ON SEQUENCE-COVERING mssc-IMAGES OF LOCALLY SEPARABLE METRIC SPACES

Nguyen Van Dung

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Abstract. We characterize sequence-covering (resp., 1-sequence-covering, 2-sequence-covering) mssc-images of locally separable metric spaces by means of \( \sigma \)-locally finite \( cs \)-networks (resp., \( sn \)-networks, \( so \)-networks) consisting of \( \aleph_0 \)-spaces (resp., \( sn \)-second countable spaces, \( so \)-second countable spaces). As the applications, we get characterizations of certain sequence-covering, quotient mssc-images of locally separable metric spaces.

1. Introduction

A study of some images of metric spaces under certain mappings is an important task on general topology. In [12], Li characterized sequence-covering (pseudo-sequence-covering) mssc-images of metric spaces by means of \( \aleph \)-spaces as follows.

**Theorem 1.1.** [12, Theorem 4] The following are equivalent for a space \( X \).

1. \( X \) is an \( \aleph \)-space.
2. \( X \) is a sequence-covering mssc-image of a metric space.
3. \( X \) is a pseudo-sequence-covering mssc-image of a metric space.

In [18], Lin and Yan characterized compact-covering, quotient \( \pi \)- and mssc-images of metric spaces by means of \( g \)-metrizable spaces, and this result has been proved by a quick and systematic proof in [25].

**Theorem 1.2.** [18, Corollary 18] The following are equivalent for a space \( X \).

1. \( X \) is a \( g \)-metrizable space.
2. \( X \) is a compact-covering, quotient compact and mssc-image of a metric space.
3. \( X \) is a compact-covering, quotient \( \pi \)- and mssc-image of a metric space.
4. \( X \) is a compact-covering, quotient \( \pi \)- and \( \sigma \)-image of a metric space.

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Related to the characterizations of images of metric spaces, many topologists were engaged in characterizing images of locally separable metric spaces, and some noteworthy results have been shown. In [16], Lin, Liu, and Dai characterized quotient $\sigma$-images of locally separable metric spaces. After that, Lin and Yan characterized sequence-covering $\sigma$-images of locally separable metric spaces [17]. Ikeda, Liu and Tanaka characterized quotient compact images of locally separable metric spaces [11]. Ge characterized pseudo-sequence-covering compact images of locally separable metric spaces [8]. An and Dung characterized quotient $\pi$-images of locally separable metric spaces [1]. In general, it is difficult to obtain nice characterizations of images of locally separable metric spaces (under covering-mappings) instead of metric domains.

Take the above into account, note that $\aleph_0$-spaces and $\mathcal{G}$-metrizable spaces are spaces having certain $\sigma$-locally finite networks, the following question arises naturally.

**Question.** How are sequence-covering (1-sequence-covering, 2-sequence-covering) mssc-images of locally separable metric spaces characterized by means of $\sigma$-locally finite networks?

In this paper, we characterize sequence-covering (resp., 1-sequence-covering, 2-sequence-covering) mssc-images of locally separable metric spaces by means of $\sigma$-locally finite $cs$-networks (resp., $sn$-networks, $so$-networks) consisting of $\aleph_0$-spaces (resp., $sn$-second countable spaces, $so$-second countable spaces). As the applications, we get characterizations of certain sequence-covering, quotient mssc-images of locally separable metric spaces. These results make the study of images of locally separable metric spaces more completely.

Throughout this paper, all spaces are regular and $T_1$, all mappings are continuous and onto, a convergent sequence includes its limit point, and $\mathbb{N}$ denotes the set of all natural numbers. Let $f : X \to Y$ be a mapping, and $\mathcal{P}$ be a family of subsets of $X$, we denote $\bigcup \mathcal{P} = \bigcup \{ P : P \in \mathcal{P} \}$, $\bigcap \mathcal{P} = \bigcap \{ P : P \in \mathcal{P} \}$, and $f(\mathcal{P}) = \{ f(P) : P \in \mathcal{P} \}$. We say that a convergent sequence $\{ x_n : n \in \mathbb{N} \} \cup \{ x \}$ converging to $x$ is eventually in $A$ if $\{ x_n : n \geq n_0 \} \cup \{ x \} \subset A$ for some $n_0 \in \mathbb{N}$, and it is frequently in $A$ if $\{ x_{n_k} : k \in \mathbb{N} \} \cup \{ x \} \subset A$ for some subsequence $\{ x_{n_k} : k \in \mathbb{N} \}$ of $\{ x_n : n \in \mathbb{N} \}$.

**Definition 1.1.** Let $\mathcal{P}$ be a family of subsets of a space $X$.

1. $\mathcal{P}$ is a network for $X$ [19] if, $\mathcal{P} = \bigcup \{ \mathcal{P}_x : x \in X \}$, where $x \in \bigcap \mathcal{P}_x$, and if $x \in U$ with $U$ open in $X$, then there exists $P \in \mathcal{P}_x$ such that $x \in P \subset U$ for every $x \in X$. Here, $\mathcal{P}_x$ is a network at $x$ in $X$.

2. $\mathcal{P}$ is a $cs$-network for $X$ [10] if, for each convergent sequence $S$ converging to $x \in U$ with $U$ open in $X$, $S$ is eventually in $P \subset U$ for some $P \in \mathcal{P}$.

