NOTE ON A QUESTION OF REINHOLD BAER ON PREGROUPS II

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Abstract. Reinhold Baer asked the relationship between certain properties in a nonempty set P with a partial operation (called an "add" by Baer [1]). The first paper in our sequence [Paper I] answered his question for a special type of an add called a pregroup by Stallings [12]. This paper [Paper II] answers an analogous question for a wider class of adds.

1. Introduction

Let P be a nonempty set with a partial operation, called an "add" by Baer [1]. Formally, a partial operation on P is a mapping \( m : D \to P \) where \( D \subseteq P \times P \). If \((a, b)\) belongs to D, we denote \( m(a, b) \) by \( ab \) and say that \( ab \) is defined or exists. [Baer [1] denoted \( m(a, b) \) by \( a + b \).] We also say that a sequence \( X = [a_1, a_2, \ldots, a_n] \) is defined if each pair \( a_1a_2, a_2a_3, \ldots, a_{n-1}a_n \) is defined. By a triple in X, we mean a subsequence \([a_i, a_{i+1}, a_{i+2}]\). The universal group \( G(P) \) of an add P is the group with presentation: \( G(P) = \text{gp}(P; \text{operation } m) \). That is, P is the set of generators, and the defining relations are of the form \( ab = c \) where \( m(a, b) = c \). P is said to be group-embeddable or simply embeddable if P can be embedded in its universal group \( G(P) \).

Next follows classical examples of embeddable adds.

Example 1.1. Let K and L be groups with isomorphic subgroups A or, equivalently, which intersect in a subgroup A. Then the amalgam \( P = K \cup_A L \) is an add which is embeddable in \( G(P) = K *_A L \), the free product of K and L with A amalgamated.

Example 1.2. Let K, H, L be groups. Suppose K and H have isomorphic subgroups A, and suppose H and L have isomorphic subgroups B. Then the amalgam \( P = K \cup_A H \cup_B L \) is an add which is embeddable in \( G(P) = K *_A H *_B L \), the free product of K, H, L with subgroups A and B amalgamated.

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Example 1.3. Let $T = (K_i; A_{rs})$ be a tree graph of groups with vertex groups $K_i$, and with edge groups $A_{rs}$. [Here $A_{rs}$ is a subgroup of vertex groups $K_r$ and $K_s$.] Let $P = \bigcup (K_i; A_{rs})$, the amalgam of the groups in $T$. Then $P$ is an add which is embeddable in $G(P) = \ast(K_i; A_{rs})$, the tree product of the vertex groups $K_i$ with the $A_{rs}$ amalgamated.

Example 1.4. Let $G = (K_i; A_{rs})$ be a graph of groups with vertex groups $K_i$ and with edge groups $A_{rs}$. Again $A_{rs}$ is a subgroup of vertex groups $K_r$ and $K_s$. Let $P = \bigcup (K_i; A_{rs})$. Then $P$ is an add but, when the graph is not a tree, $P$ need not be embeddable in $G(P) = \ast(K_i; A_{rs})$, the free product of groups $K_i$ with the $A_{rs}$ amalgamated. In fact, there are examples where $G(P) = \{e\}$.

Let $P$ be an add. Then $P$ will be called a pregroup if it satisfies the following three axioms of Stallings [12]:

P1 (Identity) There exists $1 \in P$ such that for all $a$, we have $1a$ and $a1$ are defined and $1a = a1 = a$.

P2 (Inverses) For each $a \in P$, there exists $a^{-1}$ in $P$ such that $aa^{-1}$ and $a^{-1}a$ are defined and $aa^{-1} = a^{-1}a = 1$.

P4 (Weak Associative Law) If $ab$ and $bc$ are defined, then $(ab)c$ is defined if and only if $a(bc)$ is defined, in which case $(ab)c = a(bc)$. [We then say that the triple $abc = (ab)c = a(bc)$ is defined.]

Remark. Stallings also listed the following axiom:

P3 If $ab$ is defined, then $b^{-1}a^{-1}$ is defined and $(ab)^{-1} = b^{-1}a^{-1}$.

However, one can show that [P3] follows from [P1], [P2], and [P4] (see [3]). Thus [P3] is true in a pregroup $P$.

The following is also true in a pregroup $P$ (See, e.g., Paper I [7]):

Proposition. If $ab$ is defined, then $(ab)b^{-1}$ is defined and $(ab)b^{-1} = a$. Dually, if $ab$ is defined, then $a^{-1}(ab)$ is defined and $a^{-1}(ab) = b$.

