CHARACTERIZATION OF
THE PSEUDO-SYMMETRIES
OF IDEAL WINTGEN SUBMANIFOLDS
OF DIMENSION 3

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Abstract. Recently, Choi and Lu proved that the Wintgen inequality $\rho \leq H^2 - \rho^\perp + k$, (where $\rho$ is the normalized scalar curvature and $H^2$, respectively $\rho^\perp$, are the squared mean curvature and the normalized scalar curvature) holds on any 3-dimensional submanifold $M^3$ with arbitrary codimension $m$ in any real space form $\tilde{M}^{3+m}(k)$ of curvature $k$. For a given Riemannian manifold $M^3$, this inequality can be interpreted as follows: for all possible isometric immersions of $M^3$ in space forms $\tilde{M}^{3+m}(k)$, the value of the intrinsic curvature $\rho$ of $M$ puts a lower bound to all possible values of the extrinsic curvature $H^2 - \rho^\perp + k$ that $M$ in any case can not avoid to “undergo” as a submanifold of $\tilde{M}$. From this point of view, $M$ is called a Wintgen ideal submanifold of $\tilde{M}$ when this extrinsic curvature $H^2 - \rho^\perp + k$ actually assumes its theoretically smallest possible value, as given by its intrinsic curvature $\rho$, at all points of $M$. We show that the pseudo-symmetry or, equivalently, the property to be quasi-Einstein of such 3-dimensional Wintgen ideal submanifolds $M^3$ of $\tilde{M}^{3+m}(k)$ can be characterized in terms of the intrinsic minimal values of the Ricci curvatures and of the Riemannian sectional curvatures of $M$ and of the extrinsic notions of the umbilicity, the minimality and the pseudo-umbilicity of $M$ in $\tilde{M}$.

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1. Wintgen ideal submanifolds

For surfaces $M^2$ in the Euclidean space $E^3$, the Euler inequality $K \leq H^2$, where $K$ is the (intrinsic) Gauss curvature of $M^2$ and $H^2$ is the (extrinsic) squared mean curvature of $M^2$ in $E^3$, at once follows from the fact that $K = k_1k_2$ and $H = \frac{1}{2}(k_1 + k_2)$ where $k_1$ and $k_2$ are the principal curvatures of $M^2$ in $E^3$, and, obviously, $K = H^2$ everywhere on $M^2$ if and only if the surface $M^2$ is totally umbilical in $E^3$, i.e., $k_1 = k_2$ at all points of $M^2$, or still, by a theorem of Meusnier, if and only if $M^2$ is a part of a plane $E^2$ or of a round sphere $S^2$ in $E^3$.

For surfaces $M^2$ in the 4-dimensional Euclidean space $E^4$, Wintgen proved that the Gauss curvature $K$ and the squared mean curvature $H^2$ and the (extrinsic) normal curvature $K^\perp$ always satisfy the inequality $K \leq H^2 - K^\perp$, and that actually the equality holds if and only if the curvature ellipse of $M^2$ in $E^4$ is a circle [36]; (cf. e.g. [6, 7] for studies also on the global differential geometry of submanifolds by a.o. Smale, Lashof, Chern, Chen and Willmore concerning the Euler characteristic of the normal bundle, the number of self-intersections and the total mean curvature). This fundamental inequality between the most important intrinsic and extrinsic scalar valued curvatures of surfaces $M^2$ in $E^4$ was later shown, by Rouxel and by Guadalupe and Rodriguez, to hold more generally for all surfaces $M^2$ of arbitrary codimension $m$ in the real space forms $\tilde{M}^{2+m}(k)$ of constant sectional curvature $k$, inclusive the characterisation of the equality case [21, 29]. After these extensions of the above Wintgen inequality for submanifolds of dimension 2 and of codimension 2 to submanifolds of dimension 2 and arbitrary codimension $m \geq 2$, in 1999 De Smet and Dillen and Vanransen and one of the authors proved the Wintgen inequality $\rho \leq H^2 - \rho^\perp + k$, where $\rho$ and $\rho^\perp$ respectively are the (intrinsic) normalized scalar curvature and the (extrinsic) normalized scalar normal curvature, for 2-codimensional submanifolds $M^n$ of arbitrary dimension $n \geq 2$ in the real space forms $\tilde{M}^{n+2}(k)$, and characterized the equality situation explicitly in terms of the shape operators of $M^n$ in $\tilde{M}^{n+2}(k)$ [12]. Moreover, in [12] it was conjectured that this Wintgen inequality holds for submanifolds $M^n$ of any dimension $n \geq 2$ and of any codimension $m \geq 2$ in real space forms $\tilde{M}^{n+m}(k)$, (referring to the initials of the authors of [12], Suceava recently started to call this “the DBVV conjecture” [31], and was therein followed by others, although the “conjecture on Wintgen’s inequality” may well be a more appropriate terminology). Recently, Choi and Lu proved that this conjecture is true for all 3-dimensional submanifolds $M^3$ of arbitrary codimension $m \geq 2$ in $\tilde{M}^{3+m}(k)$ and obtained characteristic expressions for the shape operators of the submanifolds $M^3$ in $\tilde{M}^{3+m}(k)$ which do realize the equality in this general inequality [8]. Concrete descriptions of some classes of 3-dimensional Wintgen ideal submanifolds were given by Bryant, Dillen, Fastenakels and Van der Veken [4, 18].