3. $\mathcal{P}$ is a $cs^*$-network for $X$ [7] if, for each convergent sequence $S$ converging to $x \in U$ with $U$ open in $X$, $S$ is frequently in $P \subset U$ for some $P \in \mathcal{P}$.

4. $\mathcal{P}$ is a $cfp$-network for $X$ [26] if, for each compact subset $H \subset U$ with $U$ open in $X$, there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup \{ C_F : F \in \mathcal{F} \} \subset U$, where $C_F$ is closed and $C_F \subset U$ for every $F \in \mathcal{F}$.
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Definition 1.2. [6] Let $X$ be a space and $P$ be a subset of $X$.

1. $P$ is a sequential neighborhood of $x$ in $X$, if whenever $S$ is a convergent sequence converging to $x$, then $S$ is eventually in $P$.

2. $P$ is a sequentially open subset of $X$, if $P$ is a sequential neighborhood of $x$ in $X$ for every $x \in P$.

Definition 1.3. Let $P = \bigcup \{ \mathcal{P}_x : x \in X \}$ be a family of subsets of a space $X$ satisfying that, for each $x \in X$, $\mathcal{P}_x$ is a network at $x$ in $X$, and if $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

1. $P$ is a weak base for $X$ [24], if $G \subset X$ such that for each $x \in G$, there exists $P \in \mathcal{P}_x$ satisfying $P \subset G$, then $G$ is open in $X$. Here, $\mathcal{P}_x$ is a weak base at $x$ in $X$.

2. $P$ is an sn-network for $X$ [15], if each member of $\mathcal{P}_x$ is a sequential neighborhood of $x$ in $X$. Here, $\mathcal{P}_x$ is an sn-network at $x$ in $X$.

3. $P$ is an so-network for $X$ [15], if each member of $\mathcal{P}_x$ is sequentially open in $X$. Here, $\mathcal{P}_x$ is an so-network at $x$ in $X$.

Definition 1.4. Let $X$ be a space.

1. $X$ is a cosmic space [20] (resp., $\aleph_0$-space [20], sn-second countable space [9], so-second countable space, second countable space [5], R-space [21], g-metrizable space [23]), if $X$ has a countable network (resp., countable cs-network, countable sn-network, countable so-network, countable base, $\sigma$-locally finite cs-network, $\sigma$-locally finite weak base).

2. $X$ is a sequential space [6], if each sequentially open subset of $X$ is open.

Remark 1.1. [17] (1) For a space, weak base $\Rightarrow$ sn-network $\Rightarrow$ cs-network.

2. An sn-network for a sequential space is a weak base.

Definition 1.5. Let $f : X \to Y$ be a mapping.

1. $f$ is an mssc-mapping [14], if $X$ is a subspace of the product space $\prod_{n \in \mathbb{N}} X_n$ of a family $\{X_n : n \in \mathbb{N}\}$ of metric spaces, and for each $y \in Y$, there exists a sequence $\{V_{y,n} : n \in \mathbb{N}\}$ of open neighborhoods of $y$ in $Y$ such that each $p_n(f^{-1}(V_{y,n}))$ is a compact subset of $X_n$, where $p_n : \prod_{i \in \mathbb{N}} X_i \to X_n$ is the projection.

2. $f$ is an 1-sequence-covering mapping [15] if, for each $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that whenever $\{y_n : n \in \mathbb{N}\}$ is a sequence converging to $y$ in $Y$ there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to $x_y$ in $X$ with each $x_n \in f^{-1}(y_n)$.

3. $f$ is a 2-sequence-covering mapping [15] if, for each $y \in Y$, $x_y \in f^{-1}(y)$, and sequence $\{y_n : n \in \mathbb{N}\}$ converging to $y$ in $Y$, there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to $x_y$ in $X$ with each $x_n \in f^{-1}(y_n)$.

4. $f$ is a sequence-covering mapping [22] if, for each convergent sequence $S$ of $Y$, there exists a convergent sequence $L$ of $X$ such that $f(L) = S$. Note that a sequence-covering mapping is a strong sequence-covering mapping in the sense of [12].

5. $f$ is a pseudo-sequence-covering mapping [11] if, for each convergent sequence $S$ of $Y$, there exists a compact subset $K$ of $X$ such that $f(K) = S$. 
(6) \( f \) is a **sequentially-quotient** mapping \([3]\) if, for each convergent sequence \( S \) of \( Y \), there exists a convergent sequence \( L \) of \( X \) so that \( f(L) \) is a subsequence of \( S \).

(7) \( f \) is a **compact-covering** mapping \([20]\) if, for each compact subset \( K \) of \( Y \), there exists a compact subset \( L \) of \( X \) such that \( f(L) = K \).

(8) \( f \) is a \( \pi \)-**mapping** \([2]\) if, for each \( y \in Y \) and for each neighborhood \( U \) of \( y \) in \( Y \), \( d(f^{-1}(y), X - f^{-1}(U)) > 0 \), where \( X \) is a metric space with a metric \( d \).

(9) \( f \) is a \( \sigma \)-**mapping** \([18]\), if there exists a base \( B \) of \( X \) such that \( f(B) \) is a \( \sigma \)-locally finite family in \( Y \).

**Definition 1.6.** \([4]\) A space \( X \) is **sequentially separable**, if \( X \) has a countable subset \( D \) such that for each \( x \in X \), there exists a sequence \( \{ x_n : n \in \mathbb{N} \} \) in \( D \) converging to \( x \). Here, the subset \( D \) is a **sequentially dense** subset of \( X \).