Each add $P$ in the above examples are prees. Example 1.4 shows that a pregroup $P$ need not be embeddable in its universal group $G(P)$.

Stallings [12] invented the name “pregroup” for a pregroup $P$ which also satisfies the following axiom:

T1= If $ab$, $bc$, and $cd$ are defined, then $abc$ or $bcd$ is defined.

Theorem A. (Stallings, 1971) A pregroup $P$ is embedded in $G(P)$.

We note that a pregroup is a generalization of the add in Example 1.1, but not of the add $P = K \cup_A H \cup_B L$ in Example 1.2. For example, let $x \in K \setminus A$, $y \in L \setminus B$, $a \in A$, $b \in B$. Then $xa$, $ab$, and $by$ are defined in $P$, but neither $xab$ nor $aby$ need be defined. However the add $P$ in Example 1.2 does satisfy the axiom:

T2 If $ab$, $bc$, $cd$, $de$ are defined, then $abc$, $bcd$, or $cde$ is defined.

Notation: Let $A$ be a set of axioms for an add $P$. We will let $A$-pree denote a pregroup $P$ which also satisfies the axioms $A$. Thus a pregroup is a 11-pree.
[We note that in Paper I [10], the term pree was used synonymously for an
add, and hence a pree did not include axioms [P1], [P2], [P4]. However here a pree
P denotes an add which does satisfy axioms [P1], [P2], [P4]. Also, in Paper I, we
denoted an A-pree by A-pregroup.]

Consider now Baer’s Postulate XI (Consists of three parts):
(a) If ab, bc, cd exist, then a(bc) or (bc)d exist.
(b) If bc, cd and a(bc) exist, then ab or (bc)d exist.
(c) If ab, bc and (bc)d exist, then a(bc) or cd exist.

Baer then states: “In certain instances it is possible to deduce properties (b),
(c) from (a); but whether or not this is true in general, the author does not know.”

The content of the following, given in four parts, appears in Paper I [10]; the
first two parts answer Baer’s question.

**Theorem T1.** In a pree P, axiom T1 is equivalent to each of the following
axioms:

1. [B1-1] If bc, cd, a(bc) are defined, then ab or (bc)d is defined.
2. [B1-2] If ab, bc, (bc)d are defined, then a(bc) or cd is defined.
3. [B1-3] If ab, (ab)c, ((ab)c)d are defined, then bc or cd is defined.
4. [B1-4] If cd, b(cd), a(b(cd)) are defined, then ab or bc is defined

Note [T1] is Baer’s Part (a), [B1-1] is Baer’s Part (b) and [B1-2] is Baer’s
Part (c).

Here we generalize Theorem T1 using the axiom [T2] instead of [T1].

**Theorem T2.** In a pree P, axiom [T2] is equivalent to each of the following
axioms:

1. [B2-1] If bc, cd, a(bc), (cd)e are defined, then ab, (bc)d, or de is defined.
2. [B2-2] If ab, (ab)c, de, (de)c are defined, then bc, cd, or (ab)c(de) is defined.

2. **Proof of Theorem T2**

First we restate axioms [T2], [B2-1], [B2-2] using different letters. Here aᵢ and
bᵢ are in the pree P.

[T2] If X = [a₁, a₂, a₃, a₄, a₅] is defined, then a triple in X is defined.

[B2-1] If b₂b₃, b₃b₄, b₁(b₂b₃), (b₂b₄)b₅ are defined, then b₁b₂, (b₂b₃)b₄, or b₂b₅ is
defined.

[B2-2] If b₁b₂, (b₁b₂)b₃, b₄b₅, b₃(b₂b₅) are defined, then b₂b₃, b₃b₄, or (b₁b₂)b₃(b₂b₅)
is defined.

**Proof that [T2] and [B2-1] are equivalent.** (1) Assume [T2] holds. Suppose
the hypothesis of [B2-1] holds, that is, suppose b₂b₃, b₃b₄, b₁(b₂b₃), (b₂b₄)b₅ are
defined. Let a₁ = b₁, a₂ = b₂b₃, a₃ = b₃⁻¹, a₄ = b₃b₄, a₅ = b₅. Then the hypothesis of [T2] holds, that is, [a₁, a₂, a₃, a₄, a₅] is defined. By [T2], one of the following is defined:

a₁a₂a₃ = b₁b₂, a₂a₃a₄ = (b₂b₃)b₄, or a₃a₄a₅ = b₂b₅. This is the conclusion of [B2-1]. Thus [T2] implies [B2-1].

