NOTES ON ANALYTIC CONVOLUTED C-SEMGROUPS

Marko Kostić

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Abstract. We establish some new structural properties of exponentially bounded, analytic convoluted C-semigroups and state a version of Kato’s analyticity criterion for such a class of operator semigroups. Our characterizations completely cover the case of analytic fractionally integrated C-semigroups.

1. Introduction and preliminaries

An important motivational factor for the genesis of this paper presents the fact that several structural properties of exponentially bounded, analytic convoluted C-semigroups have not been fully cleared in the existing literature.

The paper is organized as follows. In Proposition 2.1 and Theorem 2.1 we refine [4, Proposition 3.7(a)], [8, Theorem 10] and transfer the assertion of [9, Theorem 5.2] to analytic convoluted C-semigroups. In Theorem 2.1, we introduce the condition (H₁) which holds in the case of fractionally integrated C-semigroups. In order to better explain the importance of this condition in our investigation, let us recall that the set \( \mathcal{P}(S_K) \) consisted of all subgenerators of a (local) convoluted C-semigroup \( (S_K(t))_{t \geq 0} \) need not be finite \([8, 10, 13]\) and that, equipped with corresponding algebraic operations, \( \mathcal{P}(S_K) \) becomes a complete lattice whose partially ordering coincides with the usual set inclusion; furthermore, \( \mathcal{P}(S_K) \) is totally ordered iff \( \text{card}(\mathcal{P}(S_K)) \leq 2 \) \([10, 13]\), and in the case \( \text{card}(\mathcal{P}(S_K)) < \infty \), one can prove that \( \mathcal{P}(S_K) \) is a Boolean, which implies \( \text{card}(\mathcal{P}(S_K)) = 2^n \) for some \( n \in \mathbb{N}_0 \). In fact, the main objective in Theorem 2.1 (i) is to establish the spectral characterizations of the integral generator of an analytic convoluted C-semigroup \( (S_K(t))_{t \geq 0} \) as well as to show that such characterizations still hold for an arbitrary subgenerator of \( (S_K(t))_{t \geq 0} \) as long as the condition (H₁) holds. It is an open problem whether the statements 2.6–2.9 quoted in the formulation of Theorem 2.1 (i) remain true for an arbitrary subgenerator of \( (S_K(t))_{t \geq 0} \) if the condition (H₁)

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is neglected. Furthermore, the condition \((H_1)\) plays a crucial role in Theorem \(2.2\) which presents Kato’s analyticity criterion for convoluted \(C\)-semigroups. Even in the case of regularized semigroups, Theorem \(2.2\) and Corollary \(2.2\) improve the corresponding result of Zheng [14 Theorem]. It is well known that \(A\) generates an (exponentially) bounded, analytic \(C_0\)-semigroup of angle \(\alpha \in (0, \frac{\pi}{2})\) provided that \(e^{\pm i\alpha} A\) are generators of (exponentially) bounded \(C_0\)-semigroups \((T_{\pm \alpha}(t))_{t \geq 0}\). We transfer this assertion to analytic regularized semigroups by a slight modification of the proof of [1] Theorem 3.9.7.

By \(E\) and \(L(E)\) are denoted a complex Banach space and the Banach algebra of bounded linear operators on \(E\). For a closed linear operator \(A\) acting on \(E\), \(D(A)\), \(\text{Kern}(A)\), \(R(A)\) and \(\rho(A)\) denote its domain, kernel, range and resolvent set, respectively. By \([D(A)]\) is denoted the Banach space \(D(A)\) equipped with the graph norm. Given \(\gamma \in (0, \pi]\), put \(\Sigma_\gamma := \{\lambda \in \mathbb{C} : \lambda \neq 0, \arg(\lambda) \in (-\gamma, \gamma)\}\). In what follows, we assume \(L(E) \ni C\) is a complex-valued locally integrable function in \([0, \tau)\) and \(K\) is a strongly continuous family of bounded linear operators on \(E\) and \(\Theta\) is a complex-valued locally integrable function in \([0, \tau)\) and \(\Theta (t) = K(t)\) for a.e. \(t \in [0, \tau)\). We mainly use the following condition:

(P1): \(K\) is Laplace transformable, i.e., it is locally integrable on \([0, \infty)\) and there exists \(\beta \in \mathbb{R}\) so that

\[
\tilde{K}(\lambda) = \mathcal{L}(K)(\lambda) := \lim_{b \to \infty} \int_0^b e^{-\lambda t} K(t) \, dt := \int_0^\infty e^{-\lambda t} K(t) \, dt
\]

exists for all \(\lambda \in \mathbb{C}\) with \(\Re \lambda > \beta\). Put \(\text{abs}(K) := \inf \{\Re \lambda : \tilde{K}(\lambda) \text{ exists}\}\).

**Definition 1.1.** ([7] - [8]) Let \(A\) be a closed operator and let \(0 < \tau \leq \infty\). If there exists a strongly continuous family \((S_K(t))_{t \in [0, \tau)}\) in \(L(E)\) such that:

(i) \(S_K(t)A \subset AS_K(t)\), \(t \in [0, \tau)\),
(ii) \(S_K(t)C = CS_K(t)\), \(t \in [0, \tau)\) and
(iii) for all \(x \in E\) and \(t \in [0, \tau)\): \(\int_0^t S_K(s)x \, ds \in D(A)\) and

\[
\int_0^t S_K(s)x \, ds = S_K(t)x - \Theta(t)Cx,
\]

then it is said that \(A\) is a subgenerator of a (local) \(K\)-convoluted \(C\)-semigroup \((S_K(t))_{t \in [0, \tau)}\). If \(\tau = \infty\), then we say that \((S_K(t))_{t \geq 0}\) is an exponentially bounded \(K\)-convoluted \(C\)-semigroup with a subgenerator \(A\) if, additionally, there exist \(M > 0\) and \(\omega \geq 0\) such that \(\|S_K(t)\| \leq Me^{\omega t}, \ t \geq 0\).

