ON BOUNDED DUAL-VALUED DERIVATIONS
ON CERTAIN BANACH ALGEBRAS

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Abstract. We consider the class $\mathcal{D}(\mathcal{U})$ of bounded derivations $\mathcal{U} \xrightarrow{d} \mathcal{U}^*$ defined on a Banach algebra $\mathcal{U}$ with values in its dual space $\mathcal{U}^*$ so that $(x, d(x)) = 0$ for all $x \in \mathcal{U}$. The existence of such derivations is shown, but lacking the simplest structure of an inner one. We characterize the elements of $\mathcal{D}(\mathcal{U})$ if span$(\mathcal{U}^2)$ is dense in $\mathcal{U}$ or if $\mathcal{U}$ is a unitary dual Banach algebra.

1. Introduction

Throughout this article let $\mathcal{U}$ be a complex Banach algebra endowed with a norm $\|\cdot\|$. Let $\mathcal{U}^\sharp$ be the algebra $\mathcal{U}$ plus an adjoined unit element $e$ with the usual Banach algebra structure. As usual, by $\mathcal{U}^*$ and $(\mathcal{U}^\sharp)^*$ we will denote the dual spaces of $\mathcal{U}$ and $\mathcal{U}^\sharp$ respectively. Let $j : \mathcal{U} \hookrightarrow \mathcal{U}^\sharp$ and $p : \mathcal{U}^\sharp \rightarrow \mathcal{U}$ be the natural injection and the corresponding projection of $\mathcal{U}$ into $\mathcal{U}^\sharp$ and of $\mathcal{U}^\sharp$ onto $\mathcal{U}$ respectively. Then $\mathcal{U}^\sharp = \mathbb{C} \cdot e \bigoplus j(\mathcal{U})$, i.e., any element $\eta \in \mathcal{U}^\sharp$ can be written in a unique way as $\eta = ae + j(x)$, with $x \in \mathcal{U}$ and $a \in \mathbb{C}$, and its $\mathcal{U}^\sharp$-norm is given as $\|\eta\|_{\mathcal{U}^\sharp} = |a| + \|x\|$. Indeed, since $j$ is an isometric homomorphism then $j(\mathcal{U})$ becomes a closed ideal of $\mathcal{U}^\sharp$. Further, $p \circ j = \text{Id}_\mathcal{U}$ while $j \circ p$ is the linear projection of $\mathcal{U}^\sharp$ onto $j(\mathcal{U})$. Thus, let $e^* \in (\mathcal{U}^\sharp)^*$ be defined as $\langle \eta, e^* \rangle \triangleq a$ if $\eta = ae + j(x)$ in $\mathcal{U}^\sharp$. Then

$$(\mathcal{U}^\sharp)^* = \mathbb{C} \cdot e^* \bigoplus \text{rank}(p^*),$$

where $p^* : \mathcal{U}^* \rightarrow (\mathcal{U}^\sharp)^*$ is the adjoint operator of $p$. It is well known that $\mathcal{U}^*$ admits a Banach $\mathcal{U}$-bimodule structure if for $x, y \in \mathcal{U}$ and $x^* \in \mathcal{U}^*$ we write

$$(y, x x^*) \triangleq \langle y x, x^* \rangle \quad \text{and} \quad (y, x^* x) \triangleq \langle y x, x^* \rangle.$$
The $\mathcal{U}^\ell$-bimodule structure on $(\mathcal{U}^\ell)^*$ is given as

\[
(ae + j(x))(be^* + p^*(x^*)) \triangleq (ab + \langle x, x^* \rangle) e^* + p^*(ax^* + xx^*),
\]
\[
(be^* + p^*(x^*))\langle ae + j(x) \rangle \triangleq (ab + \langle x, x^* \rangle) e^* + p^*(ax^* + x^* x),
\]

where $a, b \in \mathbb{C}$, $x \in \mathcal{U}$, $x^* \in \mathcal{U}^*$.