(2) Assume [B2-1] holds. Suppose the hypothesis of [T2] holds, that is, suppose
[a₁, a₂, a₃, a₄, a₅] is defined. Let b₁ = a₁, b₂ = a₂a₃, b₃ = a₃⁻¹, b₄ = a₃a₄, b₅ = a₅.
Then the hypothesis of [B2-1] holds, that is, \(b_2b_3, b_3b_4, b_4(b_2b_3), (b_2b_4)b_5\) are defined. By [B2-1], one of the following is defined: \(b_1b_2 = a_1a_2a_3, (b_2b_3)b_4 = a_2a_3a_4\), or \(b_2b_5 = a_3a_4a_5\) This is the conclusion of [T2]. Thus [B2-1] implies [T2].

By (1) and (2), [T2] and [B2-1] are equivalent in a pree \(P\).

Proof that [T2] and [B2-2] are equivalent: (1) Assume [T2] holds. Suppose the hypothesis of [B2-2] holds, that is, suppose \(b_1b_2, (b_1b_2)b_3, b_4(b_2b_3)\) are defined. Let \(a_1 = b_1^{-1}, a_2 = b_1b_2, a_3 = b_3, a_4 = b_4b_5, a_5 = b_5^{-1}\). Then the hypothesis of [T2] holds, that is, \([a_1,a_2,a_3,a_4,a_5]\) is defined. By [T2], one of the following is defined: \(a_1a_2a_3 = b_2b_3, a_2a_3a_4 = (b_1b_2)b_3(b_2b_4),\) or \(a_3a_4a_5 = b_3b_4\). This is the conclusion of [B2-2]. Thus [T2] implies [B2-2].

(2) Assume [B2-2] holds. Suppose the hypothesis of [T2] holds, that is, suppose \([a_1,a_2,a_3,a_4,a_5]\) is defined. Let \(b_1 = a_1^{-1}, b_2 = a_1a_2, b_3 = a_3, b_4 = a_4a_5, b_5 = a_5^{-1}\). Then the hypothesis of [B2-2] holds, that is, \(b_1b_2, (b_1b_2)b_3, b_4(b_2b_3)\) are defined. By [B2-2], one of the following is defined: \(b_2b_3 = a_1a_2a_3, b_3b_4 = a_3a_4a_5,\) or \((b_1b_2)b_3(b_2b_4) = a_2a_3a_4\). This is the conclusion of [T2]. Thus [B2-2] implies [T2].

By (1) and (2), [T2] and [B2-1] are equivalent in a pree \(P\).

Accordingly, Theorem T2 is proved.

3. Previous Results

Many authors have generalized the Stallings pregroup \([T1-pree]\) by giving a weaker set of axioms than \([P5]=\[T1]\) which also guarantees that a pree \(P\) is embeddable in \(G(P)\). First we restate these axioms, which also appear in Paper I, and then we restate the relevant Theorem B which also appears in Paper I.

\([\text{Sn}, n \geq 4]\) (Baer 1953) Suppose \(a_1a_2^{-1}, a_2a_3^{-1}, \ldots , a_na_1^{-1}\) are defined. Then, for some \(i, j, a_{i+j-2}^{-1} (\mod n)\) is defined.

\([K]\) (Kushner 1988) If \(ab, bc, cd\) and \((ab)(cd)\) are defined, then \(abc\) or \(bcd\) is defined.

\([\text{Tu}]\) (Kushner and Lipschutz 1993) If \(X = [a_1,a_2, \ldots , a_{n+3}]\) is defined, then a triple in \(X\) is defined.

\([L]\) (Lipschutz 1994) Suppose \(ab, bc, cd\) are defined, but \([ab, cd]\) and \([a, bc, d]\) are reduced. If \((ab)z\) and \(z^{-1}(cd)\) are defined, then \(bz\) and \(z^{-1}c\) are defined.

\([M]\) (Baer 1950 and Lipschutz 1994) Equivalent fully reduced words have the same length.

We note that the axiom \([\text{Tu}]\) holds in the tree pree \(P\) in Example 1.3 when the tree has diameter \(\leq n\). Thus [T2] holds for a star graph, that is, a graph of diameter 2. We also note that Axiom [M] is analogous to the following axiom of Baer [1, page 684]: “Similar irreducible vectors have the same length”.

Theorem B. Each of the following prees \(P\) is embeddable in \(G(P)\):

(1) Sn-pree (Baer 1953, [1]);
(2) KT2-pree (Kushner and Lipschutz 1988, [7]);
(3) T2-pree (Kushner 1978, [6], and Hoare 1992, [4]);
(4) KT3-pree (Kushner and Lipschutz 1993, [8]);
(5) KLM-pree (Lipschutz 1994, [9]);
(6) KL-pree = S4S5-pree (Gilman 1998, [2], and Hoare 1998, [5]).