At this stage we would like further finally to mention that De Smet, Dillen, Fastenakels, Van der Veken and one of the authors studied the Wintgen inequality for invariant submanifolds in Kaehler, nearly Kaehler and Sasakian spaces [11, 17], and that Gmira, Haesen, Dillen and two of the authors studied this inequality for submanifolds in semi-Riemannian spaces [19, 20].
2. Pseudo-symmetric spaces

Let $M^n$ be an $n$-dimensional Riemannian manifold with metric $(0,2)$ tensor $g$ and Levi-Civita connection $\nabla$. Let $R$ denote the $(0,4)$ Riemann–Christoffel curvature tensor of $M$ as well as the curvature $(1,1)$ operator $R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$, thus having
\[
R(X,Y,Z,W) := g(R(X,Y)Z,W),
\]
where $X,Y$, etc. denote arbitrary vector fields on $M$ and $[\ldots]$ stands for the Lie bracket. By the action of the curvature operator working as a derivation on the curvature tensor $R$, the following $(0,6)$ tensor $R \cdot R$ is obtained:
\[
(R \cdot R)(X_1, X_2, X_3, X_4; X,Y) := (R(X,Y) \cdot R)(X_1, X_2, X_3, X_4)
= -R(R(X,Y)X_1, X_2, X_3, X_4) - R(X_1, R(X,Y)X_2, X_3, X_4)
- R(X_1, X_2, R(X,Y)X_3, X_4) - R(X_1, X_2, X_3, R(X,Y)X_4).
\]

As was recently shown by Haesen and one of the authors [22], the tensor $R \cdot R$ can be geometrically interpreted as giving the second-order measure of the change of the sectional curvatures $K(p, \pi)$ for tangent 2-planes $\pi$ at points $p$ after the parallel transport of $\pi$ around infinitesimal co-ordinate parallelograms in $M$ cornered at $p$.

Thus, the semi-symmetric or Szabó symmetric spaces [32, 33], i.e., the manifolds $M$ for which $R \cdot R = 0$, are those Riemannian manifolds for which all sectional curvatures remain preserved after the parallel transport of their planes around all infinitesimal co-ordinate parallelograms. The locally symmetric or Cartan symmetric spaces, i.e., the manifolds $M$ for which $\nabla R = 0$, constitute a proper subclass of the class of the Szabó symmetric spaces.

We recall that the definition (1) of the curvature tensor goes back to Schouten’s geometrical interpretation of $R$ as the second order measure of the change of the direction of vector fields after their parallel transport around closed infinitesimal curves on $M$ [30]. Then the locally flat or locally Euclidean spaces, thus the manifolds $M$ for which $R = 0$, are those Riemannian manifolds for which all directions remain preserved after parallel transport around all closed infinitesimal curves. The simplest nonflat Riemannian manifolds $M$ are the spaces of constant curvature $K = k$, i.e., the spaces whose function $K$ is isotropic (meaning that, at each point $p$, the Gauss curvature $K(p, \pi)$ at $p$ of the local surface formed by the geodesics of $M$ which pass through $p$ and whose tangent vector at $p$ lies in $\pi$, has the same value for all choices of planes $\pi$ at $p$, thus $K$ becoming a real function on $M$, which by the lemma of Schur, for $n > 2$, then necessarily has to be constant). These real space forms $M^n(k)$, by a theorem of Beltrami, can be obtained from the locally Euclidean spaces by projective transformations and their class is closed under such transformations. Further, we also recall that the knowledge of the curvature tensor $R$ is equivalent to the knowledge of the sectional or Riemannian curvatures $K$, as was shown by Cartan. Finally, as is well known, the curvature tensor $R$ of a space of constant curvature $k$ is given by
\[
R(X,Y,Z,W) = k g((X \wedge_g Y)Z,W),
\]
where the $\wedge_g$ stands for the **metrical endomorphism** $(X \wedge_g Y)Z := g(Y,Z)X - g(X,Z)Y$. Thus for the real space forms $M^n(k)$, $n > 2$, there exists a real valued function $K$ on $M$ such that $R(X,Y,Z,W) = KG(X,Y,Z,W)$, where the $(0,4)$-tensor $G$ is defined by $G(X,Y,Z,W) := g((X \wedge_g Y)Z,W)$.