Given a Banach $\mathcal{U}$-bimodule $\mathfrak{X}$ let $Z^1(\mathcal{U}, \mathfrak{X})$ be the Banach space of bounded derivations $d : \mathcal{U} \to \mathfrak{X}$, i.e., those $d \in B(\mathcal{U}, \mathfrak{X})$ that satisfy the Leibnitz rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{U}$. A bounded derivation $d$ is said to be inner if there is an element $\phi \in \mathfrak{X}$ so that $d(x) = x\phi - \phi x$ if $x \in \mathcal{U}$. In that case we write $d = ad_\phi$ and the class of inner derivations from $\mathcal{U}$ into $\mathfrak{X}$ is denoted as $\mathcal{N}^1(\mathcal{U}, \mathfrak{X})$. Kamowitz lay the functional analytic overtones required to adapt the theory of Banach algebras to the Hochschild algebraic setting (cf. [3]; see also [4]).

The theory of *amenable Banach algebras* was greatly influenced by Johnson’s memoire in 1972 (cf. [5]). A Banach algebra $\mathcal{U}$ is called *amenable* if $\mathcal{H}^1(\mathcal{U}, \mathfrak{X}^*) = \{0\}$ for any Banach $\mathcal{U}$-bimodule $\mathfrak{X}$. A Banach algebra $\mathcal{U}$ it is called *weakly amenable* if $\mathcal{H}^1(\mathcal{U}, \mathfrak{X}^*) = \{0\}$. This last notion generalizes that introduced by Bade, Curtis and Dales in [2].

In [3] it was proved that a non-unital abelian Banach algebra $\mathcal{U}$ is weakly amenable if and only if $\mathcal{U}^\ell$ is weakly amenable but the general case still remains open. Our goal in this article is to seek relationships between derivations on a non-abelian non-unital Banach algebra $\mathcal{U}$ with values in $\mathcal{U}^\ell$ and derivations in $\mathcal{U}^\ell$ with values in $(\mathcal{U}^\ell)^*$. Our investigation naturally bring us to introduce the notion of $\mathcal{D}$-derivations on $\mathcal{U}$ in Definition 2.1. Although any element of $\mathcal{N}^1(\mathcal{U}, \mathfrak{X}^*)$ is a $\mathcal{D}$-derivation on $\mathcal{U}$ sometimes there exist non-inner $\mathcal{D}$-derivations as we will see in the examples 2.2 and 2.3. In Proposition 2.1 we consider certain Banach projective tensor products all of whose derivations are $\mathcal{D}$-derivations. In Theorem 2.1 we characterize $\mathcal{D}$-derivations on $\mathcal{U}$ on Banach algebras $\mathcal{U}$ so that $\mathcal{U}^\ell$ is dense in $\mathcal{U}$, where $\mathcal{U}^\ell = \text{span}\{xy : x, y \in \mathcal{U}\}$. The relationship between $\mathcal{D}$-derivations on $\mathcal{U}$ and their extensions to the unitization $\mathcal{U}^\ell$ are studied in Proposition 2.2. In this context, inner $\mathcal{D}$-derivations on $\mathcal{U}$ are characterized in Corollary 2.1. Finally, in Proposition 2.3 we characterize $\mathcal{D}$-derivations on dual Banach algebras.

2. On $\mathcal{D}$-derivations

**Definition 2.1.** A derivation $d \in Z^1(\mathcal{U}, \mathcal{U}^*)$ is called a $\mathcal{D}$-derivation on $\mathcal{U}$ if $\langle x, d(x) \rangle = 0$ for all $x \in \mathcal{U}$. Let $\mathcal{D}(\mathcal{U})$ be the set of $\mathcal{D}$-derivations on $\mathcal{U}$.

**Example 2.1.** All inner $\mathcal{U}^\ell$-valued derivations on $\mathcal{U}$ are $\mathcal{D}$-derivations on $\mathcal{U}$.

**Example 2.2.** Let $\mathcal{U} \triangleq C^1_0[a, b]$ be the commutative Banach algebra of functions $x : [a, b] \to \mathbb{C}$ with continuous derivative $\dot{x}$ so that $x(a) = x(b)$ endowed with the norm $\|x\| \triangleq \|x\|_\infty + \|\dot{x}\|_\infty$. Then we define $\delta : \mathcal{U} \to \mathcal{U}^\ell$ as