4. Generalizations

One of the purposes in this paper is to generalize Theorem T2. We have the following axioms:

\[ [T6] \] Suppose \( X = [a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9] \) is defined. Then a triple in \( X \) is defined.

\[ [B6-1] \] Suppose all the following are defined: (1) \( b_2 b_3, b_3 b_4, b_1 (b_2 b_3), (b_3 b_4) b_5 \), (2) \( b_6 b_7, b_2 b_6, (b_6 b_7) b_6 \). Then one of the following is defined: \( b_1 b_2, (b_2 b_3) b_4, b_2 b_5, (b_4 b_5) b_5, b_3 b_6, (b_6 b_7) b_7, b_5 b_6, (b_6 b_7) b_8 \) or \( b_8 b_9 \).

\[ [B6-2] \] Suppose all the following are defined: (1) \( b_1 b_2, (b_1 b_2) b_3, b_1 b_5, b_1 (b_4 b_5) \), (2) \( b_5 b_6, (b_3 b_5) b_7, b_7 b_6, b_7 b_8, (b_7 b_8) b_7 \). Then one of the following is defined: \( b_2 b_3, (b_2 b_3) b_4, b_3 b_6, (b_5 b_6) b_5, b_5 b_7, (b_5 b_7) b_7, b_7 b_8 \), or \( b_7 b_8 \).

(Note that (2), in both cases, can be obtained from (1) by adding 4 to each subscript.)

**Theorem T6.** (1) In a pree \( P \), axiom [T6] is equivalent to [B6-1]. (2) In a pree \( P \), axiom [T6] is equivalent to [B6-2].

**Proof of Theorem T6(1).** (1) Proof that [T6] implies [B6-1]. Assume [T6] holds. Suppose the hypothesis of [B6-1] holds, that is, the following are defined: \( b_2 b_3, b_3 b_4, b_1 (b_2 b_3), (b_3 b_4) b_5, b_6 b_7, b_2 b_6, (b_6 b_7) b_6 \). Let \( a_1 = b_1, a_2 = b_2 b_3, a_3 = b_3, a_4 = b_4, a_5 = b_5, a_6 = b_6 b_7, a_7 = b_7, a_8 = b_8 b_7, a_9 = b_9 \). Then the hypothesis of [T6] holds, that is, \( [a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9] \) is defined. By [T6], one of the following is defined: \( a_1 a_2 a_3 = b_1 b_2, a_2 a_3 a_4 = (b_2 b_3) b_4, a_3 a_4 a_5 = b_4 b_5, a_4 a_5 a_6 = (b_3 b_4) b_5 (b_6 b_7), a_5 a_6 a_7 = b_6 b_7, a_6 a_7 a_8 = (b_6 b_7) b_8 \), or \( a_7 a_8 a_9 = b_8 b_9 \). This is the conclusion of [B6-1]. Thus [T6] implies [B6-1].

(2) Proof that [B6-1] implies [T6]. Assume [B6-1] holds. Suppose the hypothesis of [T6] holds, that is, suppose \( [a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9] \) is defined. Let \( b_1 = a_1, b_2 = a_2 a_3, b_3 = a_3, b_4 = a_4, b_5 = a_5, b_6 = a_6 a_7, b_7 = a_7, b_8 = a_8 a_9, b_9 = a_9 \). Then the hypothesis of [T6] holds, that is, the following are defined: \( b_2 b_3, b_3 b_4, b_1 (b_2 b_3), (b_3 b_4) b_5, b_6 b_7, b_2 b_6, (b_6 b_7) b_6 \). By [B6-1], one of the following is defined: \( b_1 b_2 = a_1 a_2 a_3, (b_2 b_3) b_4 = a_2 a_3 a_4, b_5 b_6 = a_3 a_4 a_5, (b_6 b_7) b_6 = a_4 a_5 a_6, b_5 b_6 = a_5 a_6 a_7, (b_6 b_7) b_8 = a_6 a_7 a_8, or b_8 b_9 = a_7 a_8 a_9 \). This is the conclusion of [T2]. Thus [B2-1] implies [T2].

By (1) and (2), Theorem T6(1) is proved.