The integral generator of \((S_K(t))_{t \in [0, \tau)}\) is defined by

\[
\hat{A} := \left\{(x, y) \in E^2 : S_K(t)x - \Theta(t)Cx = \int_0^t S_K(s)yds, \ t \in [0, \tau)\right\},
\]
and it is a closed linear operator which is an extension of any subgenerator of \((S_K(t))_{t\in[0,\tau]}\). Suppose \(\{A,B\} \subset \varphi(S_K)\). By [10] Proposition 1.1, the following holds:

(a) \(C^{-1}AC = C^{-1}A = A \in \varphi(S_K)\),
(b) \(A \) and \(B\) have the same eigenvalues,
(c) \(\rho_C(A) \subseteq \rho_C(B)\) if \(A \subseteq B\),
(d) \(A = B = A\), if \(\rho(A) \neq \emptyset\) or \(C = I\).

The proof of the following auxiliary lemma is similar to those of [7] Theorem 2.2 and [9] Theorem 3.3.

**Lemma 1.1.** Suppose \(K\) satisfies (P1) and \(A\) is a closed linear operator.

(i) Suppose \(M > 0, \omega > 0, A\) is a subgenerator of an exponentially bounded, \(K\)-convoluted \(C\)-semigroup \((S_K(t))_{t\geq 0}\) satisfying \(\|S_K(t)\| \leq Me^{\omega t}, t \geq 0\) and \(\omega = \max(\omega, \text{abs}(K))\). Then \(\{\lambda \in \mathbb{C} : \text{Re} \lambda > \omega\}, \hat{K}(\lambda) \neq 0\} \subset \rho_C(A)\) and \((\lambda - A)^{-1}C\) \(e^{-\lambda t}S_K(t)x dt\) for all \(x \in E\) and \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda > \omega\) and \(\hat{K}(\lambda) \neq 0\).

(ii) Suppose \(M > 0, \omega > 0, (S_K(t))_{t\geq 0}\) is a strongly continuous operator family, \(\|S_K(t)\| \leq Me^{\omega t}, t \geq 0\) and \(\omega = \max(\omega, \text{abs}(K))\). If \(\{\lambda \in (\omega_1, \infty) : \hat{K}(\lambda) \neq 0\} \subset \rho_C(A)\) and \((\lambda - A)^{-1}C\) \(e^{-\lambda t}S_K(t)x dt\) for all \(x \in E, \lambda > \omega_1, \hat{K}(\lambda) \neq 0\), then \((S_K(t))_{t\geq 0}\) is an exponentially bounded, \(K\)-convoluted \(C\)-semigroup with a subgenerator \(A\).

(iii) Let \(A\) be densely defined. Then \(A\) is a subgenerator of an exponentially bounded \(C\)-semigroup \((T(t))_{t\geq 0}\) satisfying \(\|T(t)\| \leq Me^{\omega t}, t \geq 0\) for appropriate constants \(M > 0\) and \(\omega \in \mathbb{R}\) iff \((\omega, \infty) \subset \rho_C(A)\), the mapping \(\lambda \mapsto (\lambda - A)^{-1}C\), \(\lambda > \omega\) is infinitely differentiable and

\[
\left\| \frac{d^k}{d\lambda^k}[(\lambda - A)^{-1}C] \right\| \leq \frac{Mk!}{(\lambda - \omega)^{k+1}}, \quad k \in \mathbb{N}_0, \lambda > \omega.
\]

**Definition 1.2.** [8] Let \(\alpha \in (0, \frac{\pi}{2}]\) and let \((S_K(t))_{t\geq 0}\) be a \(K\)-convoluted \(C\)-semigroup. Then we say that \((S_K(t))_{t\geq 0}\) is an analytic \(K\)-convoluted \(C\)-semigroup of angle \(\alpha\) if there exists an analytic function \(S_K : \Sigma_\alpha \rightarrow L(E)\) which satisfies

(i) \(S_K(t) = S_K(t), t > 0,\)

(ii) \(\lim_{t \rightarrow 0, \gamma \in \Sigma, z \rightarrow \gamma} S_K(z)x = 0\) for all \(\gamma \in (0, \alpha)\) and \(x \in E\).

It is said that \((S_K(t))_{t\geq 0}\) is an exponentially bounded, analytic \(K\)-convoluted \(C\)-semigroup, resp. bounded analytic \(K\)-convoluted \(C\)-semigroup, of angle \(\alpha\), if for every \(\gamma \in (0, \alpha)\), there exist \(M_\gamma > 0\) and \(\omega_\gamma \geq 0\), resp. \(\omega_\gamma > 0\), such that \(\|S_K(z)\| \leq M_\gamma e^{\omega_\gamma \text{Re} z}, z \in \Sigma_\gamma\).