A main interest of Riemann, Helmholtz, Lie, Klein, . . . in the spaces of constant curvature was related to the fact that these are precisely the Riemannian manifolds which satisfy the **axiom of free mobility**.

Now, similarly as proceeding from the locally Euclidean spaces to the real space forms, one can proceed from the Szabo symmetric spaces to the **pseudo-symmetric** or **Deszcz symmetric spaces** [1, 13, 22, 35]. The pseudo-symmetric spaces were defined as the manifolds $M$ for which the $(0,6)$ tensor $R \cdot R$ and the $(0,6)$ Tachibana tensor $Q(g,R) := - \wedge_g \cdot R$, where the metrical endomorphism $\wedge_g$ acts on the $(0,4)$ curvature tensor $R$ as a derivation, are proportional, say $R \cdot R = L(- \wedge_g \cdot R)$ for some real valued function $L$ on $M$:

$$Q(g,R)(X_1, X_2, X_3, X_4; X, Y) := -((X \wedge_g Y) \cdot R)(X_1, X_2, X_3, X_4)$$
$$= R((X \wedge_g Y)X_1, X_2, X_3, X_4) + R(X_1, (X \wedge_g Y)X_2, X_3, X_4)$$
$$+ R(X_1, X_2, (X \wedge_g Y)X_3, X_4) + R(X_1, X_2, X_3, (X \wedge_g Y)X_4).$$

A classical result states that the identical vanishing of this Tachibana tensor, $Q(g,R) = 0$, characterizes the real space forms. Further, results of Mikesh, Venzi, Defever and Deszcz learn that pseudo-symmetric spaces are obtained by applying **projective transformations** to the semi-symmetric spaces and that the class of the pseudo-symmetric spaces is closed under such transformations. Two 2-planes $\pi$ and $\bar{\pi}$, spanned by vectors $u, v$ and $x, y$ respectively, at the same point $p$ of $M$, are said to be **curvature dependent** if $Q(g,R)(u, v, v, u; x, y) \neq 0$, which is independent of the choices of bases for $\pi$ and $\bar{\pi}$. For such planes, the **double sectional curvature** or the **sectional curvature of Deszcz** or the **Riemann curvature of Deszcz** $L(p, \pi, \bar{\pi})$ is defined as the real number given by

$$L(p, \pi, \bar{\pi}) := \frac{(R \cdot R)(u, v, v, u; x, y)}{Q(g,R)(u, v, v, u; x, y)},$$

(which is independent of the choices of bases for $\pi$ and $\bar{\pi}$): it is a scalar valued Riemannian invariant. The knowledge of the tensor $R \cdot R$ is equivalent to the knowledge of the sectional curvatures $L(p, \pi, \bar{\pi})$ of Deszcz. And just like the geometrical interpretation of the sectional curvatures $K(p, \pi)$ of Riemann in terms of the parallelogramoids of Levi-Civita [27], also the sectional curvatures $L(p, \pi, \bar{\pi})$ of Deszcz can be interpreted in these terms (in this respect, we refer to [23] and [24] where in particular such interpretations are obtained for the sectional curvatures as well as for the **Ricci** and **conformal Weyl curvatures of Deszcz** in terms of the squaroïds of Levi-Civita). Finally the Deszcz symmetric spaces are characterized by the **isotropy** of the curvatures $L(p, \pi, \bar{\pi})$, i.e., by the property that at every point $p$ of $M$ the scalars $L(p, \pi, \bar{\pi})$ are the same for all possible pairs of curvature dependent tangent planes $\pi$ and $\bar{\pi}$ at $p$. In the present situation however there is no lemma of Schur, which then would further force this real valued function.
Let \( M \to R \) automatically to be constant; therefore, Kowalski and Sekizawa called the pseudo-symmetric spaces for which the double sectional curvature \( L \) is indeed a constant, independent of the planes \( \pi \) and \( \tilde{\pi} \) as well as of the points \( p \) of \( M \), the pseudo-symmetric spaces of constant type \( L \) [26]. By way of examples in this respect we would like to mention here that the 3-dimensional Thurston geometries [34], which in a kind of axiomatic way originated as natural anisotropic extensions of the spaces of constant Riemannian curvature \( K \) with their typical free mobility, all do have constant sectional curvature \( L \) of Deszcz (we set \( L = 0 \) for \( E^4 \) since \( (K = c = 0) \), \( S^3 \) \( (K = c > 0) \), \( H^3 \) \( (K = c < 0) \), \( S^2 \times E^1 \) and \( H^2 \times E^1 \); \( L = 1 \) for \( SL(2, R) \) and for the 3-dimensional Heisenberg group \( H_3 \); and \( L = -1 \) for the Lie group Sol) [2].