\[
\langle y, \delta(x) \rangle = \int_a^b y(x) \, dx \quad \text{if} \quad x, y \in \mathcal{U}.
\]
The above Riemann–Stieltjes integral is well defined, \( d \) becomes clearly a \( \mathbb{C} \)-linear functional and
\[
|\langle y, d(x) \rangle| \leq \int_a^b |y| |d(x)| \, dx \leq \|y\| \|x\| \leq \|y\| \|x\|,
\]
i.e., \( d \in \mathcal{B}(\mathcal{U}, \mathcal{U}^*) \) and \( \|d(x)\| \leq \|x\| \) for all \( x \in \mathcal{U} \). Indeed,
\[
\langle y, d(x_1 x_2) \rangle = \int_a^b y \left( \frac{dx_1}{dt} x_2 + x_1 \frac{dx_2}{dt} \right) dt
\]
\[
= \int_a^b y x_2 dx_1 + \int_a^b y x_1 dx_2
\]
\[
= (yx_2, d(x_1)) + (yx_1, d(x_2))
\]
\[
= (y, d(x_1) x_2 + x_1 d(x_2)),
\]
i.e., \( d \in \mathcal{Z}^1(\mathcal{U}, \mathcal{U}^*) \). Certainly, it is a nonzero \( \mathfrak{D} \)-derivation since for \( x \in \mathcal{U} \) we see that
\[
\langle x, d(x) \rangle = \int_a^b x \frac{dx}{dt} dt = \frac{x^2}{2} \bigg|_a^b = 0.
\]
Further, \( d \) is not inner because \( \mathcal{U} \) is abelian and so \( \mathcal{N}^1(\mathcal{U}, \mathcal{U}^*) = \{0\} \).

**Remark 2.1.** Given a dual Banach pair \((\mathfrak{X}, \mathfrak{Y}), (\circ, \circ)\) by the universal property characteristic of general tensor products there is a unique operation on \( \mathfrak{X} \otimes \mathfrak{Y} \) so that
\[
(x_1 \otimes y_1)(x_2 \otimes y_2) = (x_2, y_1)(x_1 \otimes y_2) \quad \text{if} \quad x_1, x_2 \in \mathfrak{X}, \ y_1, y_2 \in \mathfrak{Y}.
\]
Then \( \mathfrak{X} \otimes \mathfrak{Y} \) becomes an algebra. Further, if for \( u \in \mathfrak{X} \otimes \mathfrak{Y} \) we write
\[
\|u\|_\pi = \inf \left\{ \sum_{j=1}^n \|x_j\| \|y_j\| : u = \sum_{j=1}^n x_j \otimes y_j \right\}
\]
then \( (\mathfrak{X} \otimes \mathfrak{Y}, \|\cdot\|_\pi) \) becomes a normed algebra. The completion of this algebra is the well known projective Banach tensor algebra \( \hat{\mathfrak{X}} \otimes \hat{\mathfrak{Y}} \) (cf. [8, B.2.2, p. 250]). Then, \( \hat{\mathfrak{X}} \otimes \hat{\mathfrak{Y}} \) is amenable if and only if \( \dim(\mathfrak{X}) = \dim(\mathfrak{Y}) < \infty \) (cf. [8, Th. 4.3.5, p. 98]). So, if \( \mathfrak{X} \) is an infinite dimensional Banach space the determination of structure theorems of bounded derivations on \( \mathfrak{X} \otimes \mathfrak{X}^* \) has its own interest. Moreover, several Banach operator algebras can be represented as certain tensor products of the above type. For instance, if the Banach space \( \mathfrak{X} \) has the approximation property, then \( \mathcal{N}(\mathfrak{X}, \mathfrak{X}^*) \cong \mathfrak{X} \otimes \mathfrak{X}^* \), where \( \approx \) denotes an isometric isomorphism and \( \mathcal{N}(\mathfrak{X}, \mathfrak{X}^*) \) is the Banach space of \( \mathfrak{X}^* \)-nuclear operators on \( \mathfrak{X} \) (cf. [8, Th. C.1.5, p. 256]). In this setting the authors recently researched on structure theorems and properties of derivations on some non-amenable nuclear Banach algebras (see [1]).