Since no confusion seems likely, we will also denote \(S_K\) by \(S_K\). Plugging \(K(t) = \frac{t^{r-1}}{(r-1)!}, t > 0\) in Definition 1.1 and Definition 1.2, where \(r > 0\) and \(\Gamma(\cdot)\) denotes the Gamma function, we obtain the well-known classes of (analytic) \(r\)-times integrated \(C\)-semigroups; an (analytic) 0-times integrated \(C\)-semigroup is defined to be an (analytic) \(C\)-semigroup (cf. [3] Definition 21.3)). The notion of (exponential) boundedness of an analytic \(r\)-times integrated \(C\)-semigroup, \(r \geq 0\), is understood in the sense of Definition 1.2.
2. Analytic convoluted $C$-semigroups

We start this section with the following proposition.

**Proposition 2.1.** Suppose $K$ satisfies (P1), $\alpha \in (0, \frac{\pi}{2}]$ and $A$ is a subgenerator of an exponentially bounded, analytic $K$-convoluted $C$-semigroup $(S_K(t))_{t \geq 0}$ of angle $\alpha$. Suppose, further, that the condition (H) holds, where:

(H) There exist functions $c : (\alpha, \alpha) \to \mathbb{C} \setminus \{0\}$, $\omega_0 : (\alpha, \alpha) \to [0, \infty)$ and a family of functions $(K_\theta)_{\theta \in (\alpha, \alpha)}$ satisfying (P1) so that: $\text{abs}(K_\theta) \leq \omega_0(\theta)$, $\frac{\text{abs}(K_\theta)}{\cos \theta} \leq \omega_0(\theta)$.

Then, for every $\theta \in (\alpha, \alpha)$, the operator $e^{i\theta}A$ is a subgenerator of an exponentially bounded, analytic $K_\theta$-convoluted $C$-semigroup $(c(\theta)S_K(te^{i\theta}))_{t \geq 0}$ of angle $\alpha - |\theta|$. Furthermore,

(i) $S_K(te^{i\theta})A \subset AS_K(te^{i\theta})$, $t \geq 0$ and

(ii) $A \int_0^t S_K(s)x \, ds = S_K(te^{i\theta})x - \int_0^t K_\theta(s) \, dsCx$, $t \geq 0$, $x \in E$, $\theta \in (\alpha, \alpha)$.

**Proof.** Let $\theta \in (\alpha, \alpha)$ and let $\lambda \in \mathbb{R}$ be sufficiently large with $\overline{K}_\theta(\lambda) \neq 0$. Denote $\Gamma_\theta := \{te^{i\theta} : t \geq 0\}$ and notice that $(c(\theta)S_K(te^{i\theta}))_{t \geq 0}$ is a strongly continuous, exponentially bounded operator family. Clearly, $\overline{K}(\lambda e^{-i\theta}) \neq 0$ and Lemma 11 yields

\[
\overline{K}_\theta(\lambda)(\lambda - e^{i\theta}A)^{-1}Cx = \overline{K}_\theta(\lambda)e^{-i\theta}(\lambda e^{-i\theta} - A)^{-1}Cx
\]

\[
= e^{-i\theta} \frac{\overline{K}_\theta(\lambda)}{K(\lambda e^{-i\theta})} \int_0^\infty e^{-\lambda e^{-i\theta}t}S_K(t)x \, dt = e^{-i\theta}c(\theta) \int_{\Gamma_\theta} e^{-\lambda t}e^{i\theta}S_K(te^{i\theta})x \, dt
\]

\[
= \int_0^\infty e^{-\lambda t}(c(\theta)S_K(te^{i\theta})x) \, dt, \quad x \in E,
\]

where (2.3) follows from an elementary application of the Cauchy theorem. Keeping in mind Definition 11 and Lemma 11(ii), the assertion automatically follows.

Now we state the following generalization of [8, Theorem 10] and [9, Theorem 5.2].

**Theorem 2.1.** (i) Suppose $K$ satisfies (P1), $\omega \geq \max(0, \text{abs}(K))$, $\alpha \in (0, \frac{\pi}{2}]$, and $\overline{K}(\cdot)$ can be analytically continued to a function $g : \omega + \Sigma_{\alpha} \to \mathbb{C}$. Suppose, further, that $A$ is a subgenerator of an analytic $K$-convoluted $C$-semigroup $(S_K(t))_{t \geq 0}$ of angle $\alpha$ and that

\[
\sup_{z \in \Sigma_{\alpha}} \|e^{-\omega z}S_K(z)\| < \infty \text{ for all } \gamma \in (0, \alpha).
\]
Let us denote by $\hat{A}$ the integral generator of $(S_K(t))_{t \geq 0}$ and put
\[(2.5) \quad N := \{ \lambda \in \omega + \Sigma_{\frac{\pi}{2} + \alpha} : g(\lambda) \neq 0 \}.
\]

Then:
\[(2.6) \quad N \subset \rho_C(\hat{A}),
\]
\[(2.7) \quad \sup_{\lambda \in N \cap (\omega + \Sigma_{\frac{\pi}{2} + \gamma})} \| (\lambda - \omega)g(\lambda)(\lambda - \hat{A})^{-1}C \| < \infty \text{ for all } \gamma \in (0, \alpha),
\]
\[(2.8) \quad \lim_{\lambda \to +\infty, K(\lambda) \neq 0} \lambda \hat{K}(\lambda)(\lambda - A)^{-1}Cx = 0, \quad x \in E,
\]
\[(2.9) \quad \text{the mapping } \lambda \mapsto (\lambda - \hat{A})^{-1}C, \lambda \in N \text{ is analytic.}
\]

Suppose, additionally, that the following condition holds:
\[(H_1): (H) \text{ holds with } c(\cdot), \omega_0(\cdot), (K_\theta)_{\theta \in (-\alpha, \alpha)}, \text{ and additionally, } \text{abs}(K_\theta) \leq \omega \cos \theta, \theta \in (-\alpha, \alpha).
\]

Then \[2.6, 2.7\] and \[2.9\] hold with $\hat{A}$ replaced by $A$ therein.