A similar study concerning the geometrical meaning of Ricci pseudo-symmetry in the sense of Deszcz, i.e., of the manifolds \( M \) satisfying the curvature condition \( R \cdot S = L_S Q(g, S) = L_S (\Lambda \cdot S) \), where \( S \) denotes the \((0, 2)\) Ricci curvature tensor and \( Q(g, S) = -\Lambda \cdot S \) the Ricci–Tachibana tensor of \( M \) and \( L_S \) is a real valued function on \( M \), was carried out by Jahanara, Haesen and two of the authors in [25], (in this respect, see also [9] and [15]). As shown in [16], a 3-dimensional Riemannian manifold \( M \) is pseudo-symmetric if and only if it is quasi-Einstein, i.e., if its Ricci tensor \( S \) has an eigenvalue of multiplicity \( \geq 2 \). The class of the Riemannian manifolds \( M \) with pseudo-symmetric Ricci tensor \( S \) as such is considerably larger in general than the class of the manifolds \( M \) with pseudo-symmetric Riemann–Christoffel tensor \( R \), (which it obviously contains as a subclass). However, as shown in [10], for manifolds of dimension 3, these two pseudo-symmetry conditions are equivalent. As is well known, Schouten and Struik showed that the 3-dimensional Riemannian manifolds \( M \) are Einstein if and only if they have constant curvature \( K \). In [28] two of the authors made a study of the pseudo-symmetry in the sense of Deszcz of the tensors \( R \) and \( S \) of the Wintgen ideal submanifolds \( M^n \) of dimension \( n > 3 \) and of codimension 2 in the real space forms \( M^{n+2}(k) \). In particular, they showed that for those Wintgen ideal submanifolds these two, a priori distinct, curvature conditions are equivalent and occur if and only if those submanifolds are either totally umbilical or minimal. In comparison, the 3-dimensional case will show to offer two additional kinds of pseudo-symmetric Wintgen ideal submanifolds.

3. On the symmetry of ideal submanifolds

Let \( M^n \) be a submanifold of a real space form \( \tilde{M}^{n+m}(k) \) of constant curvature \( k \). Let \( g, \nabla \) and \( R \), and, respectively, \( \tilde{g}, \tilde{\nabla} \) and \( \tilde{R} \), denote the Riemannian metric, the Levi-Civita connection and the Riemann–Christoffel \((0, 4)\) curvature tensor of \( M \) and \( \tilde{M} \).

The formulae of Gauss and Weingarten then are
\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \text{and} \quad \tilde{\nabla}_X \xi = -A_{\xi} X + \nabla^\perp_X \xi,
\]
where \( h \), \( A_{\xi} \) and \( \nabla^\perp \) denote the second fundamental form, the shape operator or the Weingarten map with respect to \( \xi \) and the normal connection of \( M \) in \( \tilde{M} \), respectively, systematically using here and hereafter \( X, Y \), etc. for tangent vector.
fields on $M$ and $\xi$ etc. for normal vector fields on $M$ in $\tilde{M}$, (as basic references for Riemannian submanifolds, see [5] and [7]).

From (3) it follows that $\tilde{g}(h(X,Y),\xi) = g(A_{\xi}(X), Y)$, such that, for any orthonormal local normal frame $\xi_{\alpha}$ on $M$ in $\tilde{M}$, ($\alpha$ etc. running from 1 till the codimension $m$),

$$h(X,Y) = \sum_{\alpha} g(A_{\alpha}(X), Y) \xi_{\alpha},$$

where $A_{\alpha} = A_{\xi_{\alpha}}$. The mean curvature vector field $\vec{H}$ of $M$ in $\tilde{M}$ is defined as $\vec{H} = \frac{1}{n} \text{trace} h$ and its length $H = ||\vec{H}||$ is the mean curvature of $M$ in $\tilde{M}$. By the equation of Ricci, the normal curvature tensor $R^{\perp}$ of $M$ in $\tilde{M}$ is given as follows:

$$R^{\perp}(X,Y;\xi,\eta) := \tilde{g}(R^{\perp}(X,Y)\xi, \eta) = g([A_{\xi}, A_{\eta}](X), Y),$$

where $R^{\perp}(X,Y) := \nabla_{X}^{\perp} \nabla_{Y}^{\perp} - \nabla_{Y}^{\perp} \nabla_{X}^{\perp} - \nabla_{[X,Y]}^{\perp}$ and $[A_{\xi}, A_{\eta}] := A_{\xi} A_{\eta} - A_{\eta} A_{\xi}$.

