SHARP FUNCTION INEQUALITIES AND BOUNDNESS FOR TOEPLITZ TYPE OPERATOR RELATED TO GENERAL FRACTIONAL SINGULAR INTEGRAL OPERATOR

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Abstract. We establish some sharp maximal function inequalities for the Toeplitz type operator, which is related to certain fractional singular integral operator with general kernel. These results are helpful to investigate the boundedness of the operator on Lebesgue, Morrey and Triebel–Lizorkin spaces respectively.

1. Introduction

In recent decades, commutators have attracted a rapidly growing attention of the researchers in the field of harmonic analysis and have been widely studied by many authors [7, 19, 20]. In [4, 17, 18], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo proves a similar result when singular integral operators are replaced by the fractional integral operators in [2]. The boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel–Lizorkin and $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) spaces are obtained in [3, 8, 14]. Some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by BMO and Lipschitz functions are obtained [1, 11]. In [9, 10], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are established. The main purpose of this paper is to study the Toeplitz type operators generated by some fractional singular integral operators with general kernel and the Lipschitz and BMO functions. We will prove the
sharp maximal inequalities for the Toeplitz type operator $T_k^p$. These results are helpful to investigate the the $L^p$-norm inequality, the boundedness of the operator on Lebesgue, Morrey and Triebel–Lizorkin spaces respectively.

2. Preliminaries

At first, we should introduce some notations in the following. Throughout this paper, $Q$ denotes a cube of $\mathbb{R}^n$ with sides parallel to the axes. For any locally integrable function $f$, the sharp maximal function of $f$ is defined by

$$M^#(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|dy,$$

where, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well known that [7, 19]

$$M^#(f)(x) \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - c|dy.$$

We say that $f$ belongs to $\text{BMO}(\mathbb{R}^n)$ if $M^#(f)$ belongs to $L^\infty(\mathbb{R}^n)$ and define $\|f\|_{\text{BMO}} = \|M^#(f)\|_{L^\infty}$. It is known [19] that $\|f - f_Q\|_{\text{BMO}} \leq C \|f\|_{\text{BMO}}$. Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy.$$

For $\eta > 0$ we denote $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$. For $0 < \eta < n$ and $1 \leq r < \infty$ set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

According to [7], the $A_p$ weight can be defined as follows, for $1 < p < \infty$,

$$A_p = \left\{ w \in L_{\text{loc}}^1(\mathbb{R}^n) : \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}$$

and $A_1 = \{ w \in L^p_{\text{loc}}(\mathbb{R}^n) : M(w)(x) \leq C w(x), \text{ a.e.} \}$.

For $\beta > 0$ and $p > 1$ let $F_p^{\beta, \infty}(\mathbb{R}^n)$ be the homogeneous Triebel–Lizorkin space [14]. For $\beta > 0$ the Lipschitz space $\text{Lip}_\beta(\mathbb{R}^n)$ is the space of functions $f$ such that

$$\|f\|_{\text{Lip}_\beta} = \sup_{x, y \in \mathbb{R}^n \atop x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

**Definition 2.1.** Let $\varphi$ be a positive, increasing function on $\mathbb{R}^+$ and there exists a constant $D > 0$ such that $\varphi(2t) \leq D \varphi(t)$ for $t \geq 0$. Let $f$ be a locally integrable function on $\mathbb{R}^n$. Set, for $1 \leq p < \infty$,

$$\|f\|_{L^p, \varphi} = \sup_{x \in \mathbb{R}^n, d > 0} \left( \frac{\varphi(d)}{Q(x,d)} \int_{Q(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $Q(x, d) = \{ y \in \mathbb{R}^n : |x - y| < d \}$. As usual, the generalized Morrey space can be defined by $L^{p, \varphi}(\mathbb{R}^n) = \{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^p, \varphi} < \infty \}$. 
If \( \varphi(d) = d^\delta, \delta > 0 \), then \( L^{p, \varphi}(\mathbb{R}^n) = L^{p, \delta}(\mathbb{R}^n) \) are classical Morrey spaces [15, 16]. If \( \varphi(d) = 1 \), then \( L^{p, \varphi}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \) are Lebesgue spaces [13].