**Proposition 2.1.** Let \( \mathfrak{X} \) is an infinite dimensional Banach space endowed with and shrinking basis \( \{x_n\}_{n=1}^\infty \) and an associated sequence of coefficient functionals \( \{x_n^*\}_{n=1}^\infty \) and let \( \mathcal{U} \cong \mathfrak{X} \otimes \mathfrak{X}^* \). Then \( \mathcal{D}(\mathcal{U}) \supseteq \mathcal{Z}^1(\mathcal{U}, \mathcal{U}^*) \).
Proof. Let \( d \in \mathcal{Z}^1(\mathcal{U}, \mathcal{U}^*) \). The system of basic tensor products \( z_{n,m} \triangleq x_n \otimes x_m^* \) can be arranged into a basis \( \{ z_{n,m} \}_{n,m=1}^{\infty} \) of \( \mathfrak{X} \otimes \mathfrak{X}^* \) (the reader can see [9], or else [10] Th. 18.1, p. 172]). Given \( p, q, r, s, t \in \mathbb{N} \) we have

\[
(2.1) \quad \langle z_{p,q}, d(z_{r,t}) \rangle = \langle z_{p,q}, d(z_{r,s}) \cdot z_{s,t} + z_{s,r} \cdot d(z_{s,t}) \rangle = \langle z_{s,t} \cdot z_{p,q}, d(z_{r,s}) \rangle + \langle z_{p,q} \cdot z_{r,s}, d(z_{s,t}) \rangle = \delta_{p,t} \langle z_{s,t}, d(z_{r,s}) \rangle + \delta_{q,r} \langle z_{p,q}, d(z_{s,t}) \rangle,
\]

where \( \delta \) denotes the usual Kronecker’s symbol. By \( (2.1) \), \( \langle z_{p,q}, d(z_{r,t}) \rangle = 0 \) if \( p \neq t \) and \( q \neq r \). Using \( (2.1) \) we also get

\[
(2.2) \quad \langle z_{p,q}, d(z_{p,q}) \rangle = \langle z_{s,t}, d(z_{r,s}) \rangle + \langle z_{p,q}, d(z_{s,t}) \rangle \quad \text{if} \quad p, q, s \in \mathbb{N}.
\]

By \( (2.1,2.2) \) we see that

\[
(2.3) \quad \langle z_{p,p}, d(z_{p,p}) \rangle = 0 \quad \text{if} \quad p \in \mathbb{N}.
\]

On the other hand, by \( (2.1) \) we obtain

\[
(2.4) \quad \langle z_{p,q}, d(z_{p,q}) \rangle = 0 \quad \text{if} \quad p, q \in \mathbb{N}, \quad p \neq q.
\]

Now, let \( F \) be a finite subset of \( \mathbb{N} \times \mathbb{N} \), \( \{ \lambda_{(n,m)} \}_{(n,m) \in F} \subseteq \mathbb{C} \) and let

\[
u = \sum_{(n,m) \in F} \lambda_{(n,m)} z_{n,m}.
\]

By \( (2.3) \) and \( (2.4) \) we see that

\[
(2.5) \quad \langle u, d(u) \rangle = \sum_{(n,m),(p,q) \in F} \lambda_{(n,m)} \lambda_{(p,q)} [(z_{n,m}, d(z_{p,q})) + \langle z_{p,q}, d(z_{n,m}) \rangle].
\]

As we already observed, those summands in \( (2.5) \) so that \( n \neq q \) and \( m \neq p \) are zero. By symmetry, it suffices to consider \( n = q \) and then

\[
\langle z_{n,m}, d(z_{p,n}) \rangle + \langle z_{p,n}, d(z_{n,m}) \rangle = \langle z_{n,n}, z_{n,m} \cdot d(z_{p,n}) + d(z_{n,m}) \cdot z_{p,n} \rangle = \langle z_{n,n}, d(z_{n,n}) \rangle = 0.
\]

Consequently \( \langle u, d(u) \rangle = 0 \). Finally, the result holds since \( u \to \langle u, d(u) \rangle \) is continuous on \( \mathcal{U} \) and \( \mathfrak{X} \otimes \mathfrak{X}^* \) is dense in \( \mathcal{U} \). \( \square \)

**Example 2.3.** Let \( \mathcal{U} \triangleq \ell^p \hat{\otimes} \ell^q \), with \( 1 < p, q < \infty \), \( 1/p + 1/q = 1 \). If \( x \in \ell^p \), \( x^* \in \ell^q \) let

\[
d_{x,x^*} : \ell^p \times \ell^q \to C, \quad d_{x,x^*}(y, y^*) \triangleq \langle x, y^* \rangle - \langle y, x^* \rangle.
\]