For undefined terms, refer to \([5]\) and \([24]\).

## 2. Results

First, we characterize sequence-covering mssc-images of locally separable metric spaces by means of \( \sigma \)-locally finite cs-networks.

**Theorem 2.1.** The following are equivalent for a space \( X \).

1. \( X \) is a sequence-covering mssc-image of a locally separable metric space.
2. \( X \) has a \( \sigma \)-locally finite cs-network consisting of cosmic spaces.
3. \( X \) has a \( \sigma \)-locally finite cs-network consisting of \( \aleph_0 \)-spaces.

**Proof.** \((1) \Rightarrow (2)\). Let \( f : M \to X \) be a sequence-covering mssc-mapping from a locally separable metric space \( M \) onto \( X \), and \( \{ X_n : n \in \mathbb{N} \} \) be the family of metric spaces satisfying that \( M \) is a subspace of \( \prod_{n \in \mathbb{N}} X_n \), and for each \( x \in X \), there exists a sequence \( \{ V_{x,n} : n \in \mathbb{N} \} \) of open neighborhoods of \( x \) in \( X \) such that each \( \overline{p_n(f^{-1}(V_{x,n}))} \) is a compact subset of \( X_n \), where \( p_n : \prod_{i \in \mathbb{N}} X_i \to X_n \) is the projection. Since \( M \) is locally separable metric, \( M = \bigoplus_{\lambda \in \Lambda} M_\lambda \), where each \( M_\lambda \) is a metric space by \([5]\). Since each \( X_n \) is a metric space, \( X_n \) has a \( \sigma \)-locally finite base \( \mathcal{C}_n = \bigcup \{ \mathcal{C}_{n,i} : i \in \mathbb{N} \} \), where each \( \mathcal{C}_{n,i} \) is locally finite. Assume, if necessary, that \( \mathcal{C}_{n,i} \subseteq \mathcal{C}_{n,i+1} \) for every \( i \in \mathbb{N} \). For each \( n \in \mathbb{N} \), set

9. \( \mathcal{B}_n = \left\{ M \cap \bigcap_{i \leq n} p_i^{-1}(C_i) : \right. \)

\( \left. C_i \in \bigcup_{j \leq n} \mathcal{C}_{i,j}, i \leq n, M \cap \bigcap_{i \leq n} p_i^{-1}(C_i) \subseteq M_\lambda \text{ for some } \lambda \in \Lambda \right\} \)

set \( \mathcal{P}_n = f(\mathcal{B}_n) \), and set \( \mathcal{B} = \bigcup \{ \mathcal{B}_n : n \in \mathbb{N} \} \), \( \mathcal{P} = \bigcup \{ \mathcal{P}_n : n \in \mathbb{N} \} \). Then \( \mathcal{B} \) is a base for \( M \) consisting of separable subsets. Assume, if necessary, that \( \mathcal{B} \) is closed under finite intersections. We shall show that \( \mathcal{P} \) is a \( \sigma \)-locally finite cs-network for \( X \) consisting of cosmic spaces by the following facts (a), (b), and (c).

(a) \( \mathcal{P} \) is a cs-network for \( X \).

Let \( S \) be a convergent sequence being eventually in \( U \) with \( U \) open in \( X \). Since \( f \) is sequence-covering, there exists a convergent sequence \( L \) in \( M \) such that

(b) \( \mathcal{P} \) is a cs-network for \( X \).

Let \( S \) be a convergent sequence being eventually in \( U \) with \( U \) open in \( X \). Since \( f \) is sequence-covering, there exists a convergent sequence \( L \) in \( M \) such that

(c) \( \mathcal{P} \) is a cs-network for \( X \).

Let \( S \) be a convergent sequence being eventually in \( U \) with \( U \) open in \( X \). Since \( f \) is sequence-covering, there exists a convergent sequence \( L \) in \( M \) such that
Therefore, \( x \in \bigcup \{ f_i^{-1}(V_{a,i}) : i \in \mathbb{N} \} \), which is a compact subset of \( X \). If \( f_i^{-1}(V_{a,i}) \) is locally finite, then \( f_i^{-1}(V_{a,i}) \) meets only finitely many members of \( C_i \). Then \( f^{-1}(V_a) \) meets only finitely many members of \( \bigcup_{i \in \mathbb{N}} C_i \). Therefore, \( f^{-1}(V_a) \) meets only finitely many members of \( \bigcup_{i \in \mathbb{N}} C_i \). It follows that \( f^{-1}(V_a) \) meets only finitely many members of \( \mathcal{B}_n \). Hence \( V_a \) meets only finitely many members of \( f(\mathcal{B}_n) \), i.e., \( \mathcal{P}_n \) is locally finite. It follows that \( \mathcal{P} \) is \( \sigma \)-locally finite.

(c) Each \( P \in \mathcal{P} \) is a cosmic space.