(ii) Assume $\alpha \in (0, \frac{\pi}{2})$, $K$ satisfies (P1) and $\omega \geq \max(0, \text{abs}(K))$. Suppose that $A$ is a closed linear operator with $\{ \lambda \in \mathbb{C} : \text{Re } \lambda > \omega, \hat{K}(\lambda) \neq 0 \} \subset \rho_C(A)$ and that the function $\lambda \mapsto \hat{K}(\lambda)(\lambda - A)^{-1}C$, $\text{Re } \lambda > \omega$, $\hat{K}(\lambda) \neq 0$, can be analytically extended to a function $\tilde{q}: \omega + \Sigma_{\frac{\pi}{2} + \alpha} \to L(E)$ satisfying
\[(2.10) \quad \sup_{\lambda \in \omega + \Sigma_{\frac{\pi}{2} + \gamma}} \| (\lambda - \omega)\tilde{q}(\lambda) \| < \infty \text{ for all } \gamma \in (0, \alpha),
\]
\[(2.11) \quad \lim_{\lambda \to +\infty, \lambda \neq 0} \lambda \tilde{q}(\lambda)x = 0, \quad x \in E, \quad \text{if } \overline{D(A)} \neq E.
\]

Then the operator $A$ is a subgenerator of an exponentially bounded, analytic $K$-convoluted $C$-semigroup of angle $\alpha$.

**Proof.** The proof of (i) can be obtained as follows. By Lemma\[2.1(i),\] we have $\{ \lambda \in \mathbb{C} : \text{Re } \lambda > \omega, \hat{K}(\lambda) \neq 0 \} \subset \rho_C(A)$ and
\[\hat{K}(\lambda)(\lambda - A)^{-1}Cx = \int_0^\infty e^{-\lambda t}S_K(t)x \, dt, \quad \text{Re } \lambda > \omega, \hat{K}(\lambda) \neq 0, x \in E.
\]

Put $q(\lambda) := \int_0^\infty e^{-\lambda t}S_K(t) \, dt$, $\text{Re } \lambda > \omega$. An application of [1] Theorem 2.6.1 gives that the function $q(\cdot)$ can be extended to an analytic function $\tilde{q}: \omega + \Sigma_{\frac{\pi}{2} + \alpha} \to L(E)$ satisfying $\sup_{\lambda \in \omega + \Sigma_{\frac{\pi}{2} + \gamma}} \| (\lambda - \omega)\tilde{q}(\lambda) \| < \infty$ for all $\gamma \in (0, \alpha)$. Further on, $N$ is an open subset of $\mathbb{C}$ and it can be easily seen that every two point belonging to $N$ can be connected with a $C^\infty$ curve lying in $N$; in particular, $N$ is a connected open subset of $\mathbb{C}$. The function $F: N \to L(E)$ given by $F(\lambda) := \frac{\tilde{q}(\lambda)}{q(\lambda)}$, $\lambda \in N$ is analytic and
\[(2.12) \quad \{ \lambda \in \mathbb{C} : \text{Re } \lambda > \omega, \hat{K}(\lambda) \neq 0 \} \subset \{ \lambda \in N \cap \rho_C(A) : F(\lambda) = (\lambda - A)^{-1}C \}.\]
Let us denote $V = \{ \lambda \in N \cap \rho_C(A) : F(\lambda) = (\lambda - A)^{-1}C \}$ and suppose $\mu \in \rho_C(A)$, $x \in D(A)$ and $y \in E$. Since

$$(2.13) \quad F(\lambda)(\lambda - A)x = (\lambda - A)^{-1}C(\lambda - A)x = Cx, \lambda \in V,$$

$$(2.14) \quad F(\lambda)Cy = CF(\lambda)y, \lambda \in V,$$

$$(2.15) \quad F(\lambda)Cy = (\lambda - A)^{-1}C^2y = (\mu - A)^{-1}C^2y - (\lambda - \mu)(\mu - A)^{-1}CF(\lambda)y, \lambda \in V,$$

the uniqueness theorem for analytic functions (cf. [1] Proposition A2, Proposition B.5) implies that (2.13)–(2.15) remain true for all $\lambda \in N$. Suppose now that $(\lambda - A)x = 0$ for some $\lambda \in N$ and $x \in D(A)$. Owing to (2.13), one gets $Cx = 0$, $x = 0$ and $\lambda - A$ is injective. By the assertion (b), we obtain that $\lambda - A$ is injective. Furthermore,

$$(\lambda - A)CF(\lambda)y = (\lambda - A)F(\lambda)Cy$$

$$= (\lambda - A)((\mu - A)^{-1}C^2y - (\lambda - \mu)(\mu - A)^{-1}CF(\lambda)y)$$

$$= C^2y + (\lambda - \mu)((\mu - A)^{-1}C^2y - CF(\lambda)y) - (\lambda - \mu)(\mu - A)^{-1}CF(\lambda)y,$$