Then \( d_{x,x^*} \in \mathcal{B}^2(\ell^p \times \ell^q, \mathbb{C}) \), i.e., \( d_{x,x^*} \) is a bounded bilinear form between \( \ell^p \times \ell^q \) and \( C \). By the universal characteristic property of the projective tensor product of Banach spaces there is a unique \( \overline{d}_{x,x^*} \in \mathcal{U}^* \) so that \( \| \overline{d}_{x,x^*} \| = \| d_{x,x^*} \| \) and \( \langle y \otimes y^*, \overline{d}_{x,x^*} \rangle = d_{x,x^*}(y, y^*) \) if \( y \in \ell^p \), \( y^* \in \ell^q \). The following map is then induced

\[
\overline{d} : \ell^p \times \ell^q \to \mathcal{U}^*, \overline{d}(x, x^*) \triangleq \overline{d}_{x,x^*}.
\]
It is readily seen that \( \overline{d} \in \mathcal{B}^2(l^p, l^q, \mathcal{U}^*) \) and so there is a unique \( d \in \mathcal{B}(\mathcal{U}, \mathcal{U}^*) \) so that  
\( \|d\| = \|\overline{d}\| \) and \( d(x \otimes x^*) = \overline{d}(x, x^*) \) if \( x \in l^p, \ x^* \in l^q \). Consequently, the following identity

\[
\langle y \otimes y^*, d(x \otimes x^*) \rangle = \langle x, y^* \rangle - \langle y, x^* \rangle
\]

holds if \( x, y \in l^p \) and \( x^*, y^* \in l^q \). It is straightforward to see that \( d \in \mathcal{Z}^1(\mathcal{U}, \mathcal{U}^*) \) and hence it is a \( \mathcal{D} \)-derivation. Let us see that \( d \notin \mathcal{N}^1(\mathcal{U}, \mathcal{U}^*) \). For, let us assume that \( d \) is inner, say \( d = ad_T \) for some \( T \in \mathcal{U}^* \). Let us consider the usual basis \( \{x_n\}_{n=1}^\infty \) of \( l^p \), \( x_n = \{\delta_{n,m}\}_{m=1}^\infty \) if \( n \in \mathbb{N} \). So, \( \{x_n\}_{n=1}^\infty \) is obviously a shrinking basis and its associated sequence of coefficient functionals are \( x_n^* = \{\delta_{n,m}\}_{m=1}^\infty \) if \( n \in \mathbb{N} \). With the notation of Proposition 2.1, since \( \langle z_{n,m}, d(z_{p,q}) \rangle = \delta_{m,p} - \delta_{n,q} \) for all \( n, m, p, q \in \mathbb{N} \) we deduce that \( T(z_{n,m}) = 1 \) if \( n, m \in \mathbb{N} \) and \( n \neq m \). However, let us write

\[
u \triangleq \frac{1}{\zeta(q)^{1/q}} \sum_{n=1}^\infty \frac{1}{n} \cdot z_{1,1+n},
\]

where \( \zeta \) denotes the Riemann zeta function. Then \( u \in \mathcal{U} \) is well defined,

\[
\|u\|_\pi = \lim_{N \to \infty} \left\| \frac{1}{\zeta(q)^{1/q}} \sum_{n=1}^N \frac{1}{n} \cdot z_{1,1+n} \right\|_\pi
\]

\[
= \frac{1}{\zeta(q)^{1/q}} \lim_{N \to \infty} \left\| x_1 \otimes \sum_{n=1}^N \frac{1}{n} x_{n+1}^* \right\|_{l^q}
\]

\[
= \frac{1}{\zeta(q)^{1/q}} \lim_{N \to \infty} \left\| \sum_{n=1}^N \frac{1}{n} x_{n+1}^* \right\|_{l^q}
\]

\[
= 1,
\]

and as

\[
T(u) = \frac{1}{\zeta(q)^{1/q}} \sum_{n=1}^\infty \frac{1}{n} = \infty
\]

then \( T \) can not be bounded.

**Theorem 2.1.** Let \( \mathcal{U} \) be a Banach algebra so that \( \mathcal{U}^2 \) is dense in \( \mathcal{U} \). Let us denote \( k_{\mathcal{U}} : \mathcal{U} \hookrightarrow \mathcal{U}^{**} \) to the usual isometric embedding of \( \mathcal{U} \) into its second dual space \( \mathcal{U}^{**} \) by means of evaluations. Given \( d \in \mathcal{Z}^1(\mathcal{U}, \mathcal{U}^*) \) the following assertions are equivalent:

(i) \( d \in \mathcal{D}(\mathcal{U}) \).