Set \( P = f(B) \) for some \( B \in \mathcal{B} \). Since \( B \) is separable, \( P \) is cosmic.

\[ \mathcal{P} = \bigcup \{ \mathcal{P}_n : n \in \mathbb{N} \} \]
\[ \mathcal{P} = \bigcup \{ \mathcal{P}_n : n \in \mathbb{N} \} \] be a \( \sigma \)-locally finite \( cs \)-network for \( X \) consisting of cosmic spaces. Every locally finite family in a Lindelöf space is countable. Hence for each \( P \in \mathcal{P} \), \( \{ P \cap P' : P' \in \mathcal{P} \} \) is countable, and obviously it is a \( cs \)-network for \( P \).

\[ \mathcal{P} = \bigcup \{ \mathcal{P}_n : n \in \mathbb{N} \} \] be a \( \sigma \)-locally finite \( cs \)-network for \( X \) consisting of \( \aleph_0 \)-spaces, where each \( \mathcal{P}_n = \{ P_{\alpha_n} : \alpha_n \in A_n \} \) is a locally finite family. For each \( n \in \mathbb{N} \), since each \( P_{\alpha_n} \) is a \( \aleph_0 \)-space, \( P_{\alpha_n} \) has a countable \( cs \)-network \( \mathcal{P}_{\alpha_n} = \{ P_{\alpha_{n,i}} : i \geq n \} \). For each \( i \geq n \), set

\[ Q_{\alpha_{n,i}} = \{ P_{\alpha_n} \cup \{ P_{\alpha_{n,j}} : n \leq j \leq i \} = \{ Q_\beta : \beta \in B_{\alpha_{n,i}} \}, \]

where \( B_{\alpha_{n,i}} \) is finite, and set

\[ Q_i = \{ X \} \cup \bigcup \{ Q_{\alpha_{j,i}} : \alpha_j \in A_j, j \leq i \} = \{ Q_\beta : \beta \in B_i \}, \]

where \( B_i = \{ \beta_0 \} \cup \bigcup \{ B_{\alpha_{j,i}} : \alpha_j \in A_j, j \leq i \} \) with \( Q_\beta = X \). Since each \( \mathcal{P}_i \) is locally finite and each \( Q_{\alpha_{j,i}} \) is finite, \( Q_i \) is locally finite. Endow \( B_i \) with the discrete topology, then \( B_i \) is a metric space. Set

\[ M = \{ b = (\beta_i) \in \prod_{i \in \mathbb{N}} B_i : \text{there exists } n \in \mathbb{N} \text{ and } \alpha_n \in A_n \text{ such that } Q_{\beta_i} = X \text{ if } i < n, Q_{\beta_i} \in Q_{\alpha_{n,i}} \text{ if } i \geq n, \text{ and } \{ Q_{\beta_i} : i \geq n \} \text{ forms a network at a point } x_b \in P_{\alpha_n} \}. \]

Then \( M \), which is a subspace of the product space \( \prod_{i \in \mathbb{N}} B_i \), is a metric space. Since \( X \) is \( T_1 \) and regular, \( x_b \) is unique for every \( b \in M \). We define \( f : M \to X \) by \( f(b) = x_b \) for every \( b \in M \).

(a) \( f \) is onto.

For each \( x \in X \), there exists \( n \in \mathbb{N} \) and \( \alpha_n \in A_n \) such that \( x \in P_{\alpha_n} \). Since \( \mathcal{P}_{\alpha_n} \) is a countable \( cs \)-network for \( P_{\alpha_n} \), \( (\mathcal{P}_{\alpha_n})_x \) is a countable network at \( x \) in \( P_{\alpha_n} \). We may assume that \( (\mathcal{P}_{\alpha_n})_x = \{ P_{x,j} : j \in \mathbb{N} \} \), where \( P_{x,j} \in Q_{\alpha_{n,i}(j)} \) with some \( i(j) \in \mathbb{N} \) satisfying \( i(j) < i(j + 1) \). For each \( i \in \mathbb{N} \), take \( Q_{\beta_i} \) as follows.
(i) $i < n$: $Q_{\beta_i} = X$,
(ii) $i \geq n$: $Q_{\beta_i} = P_{\alpha, j}$ if $i = i(j)$ for some $j \in \mathbb{N}$, and otherwise, $Q_{\beta_i} = P_{\alpha}$. Then $\{Q_{\beta_i} : i \geq n\} \subseteq \{P_{\alpha}\}$. Therefore, $\{Q_{\beta_i} : i \geq n\}$ forms a network at $x = f(b)$, i.e., $f$ is onto.

(b) $f$ is continuous.

For each $b = (\beta_i) \in M$ and $x = f(b) \in U$ open in $X$. Then $x = f(b) \in Q_{\beta_k}$ for some $k \in \mathbb{N}$. Let $U_b = \{c = (\gamma_i) \in M : \gamma_k = \beta_k\}$. Then $U_b$ is open in $M$, and $b \in U_b$. For each $c \in U_b$, we find $f(c) \in Q_{\gamma_k} \subseteq U_b$. It implies that $f(U_b) \subseteq U$, i.e., $f$ is continuous.

(c) $M$ is locally separable.