Since Morrey spaces can be regarded as extensions of Lebesgue spaces, it is natural and important to study the boundedness of operators on the Morrey spaces [5, 6, 12, 13]. Here we study some singular integral operators, defined as follows [1].

**Definition 2.2.** Fix \( 0 < \delta < n \). Let \( T_\delta : S \to S' \) be a linear operator such that \( T_\delta \) is bounded on \( L^2(\mathbb{R}^n) \) and has a kernel \( K \), that is, there exists a locally integrable function \( K(x, y) \) on \( \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\} \) such that \( T_\delta(f)(x) = \int_{\mathbb{R}^n} K(x, y)f(y) \, dy \) for every bounded and compactly supported function \( f \), where \( K \) satisfies: there is a sequence of positive constant numbers \( \{C_j\} \) such that for any \( j \geq 1 \),

\[
\int_{2|y-z|<|x-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|) \, dx \leq C,
\]

\[
(1) \quad \left( \int_{2^k|z-y|<|x-y|<2^{k+1}|z-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|)^{1/q} \, dy \right)^{1/q} \leq C_k(2^k|z-y|)^{-n/q' + \delta},
\]

where \( 1 < q' < 2 \) and \( 1/q + 1/q' = 1 \). We write \( T_\delta \in \text{GSIO}(\delta) \).

Moreover, if \( b \) is a locally integrable function on \( \mathbb{R}^n \), then the Toeplitz type operator related to \( T_\delta \) can be defined by \( T_\delta^b = \sum_{k=1}^m T_\delta^{k,1} M_b T_\delta^{k,2} \), where \( T_\delta^{k,1} \) are \( T_\delta \) or \( I \) (the identity operator), \( T_\delta^{k,2} \) are the linear operators, \( k = 1, \ldots, m \), \( M_b(f) = bf \).

**Remark 2.1.** We should point out that the classical Calderón–Zygmund singular integral operator satisfies Definition 2.2 with \( C_j = 2^{-j\delta} \) [7, 19].

**Remark 2.2.** It is obvious that the fractional integral operator with rough kernel satisfies Definition 2.2 [3], that is, for \( 0 < \delta < n \), let \( T_\delta \) be the fractional integral operator with rough kernel defined by (see [3])

\[ T_\delta f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y) \, dy, \]

where \( \Omega \) is homogeneous of degree zero on \( \mathbb{R}^n \), \( \int_{S^{n-1}} \Omega(x') \, d\sigma(x') = 0 \) and \( \Omega \in \text{Lip}_\varepsilon(S^{n-1}) \) for some \( 0 < \varepsilon \leq 1 \), and there exists a constant \( M > 0 \) such that for any \( x, y \in S^{n-1} \), \( |\Omega(x) - \Omega(y)| \leq M|x - y|^\varepsilon \). When \( \Omega \equiv 1 \), \( T_\delta \) is the Riesz potential (fractional integral operator) [2].

**Remark 2.3.** One can obtain that the commutator \([b, T_\delta](f) = bT_\delta(f) - T_\delta(bf)\) is a particular operator of the Toeplitz type operator \( T_\delta \). The Toeplitz type operators \( T_\delta^b \) are nontrivial generalizations of commutators.

### 3. Some Lemmas

We begin with some preliminary lemmas.