(ii) \( \langle x, d(y) \rangle + \langle y, d(x) \rangle = 0 \) for all \( x, y \in \mathcal{U} \).

(iii) \( d^* \circ k_{\mathcal{U}} \in \mathcal{Z}^1(\mathcal{U}, \mathcal{U}^*) \).

(iv) \( d + d^* \circ k_{\mathcal{U}} = 0 \).

**Proof.** (i) \( \Rightarrow \) (ii) Given \( x, y \in \mathcal{U} \) we have

\[
0 = \langle x + y, d(x+y) \rangle = \langle x, d(y) \rangle + \langle y, d(x) \rangle.
\]
(ii) ⇒ (iii) If \( x, y, z \in \mathcal{U} \) we have
\[
\langle z, (d^* \circ k_\mathcal{U})(xy) \rangle = \langle \langle d(z), k_\mathcal{U}(xy) \rangle \rangle \\
= \langle xy, d(z) \rangle \\
= \langle x, yd(z) \rangle \\
= \langle x, d(yz) - d(y)z \rangle \\
= \langle x, d(yz) \rangle - \langle xz, d(y) \rangle \\
= \langle d(yz), k_\mathcal{U}(x) \rangle + \langle y, d(zx) \rangle \\
= \langle yz, d^*(k_\mathcal{U}(x)) \rangle + \langle d(zx), k_\mathcal{U}(y) \rangle \\
= \langle z, d^*(k_\mathcal{U}(x))y \rangle + \langle z, x d^*(k_\mathcal{U}(y)) \rangle \\
= \langle z, (d^* \circ k_\mathcal{U})(xy) \rangle + x(d^* \circ k_\mathcal{U})(y).
\]

(iii) ⇒ (iv) For \( x, y, z \in \mathcal{U} \) we have
\[
\langle xy, d(z) \rangle = \langle \langle d(z), k_\mathcal{U}(xy) \rangle \rangle \\
= \langle z, (d^* \circ k_\mathcal{U})(xy) \rangle \\
= \langle z, (d^* \circ k_\mathcal{U})(xy) + d(x) \rangle \\
= \langle yz, d^*(k_\mathcal{U}(x)) \rangle + \langle d(zx), k_\mathcal{U}(y) \rangle \\
= \langle z, d^*(k_\mathcal{U}(x)) \rangle y + \langle z, x d^*(k_\mathcal{U}(y)) \rangle \\
= \langle z, (d^* \circ k_\mathcal{U})(xy) \rangle + x(d^* \circ k_\mathcal{U})(y).
\]
Therefore,
\[
\langle z, d(xy) \rangle = -\langle xy, d(z) \rangle = -\langle d(z), k_\mathcal{U}(xy) \rangle = -\langle z, (d^* \circ k_\mathcal{U})(xy) \rangle,
\]
i.e., \((d + d^* \circ k_\mathcal{U})(xy) = 0\) if \( x, y \in \mathcal{U} \). Since \( \mathcal{U}^2 \) is dense in \( \mathcal{U} \) the claim follows.

(iv) ⇒ (i) If \( x \in \mathcal{U} \) then
\[
0 = \langle x, d(x) \rangle + (d^* \circ k_\mathcal{U})(x) = 2\langle x, d(x) \rangle.
\]
\[\Box\]

Proposition 2.2. Let \( \mathcal{U} \) be a Banach algebra and let \( d \in \mathcal{D}(\mathcal{U}) \). There is a unique \( d^\sharp \in \mathcal{D}(\mathcal{U}^\sharp) \) so that the following diagram commutes
\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{d} & \mathcal{U}^* \\
\downarrow j & & \downarrow p^* \\
\mathcal{U}^\sharp & \xrightarrow{d^\sharp} & t(\mathcal{U}^\bigstar)
\end{array}
\]

Proof. Consider \( d^\flat \triangleq p^* \circ d \circ p \). Thus, \( d^\flat \in \mathcal{B}(\mathcal{U}^\bigstar, (\mathcal{U}^\sharp)^*) \) and \( d^\flat \circ j = p^* \circ d \).
If \( \eta, \mu \in \mathcal{U}^\sharp \) we get
\[
\langle \eta, d^\flat(\eta) \rangle = \langle p(\eta), d(p(\eta)) \rangle = 0
\]
(2.6)
and if \( \eta = ae + j(x) \), \( \mu = be + j(y) \) for uniquely determined \( a, b \in \mathbb{C} \) and \( x, y \in \mathcal{U} \) then

\[
(2.7) \quad d\delta(\eta)\mu + \eta d\delta(\mu) = p^*(d(x))(be + j(y)) + (ae + j(x))p^*(d(y))
\]

\[
= ((y, d(x)) + (x, d(y)))e + p^*(ad(y) + bd(x) + d(xy))
\]

\[
= ad\delta(y) + bd\delta(x) + p^*(d(xy))
\]

\[
= p^*(d(ay + bx + xy))
\]

\[
d\delta(\eta \mu).
\]