The normalized scalar normal curvature $\rho^{\perp}$ of $M$ in $\tilde{M}$ is then defined by

$$\rho^{\perp} := \frac{2}{n(n-1)} \left\{ \sum_{i<j} \sum_{\alpha<\beta} \left[ R^{\perp}(E_{i}, E_{j}; \xi_{\alpha}, \xi_{\beta}) \right]^{2} \right\}^{1/2},$$

for any normal frame $\xi_{\alpha}$ and for any orthonormal local tangent frame $E_{i}$ on $M$, ($i$ etc. running from 1 till the dimension $n$).

We remark that $\rho^{\perp} = 0$ if and only if the normal connection is flat, which, as follows from (5) and as was already observed by Cartan, is equivalent to the simultaneous diagonalizability of all shape operators $A_{\xi}$. The equation of Gauss of $M$ in $\tilde{M}$ is given by

$$R(X,Y,Z,W) = \tilde{g}(h(Y,Z), h(X,W))) - \tilde{g}(h(X,Z), h(Y,W)) + k \left\{ g(Y,Z) g(X,W) - g(X,Z) g(Y,W) \right\}.$$

Let $S$ be the $(0,2)$-Ricci tensor of $M$: $S(X,Y) = \sum_{i} R(E_{i}, X, Y, E_{i})$. Then from (2), (4) and (6), we obtain that

$$S(X,Y) = (n-1)k g(X,Y) + \sum_{\alpha} \text{trace} A_{\alpha} g(A_{\alpha}(X), Y)$$

$$- \sum_{\alpha} \sum_{i} g(A_{\alpha}(X), E_{i}) g(A_{\alpha}(Y), E_{i}),$$

for any choice of frames $E_{i}$ and $\xi_{\alpha}$. And the normalized scalar curvature $\rho$ of $M$ is defined by

$$\rho := \frac{2}{n(n-1)} \sum_{i<j} R(E_{i}, E_{j}, E_{j}, E_{i}).$$

Choi and Lu gave the following affirmative solution of the conjecture concerning the inequality of Wintgen for the 3-dimensional submanifolds of the real space forms.
Theorem 1. [8] For any $M^3$ in $\tilde{M}^{3+m}(k)$, with $m \geq 3$:

\begin{equation}
\rho \leq H^2 - \rho^\perp + k,
\end{equation}

and the equality holds if and only if, with respect to suitably chosen local orthonormal tangent and normal frames $E_1, E_2, E_3$ and $\xi_1, \ldots, \xi_m$, the shape operators $A_\alpha$ of $M$ in $\tilde{M}$ take the forms:

\begin{equation}
A_1 = \begin{pmatrix}
c & \mu & 0 \\
\mu & c & 0 \\
0 & 0 & c
\end{pmatrix},
A_2 = \begin{pmatrix}
b + \mu & 0 & 0 \\
0 & b - \mu & 0 \\
0 & 0 & b
\end{pmatrix},
A_3 = \begin{pmatrix}
a & 0 & 0 \\
0 & a & 0 \\
0 & 0 & a
\end{pmatrix},
\end{equation}

\[ A_4 = \cdots = A_m = 0, \]

for some real valued functions $a, b, c$ and $\mu$ on $M$.

On the other hand, from the paper of De Smet, Dillen, Vrancken and one of the authors on Wintgen’s inequality, we have the following.

Theorem 2. [12] For any $M^3$ in $\tilde{M}^5(k)$, (8) is satisfied and the equality holds if and only if, with respect to suitably chosen local orthonormal frames $E_1, E_2, E_3$ and $\xi_1, \xi_2$, the shape operators $A_1$ and $A_2$ of $M$ in $\tilde{M}$ are given by

\[ A_1 = \begin{pmatrix}
c & \mu & 0 \\
\mu & c & 0 \\
0 & 0 & c
\end{pmatrix},
A_2 = \begin{pmatrix}
\mu & 0 & 0 \\
0 & -\mu & 0 \\
0 & 0 & 0
\end{pmatrix}. \]

So, for what follows and which concerns essentially dealing with the above shape operators filled into the Gauss equation of $M^3$ in $\tilde{M}^{3+m}(k)$, the latter situation ($m = 2$) is in some sense algebraically included in the former one ($m > 2$) by considering $a = b = 0$. And, in the case of codimension 1 there is no question about a normal curvature, we can carry out a general study of the 3-dimensional Wintgen ideal submanifolds $M^3$ in arbitrary space forms $\tilde{M}^{3+m}(k)$, $m \geq 2$, by dealing with the forms of the shape operators as given in (9). Frames $E_1, E_2, E_3$ and $\xi_1, \ldots, \xi_m$ for which the corresponding shape operators $A_\alpha$ assume such forms which further on will be called Choi–Lu frames of Wintgen ideal $M^3$ in $\tilde{M}^{3+m}$.

