(x_1 - x_2)^6.
\]

Non-trivial lower bound for \(d_m\) follows.

**Theorem 2.** For \(d_m\) defined as above, we have

\[
d_m \geq \left(\frac{m}{2}\right) \frac{(d_3/3)^{m-2}}{(d_2)^{m-3}}; \ m \geq 2.
\]

Applying the standard procedure (cf. [1, p.131]), we pass from finite sums to definite integrals and obtain

**Theorem 3.** Let \(f(t), p(t)\) be non-negative, continuous and integrable functions for \(t \in [a, b]\), with \(\int_a^b p(t) \, dt = 1\). Denote

\[
D_m = D_m(a, b; f, p) := \int_a^b p(t) f^m(t) \, dt - \left(\int_a^b p(t) f(t) \, dt\right)^m.
\]

Then

(i) \(D_{m-1}D_{m+1} \geq \left(1 - \frac{2}{m(m-1)}\right)(D_m)^2, \ m \geq 3;\)

(ii) If \(f(t) \neq C, t \in [a, b]\), we have

\[
D_m \geq \left(\frac{m}{2}\right) \frac{(D_3/3)^{m-2}}{(D_2)^{m-3}}; \ m \geq 2.
\]

3. Proofs

We start with an interesting formula. For \(\tilde{p}_n, \tilde{x}_n \in S_+\), making a shift \(x_i \rightarrow x_i + t\), we obtain

\[
d_m(t) := \sum_1^n p_i(x_i + t)^m - \left(\sum_1^n p_i(x_i + t)\right)^m = \sum_1^n p_i(t + x_i)^m - \left(t + \sum_1^n p_i x_i\right)^m.
\]

Developing, we get

\[
d_m(t) = \sum_2^n d_i \binom{m}{i} t^{m-i}.
\]

Therefore \(d_m(t)\) belongs to the class of Appell polynomials i.e., \(d'_m(t) = md_{m-1}(t)\) (cf [3], [4]).
If the properties of this class of polynomials lead to the proof of Theorem 1 is left to the readers to examine. For example, by (1), \(d_4(t)\) is non-negative for each \(t \in \mathbb{R}\). Hence by (3),

\[
d_4(t) = d_4 + 4d_3t + 6d_2t^2 \geq 0.
\]

Putting \(t = -\frac{1}{3} \frac{d_3}{d_2}\), we obtain (2) with \(m = 3\).

In this article we turn the other way, noting that (2) can be rewritten in the form

\[
d_\frac{m-1}{m-1} \frac{d_{m+1}}{(m-1)(m-2)} \frac{d_{m+1}}{(m+1)m} \geq \left( \frac{d_m}{m(m-1)} \right)^2, \quad m \geq 3.
\]

Hence, (2) is equivalent to the assertion that \(\frac{d_m}{m(m-1)}\) is log-convex for \(m \geq 3\).

**Definition.** A sequence of positive numbers \(\{c_m\}\) is log-convex (\(c_m \in LC\)) if \(c_{m-1}c_{m+1} \geq (c_m)^2\).

We quote here some useful lemmas from log-convex theory (cf [3]).

**Lemma 3.1.** A positive sequence \(\{c_m\}\) is log-convex if and only if the inequality

\[
c_{m-1}u^2 + 2c_muv + c_{m+1}v^2 \geq 0
\]

holds for each real \(u,v\).

**Lemma 3.2.** Let \(a_m,b_m \in LC\) and \(A,B,C\) be arbitrary positive constants. Then:

(i) \(AC^{m+B}a_m \in LC\); (ii) \(Aa_m + Bb_m \in LC\).

Now we are able to produce a proof of Theorem 1 by induction on \(n\).

**Proof of Theorem 1.** For \(n = 2\) we have to prove that

\[
\frac{p_1x_1^m + p_2x_2^m - (p_1x_1 + p_2x_2)^m}{m(m-1)} \in LC,
\]

holds for each positive \(x_1,x_2,p_1,p_2\) with \(p_1 + p_2 = 1\). To this end, we need the following simple assertion

**Lemma 3.3.** If \(A \geq B > 0\), then \(\frac{A^{m-1} - B^{m-1}}{m} \in LC\), holds for \(m \geq 2\).

Now, for fixed \(x_1,x_2,p_1,p_2\) and arbitrary \(\xi \geq 1\) put \(A = \xi, B = p_1\xi + p_2\); note that \(A \geq B\) since \(p_1 + p_2 = 1\). By lemmas 1, 3 and 2(i), for arbitrary \(u,v \in \mathbb{R}, m \geq 3\), we get

\[
p_1x_1^{m-1} \left( \frac{\xi^{m-1} - (p_1\xi + p_2)^{m-1}}{m} \right) u^2 + 2p_1x_2^m \left( \frac{\xi^{m-1} - (p_1\xi + p_2)^{m-1}}{m-1} \right) uv + p_1x_2^{m+1} \left( \frac{\xi^m - (p_1\xi + p_2)^m}{m} \right) v^2 \geq 0.
\]

Integrating (5) with respect to \(\xi\) over \(\xi \in [1,x_1/x_2]\), we obtain

\[
\frac{p_1x_1^{m-1} + p_2x_2^{m-1} - (p_1x_1 + p_2x_2)^{m-1}}{(m-1)(m-2)} u^2 + 2\frac{p_1x_1^m + p_2x_2^m - (p_1x_1 + p_2x_2)^m}{m(m-1)} uv + \frac{p_1x_1^{m+1} + p_2x_2^{m+1} - (p_1x_1 + p_2x_2)^{m+1}}{(m+1)m} v^2 \geq 0.
\]