Thus, by \( (2.6) \) and \( (2.7) \) we conclude that \( d\delta \in \mathfrak{D}(\mathcal{U}^2) \). As we already observed, \( j \circ p \) projects \( \mathcal{U}^2 \) onto \( j(\mathcal{U}) \). Since \( j(\mathcal{U}) \) is complemented in \( \mathcal{U}^2 \) by \( \mathbb{C} \cdot e \) then \( d\delta \) is uniquely determined. \( \square \)

**Corollary 2.1.** A \( \mathfrak{D} \)-derivation on \( \mathcal{U} \) is inner if and only if its associated derivation \( d\delta : \mathcal{U}^2 \rightarrow (\mathcal{U}^2)^* \) by Proposition \( 2.2 \) is inner.

**Proof.** Let \( x^* \in \mathcal{U}^* \), \( a \in \mathbb{C} \). Hence, it is easy to see that \( (ad_a)^2 = ad_{p^*(x^*)} \) and if \( d\delta = ad_{ae + p^*(x^*)} \), then \( d = ad_{x^*} \). \( \square \)

**Remark 2.2.** Let us consider a dual Banach algebra \( \mathcal{U} \), i.e., \( \mathcal{U} \cong (\mathcal{U}_a)^* \), where \( \mathcal{U}_a \) is a closed submodule of \( \mathcal{U}^* \). Although \( \mathcal{U}_a \) need not be unique, we will assume that \( \mathcal{U} \) is realized as the dual space of a fixed closed submodule \( \mathcal{U}_a \) of \( \mathcal{U}^* \). It is known that a dual Banach algebra has a unit if and only if it has a bounded approximate identity (see [7] Prop. 1.2)).

**Proposition 2.3.** Let \( \mathcal{U} \) be a dual Banach algebra with unit and let \( d \in \mathcal{Z}^1(\mathcal{U}, \mathcal{U}^*) \) so that \( d(\mathcal{U}) \subseteq k_{\mathcal{U}_a}(\mathcal{U}_a) \). Then \( d \in \mathfrak{D}(\mathcal{U}) \) if and only if \( d^* + d \circ k_{\mathcal{U}_a}^* = 0 \).

**Proof.** (\( \Rightarrow \)) Given \( x \in \mathcal{U} \) let \( x_+ \in \mathcal{U}_a \) be the unique element so that \( d(x) = k_{\mathcal{U}_a}(x_+) \). If \( x^+ \in \mathcal{U}^* \) by Theorem \( 2.1 \) iv) we have

\[
\langle x, (d \circ k_{\mathcal{U}_a}^*)(x^+) \rangle = \langle d(k_{\mathcal{U}_a}^*(x^+)), k_{\mathcal{U}_a}(x) \rangle
\]

\[
= \langle k_{\mathcal{U}_a}^*(x^+), (d^* \circ k_{\mathcal{U}_a})(x) \rangle
\]

\[
= -\langle k_{\mathcal{U}_a}^*(x^+), d(x) \rangle
\]

\[
= -\langle x, k_{\mathcal{U}_a}^*(x^+) \rangle
\]

\[
= -\langle k_{\mathcal{U}_a}(x_+), x^+ \rangle
\]

\[
= -\langle d(x), x^+ \rangle
\]

\[
= -\langle x, d^*(x^+) \rangle.
\]

(\( \Leftarrow \)) If \( x, y \in \mathcal{U} \) we obtain

\[
\langle y, (d^* \circ k_{\mathcal{U}_a})(x) \rangle = -\langle y, (d \circ k_{\mathcal{U}_a}^* \circ k_{\mathcal{U}_a})(x) \rangle = -\langle y, d(x) \rangle,
\]

i.e., \( d + d^* \circ k_{\mathcal{U}_a} = 0 \) and our claim follows. \( \square \)

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References