From (7) and (9), the $(1, 1)$ Ricci operator $S$ which is metrically related to the $(0, 2)$ Ricci tensor $S$ by $g(S(X), Y) = S(X, Y)$, with respect to Choi–Lu frames $E_i$ and $\xi_\alpha$ is readily found to be given by

\begin{equation}
S = \begin{pmatrix}
S_{11} & c\mu & 0 \\
c\mu & S_{22} & 0 \\
0 & 0 & S_{33}
\end{pmatrix},
\end{equation}

where

\begin{align}
S_{11} &= 2(a^2 + b^2 + c^2 + k) + \mu(b - 2\mu), \\
S_{22} &= 2(a^2 + b^2 + c^2 + k) - \mu(b + 2\mu), \\
S_{33} &= 2(a^2 + b^2 + c^2 + k).
\end{align}
Hence $E_3$ always determines a *Ricci principal direction*, with a corresponding *Ricci curvature*
(12) \[ \rho_3 = 2(a^2 + b^2 + c^2 + k). \]
And, from (10) and (11), the other two *Ricci curvatures*, $\rho_1$ and $\rho_2$, corresponding to some orthogonal eigendirections $\tilde{E}_1$ and $\tilde{E}_2$ in the plane field $E_1 \wedge E_2$, are easily derived as
(13) \[
\begin{align*}
\rho_1 &= \rho_3 - 2\mu^2 + |\mu|\sqrt{b^2 + c^2}, \\
\rho_2 &= \rho_3 - 2\mu^2 - |\mu|\sqrt{b^2 + c^2}.
\end{align*}
\]
Since $M^3$ is *pseudo-symmetric* or still, *symmetric in the sense of Deszcz*, if and only if its Ricci tensor has an eigenvalue of multiplicity $\geq 2$, from (13) we have the following.

**Lemma 1.** A Wintgen ideal 3-dimensional submanifold in a real space form is Deszcz symmetric if and only if

(I) $\mu = 0$ or
(II) $\mu \neq 0$, $b = c = 0$ or
(III) $\mu \neq 0$, $b^2 + c^2 = 4\mu^2$.

Proceeding more straightforwardly, from (6) and (9) the components, with respect to Choi–Lu frames, of the $(0,4)$ *curvature tensor* $R$ of a Wintgen ideal $M^3$ in $\tilde{M}^{3+m}(k)$ are readily found to be either zero or else to be completely determined, via the algebraic symmetries of $R$, by the following ones:

(14) \[
\begin{align*}
\alpha := K_{12} &= R_{1221} = a^2 + b^2 + c^2 - 2\mu^2 + k, \\
\beta := K_{13} &= R_{1331} = a^2 + b^2 + c^2 + b\mu + k, \\
\gamma := K_{23} &= R_{2332} = a^2 + b^2 + c^2 - b\mu + k, \\
\delta := K_{132} &= c\mu,
\end{align*}
\]

($K_{ij} = K(E_i \wedge E_j)$ are the *sectional curvatures* of $M$ for the 2-planes $E_i \wedge E_j$ determined by a Choi–Lu frame $E_1, E_2, E_3$). And then also the components of the $(0,6)$ tensors $R \cdot R$ and $\wedge_g \cdot R$ are readily computable, and turn out either to be zero “together” or, when at least a priori non-zero, they appear in pairs which are, via algebraic symmetries of both these $(0,6)$ tensors (cf. [22]), completely determined by the following ones:

(15) \[
\begin{align*}
(R \cdot R)(E_1, E_3, E_1, E_3; E_1, E_2) &= -2\alpha\delta, \\
(\wedge_g \cdot R)(E_1, E_3, E_1, E_3; E_1, E_2) &= -2\delta; \\
(R \cdot R)(E_1, E_3, E_2, E_3; E_1, E_2) &= \alpha(\beta - \gamma), \\
(\wedge_g \cdot R)(E_1, E_3, E_2, E_3; E_1, E_2) &= \beta - \gamma; \\
(R \cdot R)(E_1, E_2, E_2, E_3; E_1, E_3) &= -\alpha\beta + \beta\gamma - \delta^2, \\
(\wedge_g \cdot R)(E_1, E_2, E_2, E_3; E_1, E_3) &= \gamma - \alpha.
\end{align*}
\]

Then, by (14) and (15), the condition for the *pseudo-symmetry* of $R$, i.e., of the existence of a function $L$: $M \to R$ for which $R \cdot R = L(\wedge_g \cdot R)$, for Wintgen ideal submanifolds $M^3$ in $\tilde{M}^{3+m}(k)$, yields, of course, the previous cases (I, II, and III)
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of Lemma 1, but moreover in each case at the point \( p \) gives the double sectional curvature function \( L \) as stated in the following.