and thanks to the validity of (2.15) for all $\lambda \in N$, one obtains that

$$(2.16) \quad (\lambda - A)CF(\lambda)y = C^2y, \lambda \in N.$$

The last equality, injectiveness of $C$ and assertion (a) taken together imply:

$$(2.17) \quad \lambda F(\lambda)y = C^{-1}AC[F(\lambda)y] + Cy = \bar{A}F(\lambda)y + Cy, \lambda \in N, i.e.,$$

$$(2.18) \quad (\lambda - \bar{A})F(\lambda)y = Cy, \lambda \in N.$$

This implies $R(C) \subset R(\lambda - \bar{A})$, $\lambda \in N$, $N \subset \rho_C(\bar{A})$, $F(\lambda) = (\lambda - \bar{A})^{-1}C$, $\lambda \in N$, (2.6) and (2.9). The estimate (2.7) is an immediate consequence of [1] Theorem 2.6.1. Let $x \in E$ be fixed. Then $z \mapsto S_K(z)x, z \in \Sigma_\alpha$ is an analytic function which satisfies the condition (i) quoted in the formulation of [1] Theorem 2.6.1. Since $\lim_{t \downarrow 0} S_K(t)x = 0$, an application of [1] Theorem 2.6.4 implies that $\lim_{t \uparrow +\infty} \lambda S_K(t)x = 0$, i.e., (2.5) and the first part of the proof is completed. Suppose now that (H$_1$) holds. Then $\text{abs}(K_0) \leq \omega \cos \theta$, $\theta \in (-\alpha, \alpha)$, and by Lemma [1, ii], we have that, for every $\theta \in (-\alpha, \alpha)$, $\{ \lambda \in \mathbb{C} : \Re\lambda > \omega \cos \theta, \tilde{K}_\theta(\lambda) \neq 0 \} \subset \rho_C(e^{i\theta}A)$ and that:

$$(2.19) \quad \bar{K}_\theta(\lambda)e^{-i\theta}(\lambda e^{-i\theta} - A)^{-1}Cx = \int_0^\infty e^{-s\lambda}(c(\theta)S_K(te^{i\theta}))x \, dt,$$

for all $x \in E$ and $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega \cos \theta$ and $\bar{K}_\theta(\lambda) \neq 0$. Fix a number $\theta \in (-\alpha, \alpha)$ and define $G_\theta : \{ \omega + te^{i\varphi} : t > 0, \varphi \in (-\pi/2 + \theta, \pi/2 - \theta) \} \cap N \to \mathbb{C}$ by $G_\theta(\lambda) := \check{K}_\theta(\lambda)e^{i\varphi}$, $\lambda \in D(G_\theta(\cdot))$. Then it is clear that $D(G_\theta(\cdot))$ is an open, connected subset of $\mathbb{C}$ and that, owing to (2.11)–(2.20), there exists $a > 0$ such that $\Psi_{\theta,a} := \{ te^{-i\theta} \cap N : t \geq a \} \subset D(G_\theta(\cdot))$ and that $G_\theta(\lambda) = c(\theta)$, $\lambda \in \Phi_{\theta,a}$. By the uniqueness theorem for analytic functions, one obtains that $G_\theta(\lambda) = c(\theta)$,
\(\lambda \in D(G_\theta(\cdot))\). Hence, \((2.20)\) implies \(\{\omega + te^{i\varphi} : t > 0, \varphi \in (-\left(\frac{\pi}{2} + \theta\right), \frac{\pi}{2} - \theta)\} \cap \mathbb{N} \subset \rho_C(A)\),

\[
(2.20) \quad (z - A)^{-1}C x = \frac{e^{i\lambda}}{g(z)} \int_0^\infty e^{-ze^{i\varphi}t} S_K(te^{i\varphi})x \, dt,
\]

for all \(z \in \{\omega + te^{i\varphi} : t > 0, \varphi \in (-\left(\frac{\pi}{2} + \theta\right), \frac{\pi}{2} - \theta)\} \cap \mathbb{N}\) and \(x \in E\), and the mapping \(z \mapsto (z - A)^{-1}C\), \(z \in \mathbb{N}\), \(\arg(z - \omega) \in (-\left(\frac{\pi}{2} + \theta\right), \frac{\pi}{2} - \theta)\) is analytic. One can apply the same argument to \(e^{-i\lambda} A\) in order to see that \(\{z \in \mathbb{N} : \arg(z - \omega) \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})\} \subset \rho_C(A)\) and that the mapping \(z \mapsto (z - A)^{-1}C\), \(z \in \mathbb{N}\), \(\arg(z - \omega) \in (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})\) is analytic. Thereby, \(\{z \in \mathbb{N} : \arg(z - \omega) < \theta + \frac{\pi}{2}\} \subset \rho_C(A)\) and the mapping \(z \mapsto (z - A)^{-1}C\), \(z \in \mathbb{N}\), \(\arg(z - \omega) < \theta + \frac{\pi}{2}\) is analytic. This completes the proof of (i). The proof of (ii) in the case \(\overline{D(A)} \neq E\) is given in \(8\). Suppose now that \(\overline{D(A)} = E\). We will prove that \((2.20)\) automatically holds for every \(x \in E\). Arguing as in the proof of \([8\) Theorem 10], one obtains that there exists an analytic function \(S_K : \Sigma_\alpha \rightarrow L(E)\) such that \(\sup_{z \in \omega + r(\frac{\pi}{2} + \theta)} \|e^{-wz}S(t,z)\| < \infty\) for all \(\beta \in (0, \alpha)\).