Therefore by Lemma 1 we conclude that (4) is true.
Let $T := \frac{1}{1-p_n} \sum_{i=1}^{n-1} p_i x_i$. Then
\[
\frac{\alpha_m(n)}{m(m-1)} = \frac{\alpha_m(n-1)}{m(m-1)} + \frac{(1 - p_n)T^m + p_n x_n - ((1-p_n)T + p_n x_n)^m}{m(m-1)}.
\]

Since $\frac{\alpha_m(n-1)}{m(m-1)} \in LC$ by induction hypothesis, by (4) and Lemma 2(ii), it follows that $\frac{\alpha_m(n)}{m(m-1)} \in LC$, and the proof is done. \qed

To see that the constant $c_m = 1 - \frac{2}{m(m-1)}$ is best possible, consider the representation (3). Since variable $t$ is independent of the sequences $\tilde{p}_n, \tilde{x}_n$, we have $d_m(t) \sim d_2(\frac{n}{m}) t^{m-2}$ ($t \to \infty$). Hence
\[
\frac{d_{m-1}(t)d_{m+1}(t)}{(d_m(t))^2} \sim \frac{\binom{m-1}{2} m^3 \binom{m}{2} m^{-1}}{\binom{m}{2} m^{-1}} = c_m \quad (t \to \infty).
\]

**Proof of Theorem 2.** From (2) we get $d_{m+1}/d_m \geq c_m(d_m/d_{m-1})$, $m \geq 3$. Hence
\[
\prod_{k=3}^{m} \left( \frac{d_{k+1}}{d_k} \right) \geq \prod_{k=3}^{m} \left( \frac{k+1}{k+2} \right) \prod_{k=3}^{m} \left( \frac{d_k}{d_{k-1}} \right).
\]
i.e.,
\[
\frac{d_{m+1}}{d_m} \geq \frac{m+1}{3(m-1)} \frac{d_3}{d_2}, \quad m \geq 2.
\]
Therefore, the conclusion follows from
\[
\frac{d_m}{d_2} = \prod_{k=2}^{m} \left( \frac{d_{k+1}}{d_k} \right) \geq \prod_{k=2}^{m} \left( \frac{k+1}{k+2} \right) \prod_{k=2}^{m} \left( \frac{d_3}{3d_2} \right) = \left( \frac{m}{2} \right) \left( \frac{d_3}{3d_2} \right)^{m-2}.
\]

**Proof of Theorem 3.** Write $\alpha_m(n)$ in the form
\[
\alpha_m(n) = \sum_{i=1}^{n} p_{ni} x_{ni} - \left( \sum_{i=1}^{n} p_{ni} x_{ni} \right),
\]
with $p_{ni} := p(a + ikx_{ni}/n), x_{ni} := f(a + ikx_{ni}/n)$. Passing to the limit, we obtain $\lim_{n \to \infty} d_m(n) = D_m$ and from Theorems 1, 2 the assertions of Theorem 3 follow. \qed

There remains a problem of inverse inequality for the sequence $d$.

**Question 1.** Is there a constant $C_m$, independent of $\tilde{p}_n, \tilde{x}_n \in S_+$, such that $d_{m-1}d_{m+1} \leq C_m(d_m)^2$, $m \geq 2$?

The answer to this question is negative.

**Proof.** We apply a special choice of the sequences $\tilde{p}_n, \tilde{x}_n \in S_+$. Namely, for fixed $n \geq 2$ let $p_t := (t^{-1})/2n^{-1}; x_t := (1-t)^{n-1}(1+t)^{n-1}, -1 < t < 1$. We obtain a sequence $d^* = \{d^*_m(t)\}$ with
\[
d^*_m(t) = \left( \frac{(1-t)^m + (1+t)^m}{2} \right)^{n-1} - 1.
\]
For sufficiently large $n$, we have

$$d_2^*(1/\sqrt{2}) \sim (3/2)^{n-1}; \quad d_4^*(1/\sqrt{2}) \sim (17/4)^{n-1}; \quad d_3^*(1/\sqrt{2}) \sim (5/2)^{n-1}.$$ 

Hence $C_3 \geq (51/50)^{n-1} \to \infty$ ($n \to \infty$).

Therefore, we have to reformulate the problem.

**Question 2.** Is there a constant $C_{m,n}$ such that $d_{m-1}^{(n)}d_{m+1}^{(n)} \leq C_{m,n}(d_m^{(n)})^2$, for each $m, n \geq 2$, independently of sequences $\tilde{p}_n, \tilde{x}_n \in S_+$?

The best possible constant (if exists) is given by

$$C_{m,n} = \sup \left\{ \frac{d_{m-1}^{(n)}d_{m+1}^{(n)}}{(d_m^{(n)})^2} \mid \tilde{p}_n, \tilde{x}_n \in S_+ \right\}$$

Examining the sequence $d^*$, we conclude that $C_{m,n} \geq (1 + C/m^2)^{n-1}$, where $C$ is an absolute constant.

**References**


Mathematical Institute SANU
Kneza Mihaila 36
Belgrade
Serbia
ssimic@mi.sanu.ac.rs