**Lemma 2.** We have

\[
L = 0 \quad \text{in case (I)}, \\
L = a^2 + k \quad \text{in case (II)}, \\
L = a^2 + 2\mu^2 + k \quad \text{in case (III)}.
\]

In case (I), \( M^3 \) is Einstein and thus a real space form and so, in particular, \( M^3 \) is then semi-symmetric and hence (also without calculations, one could know that) \( L = 0 \). In cases (II) and (III), according to the general theory in this respect, if \( M^3 \) is quasi-Einstein and has \( \lambda \) as an eigenvalue of the Ricci tensor \( S \) of multiplicity 1, then \( L = \frac{\lambda^2}{2} \), which allows to obtain Lemma 2 also from the consideration of (12) and (13).

Now we aim to geometrically characterize the cases (I), (II), and (III). Clearly, from (9), we see that (I) corresponds to the totally umbilical Wintgen ideal submanifolds \( M^3 \) in real space forms \( \tilde{M}^{3+m}(k) \); such \( M^3 \) are intrinsically itself spaces of constant curvature \( K = a^2 + b^2 + c^2 + k \). We recall that a submanifold \( M^n \) in a Riemannian manifold \( \tilde{M}^{n+m} \) is said to be pseudo-umbilical if its mean curvature vector field \( \vec{H} \) determines an umbilical normal direction on \( M \) in \( \tilde{M} \). When hereafter we call a submanifold pseudo-umbilical we mean it to be properly pseudo-umbilical, i.e., we exclude from it the trivial cases when it is minimal (\( \vec{H} = \vec{0} \)), or when it is totally umbilical (i.e., when every normal direction \( \xi \) is umbilical). The mean curvature vector field \( \vec{H} \) for the submanifolds \( M^3 \) under consideration being given by \( \vec{H} = c\xi_1 + b\xi_2 + a\xi_3 \), it further follows from (9) that the shape operator of \( M \) in \( \tilde{M} \) with respect to \( \vec{H} \) is given by

\[
A_{\vec{H}} = \begin{pmatrix}
  a^2 + b^2 + c^2 + b\mu \\
  c\mu \\
  0 \\
  a^2 + b^2 + c^2 - b\mu \\
  0 \\
  a^2 + b^2 + c^2 
\end{pmatrix}.
\]

In case \( M \) is not totally umbilical in \( \tilde{M} \), i.e., in case \( \mu \neq 0 \), and in case \( M \) is not minimal in \( \tilde{M} \), i.e., in case not \( a = b = c = 0 \) (cf. (9)), then (16) shows that \( \vec{H} \) determines an umbilical normal direction of \( M \) in \( \tilde{M} \) if and only if \( b = c = 0 \). In summary, from the above we know that cases (I) and (II) correspond to the Wintgen ideal submanifolds which are totally umbilical or minimal or pseudo-umbilical. Finally, we next aim for a geometrical characterisation of case (III): \( b^2 + c^2 = 4\mu^2 \) where \( \mu \neq 0 \). To simplify a bit the discussion, we will assume from now on that \( \mu > 0 \), which we can do without loss of generality, being always realizable in view of (9) by eventual changing orientations of \( \xi_\alpha \)'s and orderings of \( E_1 \) and \( E_2 \). From (12) and (13) we recall that the eigenvalues of the Ricci tensor
are given by
\[
\begin{align*}
\rho_1 &= 2(a^2 + b^2 + c^2 + k) - 2\mu^2 + \mu\sqrt{b^2 + c^2}, \\
\rho_2 &= 2(a^2 + b^2 + c^2 + k) - 2\mu^2 - \mu\sqrt{b^2 + c^2}, \\
\rho_3 &= 2(a^2 + b^2 + c^2 + k),
\end{align*}
\]
(17)
where \(\rho_3\) is the one in the \(E_3\)-direction of \(M\) and that \(\rho_1\) and \(\rho_2\) are the eigenvalues corresponding to certain eigendirections in the plane \(\pi = E_1 \wedge E_2\) perpendicular to \(E_3\) and of which the special character is reflected obviously, having \(\mu \neq 0\), in the form of the shape operators given in (9), and which we further will call the **Choi–Lu plane** of \(M^3\). Studying at present \(M^3\)'s which are not totally umbilical, in particular, these submanifolds are not Einstein, and so, at every point, they have a **smallest Ricci curvature**, which we’ll denote by \(\inf \text{Ric}\). From (17) it is clear this is \(\rho_2\):
\[
\inf \text{Ric} = 2(a^2 + b^2 + c^2 + k) - 2\mu^2 - \mu\sqrt{b^2 + c^2},
\]
attempted by a particular direction in the Choi–Lu plane of \(M^3\). On the other hand, we recall from (14) that the sectional curvature \(K_{\text{Choi–Lu}}\) of the Choi–Lu plane is given by
\[
K_{\text{Choi–Lu}} = K_{12} = a^2 + b^2 + c^2 + k - 2\mu^2.
\]
Hence \(\inf \text{Ric} = 2K_{\text{Choi–Lu}}\) if and only if \(-2\mu^2 - \mu\sqrt{b^2 + c^2} = -4\mu^2\), i.e., if and only if (III) holds, namely when \(b^2 + c^2 = 4\mu^2\). From (14) we moreover see that, in this case,
\[
\begin{align*}
K_{12} &= K_{\text{Choi–Lu}} = a^2 + 2\mu^2 + k, \\
K_{13} &= K_{\text{Choi–Lu}} + \mu(2\mu + b), \\
K_{23} &= K_{\text{Choi–Lu}} + \mu(2\mu - b),
\end{align*}
\]
which implies, obviously having also that \(b^2 \leq 4\mu^2\) and so, since \(\mu > 0\), that
\[
-2\mu \leq b \leq 2\mu,
\]
and thus that as well \(0 \leq 2\mu + b\) as \(0 \leq 2\mu - b\), together with the equation of Gauss and (9), that in case (III) the sectional curvature \(K_{12}\) actually equals \(\inf K\), the function on \(M\) giving the minimum of the sectional curvatures \(K\) at each point of \(M^3\). So, in this situation we can observe on the side that the \(\delta(2)\)-curvature of Chen [7], \(\delta(2) := \tau - \inf K\), where \(\tau := \sum_{i < j} K_{ij}\) is the scalar curvature of \(M^3\), is given by \(\delta(2) = \tau - K_{12} = K_{13} + K_{23} = \rho_3 = \rho_1\) which, in this case, is \(\sup \text{Ric}\), the real valued function on \(M\) giving the maximum Ricci curvature at each point of \(M^3\). Once more, in accordance with the general theory of Deszcz symmetric 3-dimensional Riemannian manifolds \(M^3\), for the properly quasi-Einstein manifolds \(M^3\) the sectional curvature \(L\) of Deszcz satisfies \(L = \frac{\lambda}{\tau}\), where \(\lambda\) is the principal curvature with multiplicity 1, so actually \(\lambda = \rho_2 = 2(a^2 + 2\mu^2 + k)\), such that
\[
L = K_{\text{Choi–Lu}} = \inf K.
\]
In this respect, coming back to case (II), from (14) and taking further into account (9) and the equation of Gauss, it follows that in this case, since \( K_{12} = a^2 + k - 2\mu^2 \),
\[
L = a^2 + k = K_{23} = K_{13} = \sup K,
\]
sup \( K \) denoting the maximum of the sectional curvature function on \( M \), i.e., the function on \( M \) which value at each point is the maximum of all the sectional curvatures of \( M \) at this point. Taking into account at last also the case (III) of Lemma 2, in summary we can formulate the following.