By \([11\) Proposition 2.6.3(b)] and the proof of \([8\) Theorem 10], it suffices to show that \(\lim_{t \downarrow 0} S_K(t)x = 0\). Suppose, for the time being, \(x \in D(A)\). Since \(\tilde{g}(\lambda)x = K(\lambda)(\lambda - A)^{-1}Cx\), \(\lambda \in \mathbb{C}\), \(\Re \lambda > \omega\), \(K(\lambda) \neq 0\) we have that \(\mathcal{L}(\int_0^t S_K(s)Ax \, ds)(\lambda) = \frac{\tilde{g}(\lambda)}{\lambda} Ax = \tilde{g}(\lambda)x - \frac{K(\lambda)}{\lambda} Cx = \mathcal{L}(S_K(t)x - \Theta(t)Cx)(\lambda), \lambda \in \mathbb{C}\), \(\Re \lambda > \omega\), \(K(\lambda) \neq 0\) and the uniqueness theorem for Laplace transforms implies \(\int_0^t S_K(s)Ax \, ds = S_K(t)x - \Theta(t)Cx\), \(t \geq 0\). Therefore \(\|S_K(t)x\| \leq \|\Theta(t)Cx + te^{\omega t}\|\|Ax\|\), \(t \geq 0\) and \(\lim_{t \downarrow 0} S_K(t)x = 0\). Combined with the exponential boundedness of \(S_K(\cdot)\), this indicates that \(\lim_{t \downarrow 0} S_K(t)x = 0\) for every \(x \in E\).

Let \(0 \neq \Omega \subset \rho_C(A)\) be open. By \([5\) Remark 2.7], we have that the continuity of mapping \(\lambda \mapsto (\lambda - A)^{-1}C\), \(\lambda \in \Omega\) implies its analyticity. Furthermore, it can be simply verified that the function \(K(t) = \frac{t^{r-1}}{1(t)}\), \(t > 0\), \(r > 0\) satisfies the condition \((H_d)\) with \(c(t) = e^{-it\theta}\), \(\omega_0(\theta) = 0\) and \(K_0(t) = K(t), \theta \in (-\alpha, \alpha), t > 0\). Keeping in mind Proposition 1.1, \(Theorem \[2.1\] and these remarks, one immediately obtains the proof of the following corollary; notice only that, in the case \(r = 0\), the equality \((2.24)\) follows from \([11\) Theorem 2.6.4] and elementary definitions.

**Corollary 2.1.** (i) Suppose \(r \geq 0\) and \(\alpha \in (0, \frac{\pi}{2}]\). Then the operator \(A\) is a subgenerator of an exponentially bounded, analytic \(r\)-times integrated \(C\)-semigroup \((S_r(t))_{t \geq 0}\) of angle \(\alpha\) if for every \(\gamma \in (0, \alpha)\), there exist \(M_\gamma > 0\) and \(\omega_\gamma \geq 0\) such that:

\[
(2.21) \quad \omega_\gamma + \Sigma_{\frac{\pi}{2} + \gamma} \subset \rho_C(A),
\]

\[
(2.22) \quad \|(\lambda - A)^{-1}C\| \leq M_\gamma (1 + |\lambda|)^{-1}, \quad \lambda \in \omega_\gamma + \Sigma_{\frac{\pi}{2} + \gamma},
\]

\[
(2.23) \quad \lim_{\lambda \rightarrow +\infty} \frac{(\lambda - A)^{-1}C x}{\lambda^{r-1}} = \chi_{(0)}(r)Cx, \quad x \in E, \quad \text{if} \quad \overline{D(A)} \neq E.
\]
(ii) Let \( \theta \in (-\alpha, \alpha) \) and let \( A \) be a subgenerator of an exponentially bounded, analytic \( r \)-times integrated \( C \)-semigroup \( (S_r(t))_{t \geq 0} \) of angle \( \alpha \). Then \( e^{i\theta}A \) is a subgenerator of an exponentially bounded, analytic \( r \)-times integrated \( C \)-semigroup \( (e^{-i\theta}S_r(te^{i\theta}))_{t \geq 0} \) of angle \( \alpha - |\theta| \). \( S_r(z)A \subset AS_r(z) \) and \( A \int_0^1 S_r(s)xds = S_r(z)x - \frac{e^{i\pi x}}{1(r+1)}Cx, \, z \in \Sigma_\alpha, \, x \in E \).

Now we state Kato’s analyticity criterion for convoluted \( C \)-semigroups.

**Theorem 2.2.** Suppose \( \alpha \in (0, \frac{\pi}{4}] \), \( K \) satisfies (P1), \( \omega \geq \max(0, \text{abs}(K)) \), there exists an analytic function \( g: \omega + \Sigma_{\omega + \alpha} \to \mathbb{C} \) such that \( g(\lambda) = \tilde{K}(\lambda), \, \lambda \in \mathbb{C}, \, \text{Re} \lambda > \omega \) and (H3) holds. Then \( A \) is a subgenerator of an analytic \( K \)-convoluted \( C \)-semigroup \( (S_K(t))_{t \geq 0} \) satisfying \(^{(2.4)}\) iff:

(i.1) For every \( \theta \in (-\alpha, \alpha) \), \( e^{i\theta}A \) is a subgenerator of a \( K_{\theta} \)-convoluted \( C \)-semigroup \( (S_\theta(t))_{t \geq 0} \), and