**Proof of Theorem T6(2).** (1) Proof that [T6] implies [B6-2]. Assume [T6] holds. Suppose the hypothesis of [B6-2] holds, that is, that the following are defined: \( b_1 b_2, (b_1 b_2) b_3, b_2 b_3, b_3 b_4, b_4 b_5, b_2 b_6, b_2 b_7, b_7 (b_6 b_7) \). Let \( a_1 = b_1 b_2, a_2 = b_1 b_2, a_3 = b_3, a_4 = b_4 b_5, a_5 = b_5, a_6 = b_6 b_7, a_7 = b_7, a_8 = b_8 b_7, a_9 = b_9 \). Then the hypothesis of [T6] holds, that is, \( [a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9] \) is defined. By [T6], one of the following is defined: \( a_1 a_2 a_3 = b_2 b_3, a_2 a_3 a_4 = (b_1 b_2) b_3 (b_4 b_5), \)
\[a_3a_4a_5 = b_3b_4, \ a_4a_5a_6 = (b_4b_5)b_6, \ a_5a_6a_7 = b_5b_7, \ a_6a_7a_8 = (b_5b_7)b_7(b_8b_9), \ \text{or} \ \ a_7a_8a_9 = b_7b_8. \] This is the conclusion of \([B6-2]\). Thus \([T6]\) implies \([B6-2]\).

(2) Proof that \([B6-2]\) implies \([T6]\). Assume \([B6-2]\) holds. Suppose the hypothesis of \([T6]\) holds, that is, suppose \([a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9]\) is defined. Let \(b_1 = a_1^{-1}, \ b_2 = a_1a_2, \ b_3 = a_3, \ b_4 = a_4a_5, \ b_5 = a_5^{-1}, \ b_6 = a_5a_6, \ b_7 = a_7, \ b_8 = a_8a_9, \ b_9 = a_9^{-1}\). Then the hypothesis of \([B6-2]\) holds, that is, the following are defined: \(b_1b_2, (b_1b_2)b_3, b_4b_5, b_5(b_4b_5), b_5b_6, (b_5b_6)b_7, b_8b_9, b_7(b_8b_9)\). By \([B6-2]\), one of the following is defined: \(b_2b_3 = a_1a_2a_3, (b_1b_2)b_3(b_4b_5) = a_2a_3a_4, b_3b_4 = a_3a_4a_5, (b_1b_2)b_5 = a_4a_5a_6, b_7b_8 = a_5a_6a_7a_8, \) or \(b_7b_8 = a_7a_8a_9\). This is the conclusion of \([T6]\). Thus \([B6-2]\) implies \([T]\). By (1) and (2), Theorem \(T6(2)\) is proved.

Accordingly, Theorem \(T6\) is proved.

5. Questions

We have shown that the proof of Theorem \(T6\) is very similar to the proof of Theorem \(T2\). Likely, one can prove an analogous Theorem \(Tm\) where \(m\equiv2 \pmod 4\).

**Question 1.** Find a generalization of \(T2\) for other \(Tm\), especially \([T3]\), \([T4]\), and \([T5]\).

The following transitive order relation on a pregroup \(P\) is due to Stallings (see [11]). Let \(L(x) = \{a \in P; ax\ \text{is defined}\}\). Put \(x \leq y\) if \(L(y) \subseteq L(x)\), and put \(x < y\) if \(L(y) \subset L(x)\) and \(L(y) \neq L(x)\). The following theorem is due to Hoare [5] and Rimlinger [11].

**Theorem C.** The following conditions on a pregroup \(P\) are equivalent:

1. \([T1]\) If \(X = [w, x, y, z]\) is defined, then \(wxy\) or \(xyz\) is defined.
2. If \(x^{-1}a\) and \(a^{-1}y\) are defined but \(x^{-1}y\) is not defined, then \(a < x\) and \(a < y\).
3. If \(x^{-1}y\) is defined, then \(x \leq y\) or \(y \leq x\).

**Question 2.** Find an analogous Theorem \(C2\) for the axiom \([T2]\).

We note that the following axioms are a direct generalization of axioms \([B1-3]\) and \([B1-4]\). [B2-3] If \(ab, (ab)c, ((ab)c)d, (((ab)c)d)e\) are defined, then \(bc, cd, \text{or} \ de\) is defined. [B2-4] If \(de, c(de), b(c(de)), a(b(c(de)))\) are defined, then \(ab, bc, \text{or} \ cd\) is defined.

**Question 3.** What role, if any, do the axioms \([B2-3]\) and \([B2-4]\), and the analogous axioms \([Bn-3]\) and \([Bn-4]\), play in the embedding of a pregroup \(P\) in its universal group \(G(P)\)?

**Question 4.** Find alternate generalizations of \([B2-3]\) and \([B2-4]\), and the role they would play in the embedding of a pregroup \(P\) in its universal group \(G(P)\).

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