**Theorem 3.** A Wintgen ideal submanifold \( M^3 \) in a real space form \( \tilde{M}^{3+m}(k) \) is Deszcz symmetric if and only if (I) \( M^3 \) is a totally umbilical submanifold with Deszcz sectional curvature \( L = 0 \) (\( M^3 \) then being a space of constant sectional curvature \( K \)), or, (II) \( M^3 \) is a minimal submanifold or a pseudo-umbilical submanifold, with Deszcz sectional curvature \( L = \sup K \), or else, (III) \( M^3 \) is characterized by the curvature condition \( \inf K = 2(\sup K) = 2(\inf K) \), with Deszcz sectional curvature \( L = \inf K \).

4. Further comments and remarks

(1) We would like to refer again to the references mentioned in Section 1 for explicit descriptions of several examples of Wintgen ideal 3-dimensional submanifolds.

(2) Refering amongst others to Berger’s discussion in his “Panorama” [3] pertaining to the extremal values of the sectional curvature function \( K \) of Riemannian manifolds, we observe from Theorem 3 that for the nontrivial Wintgen ideal submanifolds \( M^3 \) in \( \tilde{M}^{3+m}(k) \), i.e., the nontotally umbilical ones, the isotropic Deszcz sectional curvatures \( L \) are either given by the maximum or by the minimum values of \( K \) at each point. The Deszcz symmetry of those submanifolds \( M^3 \) being equivalent to being quasi-Einstein, in the above nontrivial case, \( L = \sup K \) or \( L = \inf K \) according to the geometrical fact that the eigendirection of the Ricci tensor whose eigenvalue has multiplicity 1 is either perpendicular to the plane of Choi–Lu of these Wintgen ideal submanifolds \( M^3 \) or belongs to this plane.

(3) Concerning the origin of the Ricci tensor of 3-dimensional Riemannian manifolds and some related views on the \( \delta \)-curvatures of Chen, see [25].

**References**

6. B. Y. Chen, Geometry of Submanifolds and Its Applications, Science University of Tokyo, Department of Mathematics, Tokyo, 1981.


32. Z. Szabó, Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$. I. The local version, J. Diff. Geom. 17 (1982), 531–582.