Let $b = (\beta_i) \in M$. Then there exists $n \in \mathbb{N}$ and $\alpha_n \in A_n$ such that $Q_{\beta_i} = X$ if $i < n$, $Q_{\beta_i} \subseteq Q_{\alpha_n,i}$ if $i \geq n$, and $\{Q_{\beta_i} : i \geq n\}$ forms a network at a point $x_b$ in $P_{\alpha_n}$. Set $M_b = \{c = (\gamma_i) \in M : \gamma_n = \alpha_n\}$. Then $M_b$ is open in $M$, and $b \in M_b$. For each $c = (\gamma_i) \in M_b$, there exists $m \in \mathbb{N}$ and $\alpha_m \in A_m$ such that $Q_{\gamma_i} = X$ if $i < m$, $Q_{\gamma_i} \subseteq Q_{\alpha_m,i}$ if $i \geq m$, and $\{Q_{\gamma_i} : i \geq m\}$ forms a network at a point $x_c$ in $P_{\alpha_m}$. It follows from $Q_{\gamma_i} = Q_{\beta_i}$ that $P_{\alpha_n} \cap P_{\alpha_m} \neq \emptyset$. Since $P_{\alpha_n}$ is an $\aleph_0$-space and $P_{\alpha_m}$ is locally finite, $C_m = \{\alpha_m \in A_m : P_{\alpha_m} \cap P_{\alpha_n} \neq \emptyset\}$ is countable for every $m \in \mathbb{N}$. Then $E_i = \{(\beta_i) \cup \left(\bigcup\{B_{\alpha_{j,i}} : \alpha_j \in C_j, j \leq i\}\right)\}$ is countable. It implies that $\{\beta_i\} \times \cdots \times \{\beta_{n-1}\} \times \prod_{i \geq n} E_i$, $M_b$ is separable. Therefore, $M$ is locally separable.

(d) $f$ is an mssc-mapping.

For each $x \in X$ and each $i \in \mathbb{N}$, since $P_i$ is locally finite, there exists an open neighborhood $V_{x,i}$ of $x$ in $X$ such that $D_i = \{\alpha_i \in A_i : P_{\alpha_i} \cap V_{x,i} \neq \emptyset\}$ is finite. Then $E_i = \{(\beta_i) \cup \left(\bigcup\{B_{\alpha_{j,i}} : \alpha_j \in D_j, j \leq i\}\right)\}$ is finite. Since $p_i(f^{-1}(V_{x,i})) \subseteq F_i$, $p_i(f^{-1}(V_{x,i}))$ is compact. It implies that $f$ is an mssc-mapping.

(e) $f$ is sequence-covering.

For each convergent sequence $S$ in $X$, since $P$ is a $\sigma$-locally finite cs-network for $X$, there exists $n \in \mathbb{N}$ and $\alpha_n \in A_n$ such that $S$ is eventually in $P_{\alpha_n} \subseteq P_n$. Then $L_{\alpha_n} = S \cap P_{\alpha_n}$ is a convergent sequence in $P_{\alpha_n}$. For each $i \geq n$, we find that $\bigcup\{Q_{\alpha_{n,i}} : i \geq n\}$ is a $\sigma$-locally finite cs-network for $P_{\alpha_n}$ satisfying $P_{\alpha_n} \subseteq Q_{\alpha_{n,i}} \subseteq Q_{\alpha_{n,i+1}}$. It follows from the proof (3)$\Rightarrow$(2) of [13] Theorem 5.1] that there exists a convergent sequence $H_{\alpha_n}$ in $M_{\alpha_n}$ such that $f_{\alpha_n}(H_{\alpha_n}) = L_{\alpha_n}$, where

$$M_{\alpha_n} = \{c = (\gamma_i)_{i \geq n} \in \prod_{i \geq n} B_{\alpha_{n,i}} : \{Q_{\gamma_i} : i \geq n\} \text{ forms a network at a point } x_c \text{ in } P_{\alpha_n}\},$$

and $f_{\alpha_n} : M_{\alpha_n} \to P_{\alpha_n}$ defined by $f_{\alpha_n}(c) = x_c$ for every $c \in M_{\alpha_n}$. For each $c = (\gamma_i)_{i \geq n} \in H_{\alpha_n}$, set $h_c = (\beta_i)_{i \in \mathbb{N}}$, where $Q_{\beta_i} = X$ if $i < n$ and $\beta_i = \gamma_i$ if $i \geq n$, and set $H = \{h_c : c \in H_{\alpha_n}\}$. Then $H$ is a convergent sequence in $M$ and $f(H) = L_{\alpha_n}$. Since $S$ is eventually in $P_{\alpha_n}$, $S - P_{\alpha_n}$ is finite. Then $S - P_{\alpha_n} = f(F)$ with some finite subset $F$ of $M$. Set $L = H \cup F$, then $L$ is a convergent sequence in $M$ satisfying $f(L) = S$. It implies that $f$ is sequence-covering. □
Remark 2.1. The argument for cs-networks in the proof (2) \( \Rightarrow (3) \) of Theorem 2.1 cannot apply to cfp-networks.

Corollary 2.1. The following are equivalent for a space \( X \).

1. \( X \) is a sequence-covering, quotient mssc-image of a locally separable metric space.
2. \( X \) is a sequential space having a \( \sigma \)-locally finite cs-network consisting of cosmic spaces.
3. \( X \) is a sequential space having a \( \sigma \)-locally finite cs-network consisting of \( \aleph_n \)-spaces.

Proof. (1) \( \Rightarrow (2) \). Since \( X \) is a quotient image of a locally separable metric space, \( X \) is a sequential space by [6] Proposition 1.2. Then \( X \) is a sequential space having a \( \sigma \)-locally finite cs-network consisting of cosmic spaces by Theorem 2.1. (2) \( \Rightarrow (3) \). As in the proof (2) \( \Rightarrow (3) \) of Theorem 2.1.

Next, we characterize 1-sequence-covering mssc-images of locally separable metric spaces by means of \( \sigma \)-locally finite sn-networks.

Theorem 2.2. The following are equivalent for a space \( X \).