(i.2) for every \( \beta \in (0, \alpha) \), there exists \( M_{\beta} > 0 \) such that

\[
\left\| \frac{1}{c(\theta)}S_\theta(t) \right\| \leq M_{\beta}e^{\omega t \cos \theta}, \quad t \geq 0, \quad \theta \in (-\beta, \beta).
\]

**Proof.** Suppose \( A \) is a subgenerator of an analytic \( K \)-convoluted \( C \)-semigroup \( (S_K(t))_{t \geq 0} \) satisfying \(^{(2.4)}\). By Proposition 1.1, we have that (i.1) and (i.2) hold with \( S_\theta(t) = c(\theta)S_K(te^{i\theta}), \, t \geq 0, \, \theta \in (-\alpha, \alpha) \). To prove the converse statement, notice that the argumentation given in the final part of the proof of Theorem 2.1 implies that \( (\omega + \Sigma_{\omega + \alpha}) \cap N \subset \rho c(A) \) and that there exists an analytic mapping \( G: \omega + \Sigma_{\omega + \alpha} \to L(E) \) such that \( G(\lambda) = g(\lambda)(\lambda - A)^{-1}C, \, \lambda \in (\omega + \Sigma_{\omega + \alpha}) \cap N \), where \( N \) is defined by \(^{(2.3)}\). Furthermore, for every \( \theta \in (-\alpha, \alpha) \):

\[
(2.26) \quad G(\lambda) = e^{i\theta} \int_0^\infty e^{-\lambda t e^{i\theta}} \left( \frac{1}{c(\theta)}S_\theta(t) \right) dt \quad \text{if arg} (\lambda - \omega) \in \left(-\left(\frac{\pi}{2} + \theta\right), \frac{\pi}{2} - \theta\right).
\]

\[
(2.27) \quad G(\lambda) = e^{-i\theta} \int_0^\infty e^{-\lambda t e^{-i\theta}} \left( \frac{1}{c(-\theta)}S_{-\theta}(t) \right) dt \quad \text{if arg} (\lambda - \omega) \in \left(\theta - \frac{\pi}{2}, \frac{\pi}{2}\right).
\]

Keeping in mind (i.2) as well as \(^{(2.26)}, (2.27)\), we have that, for every \( \beta \in (0, \alpha) \), \( \sup_{\lambda \in \omega + \Sigma_{\omega + \beta}} \| (\lambda - \omega) G(\lambda) \| < \infty \). By \(^{(1.1)}\) Theorem 2.6.1, one gets the existence of an analytic mapping \( S_K : \Sigma_\alpha \to L(E) \) such that \( \sup_{z \in \Sigma_\beta} \| e^{-\omega z} S_K(z) \| < \infty \) for all \( \beta \in (0, \alpha) \) and that \( G(\lambda) = \tilde{S}_K(\lambda) \) for all \( \lambda \in (\omega, \infty) \). Furthermore, the uniqueness theorem for Laplace transforms implies \( S_K(z) = \frac{1}{c(\text{arg}(z))}S_{\text{arg}(z)}(z) \), \( z \in \Sigma_\alpha \), and since \( c(0) = 1 \) and \( K_0 = K \), it suffices to show that, for every fixed \( x \in E \) and \( \beta \in (0, \alpha) \), one has \( \lim_{z \in \Sigma_{-\beta}, z \to 0} S_K(z)x = 0 \) (cf. also Lemma \(^{1.1}\)(ii)).

To this end, notice that \( \lim_{t \downarrow 0} S_K(t)x = \lim_{t \downarrow 0} S_0(t)x = 0 \) and that \(^{(1)}\) Proposition 2.6.3(b) implies \( \lim_{z \in \Sigma_{\beta}, z \to 0} e^{-\omega z} S_K(z)x = \lim_{z \in \Sigma_{\beta}, z \to 0} S_K(z)x = 0, \, z \in \Sigma_\alpha \). □

In the following corollary, we remove any density assumption from \(^{(14)}\) Theorem:
Corollary 2.2. Suppose \( r \geq 0, \alpha \in (0, \frac{\pi}{2}] \) and \( \omega \in [0, \infty) \) if \( r > 0 \), resp. \( \omega \in \mathbb{R} \) if \( r = 0 \). Then \( A \) is a subgenerator of an analytic \( r \)-times integrated \( C \)-semigroup \( (S_r(t))_{t \geq 0} \) of angle \( \alpha \) satisfying \( \sup_{\lambda \in \Sigma_{\beta}} \|e^{-\omega \cdot z}S_r(z)\| < \infty \) for all \( \beta \in (0, \alpha) \) if the following conditions hold:

(i.1) For every \( \theta \in (-\alpha, \alpha) \), \( e^{i\theta}A \) is a subgenerator of an \( r \)-times integrated \( C \)-semigroup \( (S_r(t))_{t \geq 0} \), and

(i.2) for every \( \beta \in (0, \alpha) \), there exists \( M_\beta > 0 \) such that \( \|S_\beta(t)\| \leq M_\beta e^{\omega t \cos \theta} \), \( t \geq 0, \theta \in (-\beta, \beta) \).

Now we state the following extension of [1] Theorem 3.9.7 and [1] Corollary 3.9.9:

Theorem 2.3. Suppose \( \alpha \in (0, \frac{\pi}{2}) \), \( A \) is densely defined and \( e^{\pm i\alpha}A \) are subgenerators of \((\text{exponentially})\) bounded \( C \)-semigroups \((T_{\pm \alpha}(t))_{t \geq 0}\). Then \( A \) is a subgenerator of an \((\text{exponentially})\) bounded, analytic \( C \)-semigroup of angle \( \alpha \).