1. \( X \) is an 1-sequence-covering mssc-image of a locally separable metric space.
2. \( X \) has a \( \sigma \)-locally finite sn-network consisting of cosmic spaces.
3. \( X \) has a \( \sigma \)-locally finite sn-network consisting of \( \aleph_n \)-second countable spaces.

Proof. (1) \( \Rightarrow (2) \). Let \( f : M \rightarrow X \) be an 1-sequence-covering mssc-mapping from a locally separable metric space \( M \) onto \( X \). For each \( x \in X \), let \( a_x \in f^{-1}(x) \) satisfying that whenever \( \{x_n : n \in \mathbb{N}\} \) is a sequence converging to \( x \) in \( X \) there exists a sequence \( \{a_n : n \in \mathbb{N}\} \) converging to \( a_x \) in \( M \) with each \( a_n \in f^{-1}(x_n) \). By using notations in the proof (1) \( \Rightarrow (2) \) of Theorem 2.1, again, let \( Q_x = \{P \in \mathcal{P} : P = f(B) \text{ with } a_x \in B \in \mathcal{B}\} \), and let \( Q = \bigcup \{Q_x : x \in X\} \). We shall prove that \( Q \) is a \( \sigma \)-locally finite sn-network for \( X \) consisting of cosmic spaces by the following facts (a), (b), (c) for every \( x \in X \), and (d), (e).

(a) \( Q_x \) is a network at \( x \) in \( X \).

It is clear that \( x \in \bigcap Q_x \). Let \( x \in U \) with \( U \) open in \( X \), then \( x \in f^{-1}(U) \).

(b) If \( Q_1, Q_2 \in Q_x \), then \( Q \subset Q_1 \cap Q_2 \) for some \( Q \in Q_x \).

Set \( Q_1 = f(B_1), Q_2 = f(B_2) \), where \( B_1, B_2 \in B \) with \( a_x \in B_1 \) and \( a_x \in B_2 \).

(c) Since \( B \) is a base for \( M \), \( a_x \in B \in f^{-1}(U) \) for some \( B \in B \). Set \( Q = f(B) \), then \( Q \in Q_x \) and \( Q \subset Q_1 \cap Q_2 \).
(c) Each $Q \in \mathcal{Q}_x$ is a sequential neighborhood of $x$.

Set $Q = f(B)$ with $a_x \in B \in \mathcal{B}$. For each convergent sequence $S$ converging to $x$, there exists a convergent sequence $L$ converging to $a_x$ in $M$ such that $f(L) = S$. Since $L$ is eventually in $B$, $S$ is eventually in $Q$. It implies that $Q$ is a sequential neighborhood of $x$.

(d) $\mathcal{Q}$ is $\sigma$-locally finite.

Since $Q \subset \mathcal{P}$ and $\mathcal{P}$ is $\sigma$-locally finite, $\mathcal{Q}$ is $\sigma$-locally finite.

(e) Each $Q \in \mathcal{Q}$ is a cosmic space.

Set $Q = f(B)$ for some $B \in \mathcal{B}$. Since $B$ is separable, $Q$ is cosmic.

(2) $\Rightarrow$ (3). As in the proof (2) $\Rightarrow$ (3) of Theorem 2.1

(3) $\Rightarrow$ (1). Let $\mathcal{P} = \bigcup \{P_n : n \in \mathbb{N}\}$ be a $\sigma$-locally finite sn-network for $X$ consisting of $\aleph_0$-spaces. By using notations and arguments in the proof (3) of Theorem 2.1 again, since each sn-network is also a cs-network, it suffices to prove that the mapping $f$ is 1-sequence-covering.

For each $x \in X$, since $\mathcal{P}$ is a $\sigma$-locally finite sn-network for $X$, there exists $n \in \mathbb{N}$ and $\alpha_n \in A_n$ such that $P_{\alpha_n}$ is a sequential neighborhood of $x$. Then $\bigcup \{Q_{\alpha_n,i} : i \geq n\}$ is a $\sigma$-locally finite sn-network for $P_{\alpha_n}$. It implies that $f_{\alpha_n}$ is 1-sequence-covering by [13] Theorem 2.1. Hence, there exists $c_x = (\gamma_{x,i})_{i \geq n} \in f_{\alpha_n}^{-1}(x)$ such that whenever $\{x_m : m \in \mathbb{N}\}$ is a sequence converging to $x$ in $P_{\alpha_n}$, there exists a sequence $\{c_m : m \in \mathbb{N}\}$ converging to $c_x$ in $M_{\alpha_n}$ with each $c_m \in f_{\alpha_n}^{-1}(x_m)$. Set $b_x = (\beta_{x,i})$, where $Q_{\beta_{x,i}} \supset x$ if $i < n$ and $\beta_{x,i} = \gamma_{x,i}$ if $i \geq n$, then $b_x \in f^{-1}(x)$. Let $\{y_m : m \in \mathbb{N}\}$ be a sequence in $X$ converging to $x$. Since $P_{\alpha_n}$ is a sequential neighborhood of $x$, there exists $m_0 \in \mathbb{N}$ such that $\{y_m : m \geq m_0\} \subset P_{\alpha_n}$ is a sequence converging to $x$ in $P_{\alpha_n}$. Then there exists a sequence $\{c_m : m \geq m_0\}$ in $M_{\alpha_n}$ converging to $c_x$ and $c_m \in f_{\alpha_n}^{-1}(y_m)$ for each $m \geq m_0$. For each $c_m = (\gamma_{m,i})_{i \geq n}$, set $b_m = (\beta_{m,i})$, where $Q_{\beta_{m,i}} \supset x$ if $i < n$ and $\beta_{m,i} = \gamma_{m,i}$ if $i \geq n$. Then $b_m \in M$ and $f(b_m) = y_m$ for each $m \geq m_0$. For each $m < m_0$, take some $b_m \in f^{-1}(y_m)$. Then $\{b_m : m \in \mathbb{N}\}$ is a sequence in $M$ converging to $b_x$ and $b_m \in f^{-1}(y_m)$ for each $m \in \mathbb{N}$. It implies that $f$ is 1-sequence-covering.

□

Corollary 2.2. The following are equivalent for a space $X$.

(1) $X$ is an 1-sequence-covering, quotient mssc-image of a locally separable metric space.

(2) $X$ has a $\sigma$-locally finite weak base consisting of cosmic spaces.

(3) $X$ has a $\sigma$-locally finite weak base consisting of sn-second countable spaces.

Proof. (1) $\Rightarrow$ (2). Since $X$ is a quotient image of a locally separable metric space, $X$ is a sequential space by [6] Proposition 1.2. Then $X$ is a sequential space having a $\sigma$-locally finite sn-network $\mathcal{P}$ consisting of cosmic spaces by Theorem 2.2. It follows from Remark [11] that $\mathcal{P}$ is a weak base for $X$. Therefore, $X$ has a $\sigma$-locally finite weak base consisting of cosmic spaces.

(2) $\Rightarrow$ (3). Since $X$ has a $\sigma$-locally weak base, $X$ is a sequential space. It follows from Theorem 2.2 that $X$ is a sequential space having a $\sigma$-locally finite sn-network $\mathcal{P}$ consisting of sn-second countable spaces. By Remark [11] $\mathcal{P}$ is a weak base for $X$. It implies that $X$ has a $\sigma$-locally finite weak base consisting of sn-second countable spaces.
It follows from Theorem 2.2 that $X$ is an $1$-sequence-covering mssc-image of a locally separable metric space under some mapping $f$. Since $X$ has a $\sigma$-locally finite weak base, $X$ is a sequential space. Then $f$ is an $1$-sequence-covering mapping onto a sequential space, and so $f$ is a quotient mapping by [17, Lemma 3.5]. It implies that $X$ is an $1$-sequence-covering, quotient mssc-image of a locally separable metric space.

\textbf{Remark 2.2.} We can replace "cosmic spaces" in Theorem 2.2 and Corollary 2.2 by "$\aleph_0$-spaces".

In the following, we characterize $2$-sequence-covering mssc-images of locally separable metric spaces by means of $\sigma$-locally finite so-networks.

\textbf{Theorem 2.3.} The following are equivalent for a space $X$.

1. $X$ is a $2$-sequence-covering mssc-image of a locally separable metric space.
2. $X$ has a $\sigma$-locally finite so-network consisting of cosmic spaces.
3. $X$ has a $\sigma$-locally finite so-network consisting of $\sigma$-second countable spaces.

\textbf{Proof.} \([1] \Rightarrow [2]\). Let $f : M \to X$ be a $2$-sequence-covering mssc-mapping from a locally separable metric space $M$ onto $X$. For each $x \in X$, by using notations in the proof of Theorem 2.1 again, let $\mathcal{B}_x = \{B \in \mathcal{B} : f^{-1}(x) \cap B \neq \emptyset\}$, and let $\mathcal{R}_x$ be the family of all finite intersections of members of $f(\mathcal{B}_x)$. We shall prove that $\mathcal{R} = \bigcup \{\mathcal{R}_x : x \in X\}$ is a $\sigma$-locally finite so-network for $X$ consisting of cosmic spaces by the following facts (a), (b), (c) for every $x \in X$ and (d), (e).

(a) $\mathcal{R}_x$ is a network at $x$ in $X$.
(b) This is obvious because $\mathcal{B}_x$ is a base for $f^{-1}(x)$.
(c) If $R_1, R_2 \in \mathcal{R}_x$, then $R \subseteq R_1 \cap R_2$ for some $R \in \mathcal{R}_x$.
(d) Each $R \in \mathcal{R}_x$ is sequentially open.
(e) Let $B \in \mathcal{B}_x$, $y \in f(B)$, and $S$ be a convergent sequence converging to $y$. Since $y \in f(B)$, $f^{-1}(y) \cap B \neq \emptyset$. Take some $a_y \in f^{-1}(y) \cap B$. Then there exists a convergent sequence $L$ converging to $a_y$ in $M$ such that $f(L) = S$. Since $L$ is eventually in $B$, $S$ is eventually in $f(B)$. It implies that $f(B)$ is sequentially open, i.e., every member of $f(\mathcal{B}_x)$ is sequentially open. Because $R$ is some finite intersection of members of $f(\mathcal{B}_x)$, we find that $R$ is sequentially open.
(f) $\mathcal{R}$ is $\sigma$-locally finite.
(g) Since $\bigcup \{f(\mathcal{B}_x) : x \in X\} \subseteq \mathcal{P}$ and $\mathcal{P}$ is $\sigma$-locally finite, $\bigcup \{f(\mathcal{B}_x) : x \in X\}$ is $\sigma$-locally finite. It implies that $\mathcal{R}$ is $\sigma$-locally finite.
(h) Each $R \in \mathcal{R}$ is a cosmic space.
(i) For each $B \in \mathcal{B}_x$, since $B$ is separable, $f(B)$ is cosmic, i.e., every member of $f(\mathcal{B}_x)$ is cosmic. It implies that $R$ is cosmic.