Proof. Suppose \( \|T_{\pm \alpha}(t)\| \leq Me^{\omega t}, t \geq 0 \) for appropriate constants \( M \geq 0 \) and \( \omega > 0 \). Put \( \mu := \frac{\omega}{\cos \alpha} \) and \( A_\mu := A - \mu \). Then \( e^{\pm i\alpha}A_\mu \) are subgenerators of bounded \( C \)-semigroups \((S_{\pm \alpha}(t))_{t \geq 0} \) and \( \|S_{\pm \alpha}(t)\| \leq M, t \geq 0 \).

Proceeding as in the proof of [1] Theorem 3.9.7, one gets that \( \Sigma_{\frac{\pi}{2}+\alpha} \subset \rho_C(A_\mu) \) and that the mapping \( \lambda \mapsto (\lambda - A_\mu)^{-1}C, \lambda \in \Sigma_{\frac{\pi}{2}+\alpha} \) is analytic. Then the proof of [5] Corollary 2.8 implies that, for every \( n \in \mathbb{N} \) and \( \lambda \in \Sigma_{\frac{\pi}{2}+\alpha} \):

\[(2.28) \ R(C) \subset R((\lambda - A_\mu)^{n+1}) \text{ and } \frac{d^n}{d\lambda^n}((\lambda - A_\mu)^{-1}C = (-1)^n n!(\lambda - A_\mu)^{-(n+1)}C)\]

Put now \( T_{n,k}(z) := (I - \frac{z}{n} A_\mu)^{-k}C, z \in \Sigma_{\alpha}, k \in \mathbb{N}, n \in \mathbb{N} \). By (2.28), we obtain that, for every \( r \geq 0 \):

\[(2.29) \quad \|T_{n,k}(re^{\pm i\alpha})\| = \left\| \left(I - \frac{re^{\pm i\alpha}}{n} A_\mu\right)^{-k}C \right\| = \left\| \frac{n^k}{r^k} \left(I - e^{\pm i\alpha}A_\mu\right)^{-k}C \right\|
\]

\[= \left\| \frac{n^k (\frac{d^{k-1}}{d\lambda^{k-1}}(\lambda - e^{\pm i\alpha}A_\mu)^{-1}C)\lambda = \frac{k}{\alpha}}{(-1)^{k-1}(k-1)!} \right\| = \left\| \frac{n^k (-1)^{k-1} \int_0^\infty e^{-\hat{\lambda} t} t^{k-1} S_{\pm \alpha}(t) \, dt}{(-1)^{k-1}(k-1)!} \right\| \leq M.\]

Arguing similarly, we get:

\[(2.30) \quad \|T_{n,k}(z)\| \leq \frac{M}{\cos \alpha}, z \in \Sigma_{\alpha}, k \in \mathbb{N}, n \in \mathbb{N} \]

Taking into account the Phragmén–Lindelöf principle (cf. for instance [1] Theorem 3.9.8) and (2.29)–(2.30), one obtains that \( \|T_{n,k}(z)\| \leq M, z \in \Sigma_{\alpha}, k \in \mathbb{N}, n \in \mathbb{N} \). In particular, \( \| \frac{d^n}{d\lambda^n}(\lambda - A_\mu)^{-1}C \| \leq \frac{M_\lambda}{\alpha^n}, \lambda > 0, n \in \mathbb{N} \) and Lemma 1.1(iii) implies that \( A_\mu \) is a subgenerator of a bounded \( C \)-semigroup \((T(t))_{t \geq 0}\) such that \((\lambda - A_\mu)^{-1}C x = \int_0^\infty e^{-\lambda t} T(t) x \, dt, \lambda \in \mathbb{C}, \Re \lambda > 0, x \in E \). By the Post–Wiener inversion formula [1] Theorem 1.7.7, one obtains \( T(t)x = \lim_{n \to \infty} T_{n,\alpha+1}(\frac{4}{\alpha})x, x \in E, t \geq 0 \) and Vitali's theorem [1] Theorem A.5, p. 458 implies that there exists an analytic mapping \( \tilde{T} : \Sigma_{\alpha} \to L(E) \) such that \( \tilde{T}(t) = T(t), t > 0 \) and that
$\|\tilde{T}(z)\| \leq M, \ z \in \Sigma_{\alpha}$. By [1] Proposition 2.6.3(b), one yields that the mapping $z \mapsto \tilde{T}(z)x, \ z \in \Sigma_\beta$ is continuous for every fixed $x \in E$ and $\beta \in (0, \alpha)$ and the proof of theorem completes a routine argument.

The preceding theorem has been recently generalized in [11]:

**Theorem 2.4.** Suppose $\alpha \in (0, \frac{\pi}{2})$, $r \geq 0$, and $e^{\pm i\alpha}A$ are subgenerators of exponentially bounded $r$-times integrated $C$-semigroups $(S^{1\alpha}_{r}(t))_{t\geq 0}$. Then, for every $\zeta > 0$, $A$ is a subgenerator of an exponentially bounded, analytic $(r + \zeta)$-times integrated $C$-semigroup $(S^{r+\zeta}_{r}(t))_{t\geq 0}$ of angle $\alpha$; if $A$ is densely defined, then $A$ is a subgenerator of an exponentially bounded, analytic $r$-times integrated $C$-semigroup $(S_{r}(t))_{t\geq 0}$ of angle $\alpha$.

**References**