($[2] \Rightarrow [3]$). As in the proof of Theorem 2.1 again, since each so-network is also a $cs$-network, it suffices to prove that the mapping $f$ is $2$-sequence-covering.
For each \( x \in X \) and each \( b_x \in f^{-1}(x) \), let \( b_x = (\beta_{x,i}) \). Then there exists some \( n \in \mathbb{N} \) and \( \alpha_n \in A_n \) such that \( Q_{\beta_{x,i}} = X \) if \( i < n \), \( Q_{\beta_{x,i}} \in Q_{\alpha_n,i} \) if \( i \geq n \), and \( \{Q_{\beta_{x,i}} : i \geq n\} \) forms a network at \( x \) in \( P_{\alpha_n} \). Set \( c_x = (\beta_{x,i})_{i \geq n} \), then \( c_x \in f_{\alpha_n}^{-1}(x) \). Since \( \{Q_{\alpha_n,i} : i \geq n\} \) is a \( \sigma \)-locally finite so-network for \( P_{\alpha_n} \), \( f_{\alpha_n} \) is a 2-sequence-covering by [13] Theorem 3.1. Let \( \{x_m : m \in \mathbb{N}\} \) be a sequence converging to \( x \) in \( X \). Since \( P_{\alpha_n} \) is sequentially open, there exists \( n_0 \in \mathbb{N} \) such that \( \{x_m : m \geq n_0\} \) is a sequence converging to \( x \) in \( P_{\alpha_n} \). Then there exists a sequence \( \{c_m : m \geq m_0\} \) in \( M_{\alpha_n} \) converging to \( c_x \) and \( c_m \in f_{\alpha_n}^{-1}(x_m) \) for each \( m \geq m_0 \). For each \( c_m = (\gamma_{m,i})_{i \geq n}, \) set \( b_m = (\beta_{m,i}) \), where \( Q_{\beta_{m,i}} = X \) if \( i < n \), and \( \beta_{m,i} = \gamma_{m,i} \) if \( i \geq n \). Then \( b_m \in M \) and \( f(b_m) = x_m \) for each \( m \geq m_0 \). For each \( m < m_0 \), take some \( b_m \in f^{-1}(x_m) \). Then \( \{b_m : m \in \mathbb{N}\} \) is a sequence in \( M \) converging to \( b_x \) and \( b_m \in f^{-1}(x_m) \) for each \( m \in \mathbb{N} \). It implies that \( f \) is 2-sequence-covering.

**Corollary 2.3.** The following are equivalent for a space \( X \).

1. \( X \) is a 2-sequence-covering, quotient mssc-image of a locally separable metric space.
2. \( X \) has a \( \sigma \)-locally finite base consisting of cosmic spaces.
3. \( X \) has a \( \sigma \)-locally finite base consisting of second countable spaces.

**Proof.** \([1] \Rightarrow [2] \). Since \( X \) is a quotient image of a locally separable metric space, \( X \) is a sequential space by [6] Proposition 1.2. It follows from Theorem 2.3 that \( X \) is a sequential space having a \( \sigma \)-locally finite so-network \( \mathcal{P} \) consisting of cosmic spaces. For each \( P \in \mathcal{P} \), since \( X \) is sequential and \( P \) is sequential open, \( P \) is open in \( X \). Hence \( \mathcal{P} \) is a \( \sigma \)-locally finite base for \( X \) consisting of cosmic spaces.

\([2] \Rightarrow [3] \). It follows from Theorem 2.3 that \( X \) has a \( \sigma \)-locally finite so-network \( \mathcal{P} \) consisting of so-second countable spaces. Since \( X \) has a \( \sigma \)-locally finite base, \( X \) is sequential. It implies that every \( P \in \mathcal{P} \) is open. Then \( \mathcal{P} \) is a \( \sigma \)-locally finite base consisting of so-second countable spaces.

Let \( P \in \mathcal{P} \) and \( Q \) be a countable so-network for \( P \). Since \( P \) is open, \( P \) is a sequential space by [6] Proposition 1.9. Then every \( Q \in \mathcal{Q} \) is open in \( P \). Hence \( \mathcal{Q} \) is a countable base for \( P \). It implies that \( P \) is a second countable space.

By the above, \( X \) has a \( \sigma \)-locally finite base consisting of second countable spaces.

\([3] \Rightarrow [1] \). It follows from Theorem 2.3 that \( X \) is a 2-sequence-covering mssc-image of a locally separable metric space under some mapping \( f \). Since \( X \) has a \( \sigma \)-locally finite base, \( X \) is sequential. Then \( f \) is a 2-sequence-covering mapping onto a sequential space, and so \( f \) is a quotient mapping by [17] Lemma 3.5. It implies that \( X \) is a 2-sequence-covering, quotient mssc-image of a locally separable metric space.

**Remark 2.3.** We can replace “cosmic spaces” in Theorem 2.3 and Corollary 2.3 by “\( R_0 \)-spaces”, or “\( sn \)-second countable spaces”.

**References**


Mathematics Faculty
Dongthap University
Caolanh City
Dongthap Province
Vietnam
nguyendungtc@yahoo.com
nvdung@staff.dthu.edu.vn