**Lemma 3.1.** [1] Let \( T_\delta \) be the singular integral operator as Definition 1.2. Then \( T_\delta \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^r(\mathbb{R}^n) \) for \( 1 < p < n/\delta \) and \( 1/r = 1/p - \delta/n \).
Lemma 3.2. [14] For $0 < \beta < 1$ and $1 < p < \infty$, we have
\[
\|f\|_{L_p^\beta,\infty} \approx \sup_{Q \subset \mathcal{E}} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x)-f_Q| \, dx \approx \sup_{c \in \mathbb{R}} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x)-c| \, dx. 
\]

Lemma 3.3. [7] Let $0 < p < \infty$ and $w \in \bigcup_{1 \leq r < \infty} A_r$. Then, for any smooth function $f$ for which the left-hand side is finite,
\[
\int_{\mathbb{R}^n} M(f)(x)^p w(x) \, dx \leq C \int_{\mathbb{R}^n} M^\#(f)(x)^p w(x) \, dx. 
\]

Lemma 3.4. [2] Suppose that $0 < \eta < n$, $1 \leq s < p < n/\eta$ and $1/r = 1/p - \eta/n$. Then $\|M_{\eta,s}(f)\|_{L^r} \leq C\|f\|_{L^p}$.

Lemma 3.5. Let $1 < p < \infty$, $0 < D < 2^n$. Then, for any smooth function $f$ for which the left-hand side is finite, $\|M(f)\|_{L^p\cap A_r} \leq C\|M^\#(f)\|_{L^p\cap A_r}$.

Proof. For any cube $Q = Q(x_0, d)$ in $\mathbb{R}^n$, we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [7]. Noticing that $M(\chi_Q) \leq 1$ and $M(\chi_Q)(x) \leq d^n/(|x-x_0| - d)^n$ if $x \in Q^c$, by Lemma 3.3, we have, for $f \in L^{p,\eta}(\mathbb{R}^n)$,
\[
\int_Q M(f)(x)^p \, dx = \int_{\mathbb{R}^n} M(f)(x)^p \chi_Q(x) \, dx \\
\leq \int_{\mathbb{R}^n} M(f)(x)^p M(\chi_Q)(x) \, dx \leq C \int_{\mathbb{R}^n} M^\#(f)(x)^p M(\chi_Q)(x) \, dx \\
= C \left( \int_Q M^\#(f)(x)^p M(\chi_Q)(x) \, dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^\#(f)(x)^p M(\chi_Q)(x) \, dx \right) \\
\leq C \left( \int_Q M^\#(f)(x)^p \, dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^\#(f)(x)^p \frac{|Q|}{|2^{k+1}Q|} \, dx \right) \\
\leq C \left( \int_Q M^\#(f)(x)^p \, dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} M^\#(f)(x)^p 2^{-kn} \, dy \right) \\
\leq C\|M^\#(f)\|_{L^p\cap A_r} \sum_{k=0}^{\infty} 2^{-kn} \varphi(2^{k+1}d) \leq C\|M^\#(f)\|_{L^p\cap A_r} \sum_{k=0}^{\infty} (2^{-n} D)^{k} \varphi(d) \\
\leq C\|M^\#(f)\|_{L^p\cap A_r} \varphi(d),
\]
 thus
\[
\left( \frac{1}{\varphi(d)} \int_Q M(f)(x)^p \, dx \right)^{1/p} \leq C \left( \frac{1}{\varphi(d)} \int_Q M^\#(f)(x)^p \, dx \right)^{1/p}
\]
and $\|M(f)\|_{L^p\cap A_r} \leq C\|M^\#(f)\|_{L^p\cap A_r}$. This finishes the proof. \hfill \Box

Lemma 3.6. Let $0 < D < 2^n$, $1 \leq s < p < n/\eta$ and $1/r = 1/p - \eta/n$. Then
\[
\|M_{\eta,s}(f)\|_{L^r} \leq C\|f\|_{L^p}. 
\]

The proof of Lemma 3.6 is similar to that of Lemma 3.5 by Lemma 3.4, we omit the details.
4. Theorems and Their Proofs

We shall prove the following theorems.

**Theorem 4.1.** Let the sequence \( \{C_j\} \in l^1, 0 < \beta < 1, q' \leq s < \infty \) and \( b \in \text{Lip}_\beta (\mathbb{R}^n) \). Suppose \( T_b \) is a bounded linear operator from \( L^p (\mathbb{R}^n) \) to \( L^r (\mathbb{R}^n) \) for any \( p, r \) with \( 1 < p < n/\delta \) and \( 1/r = 1/p - \delta/n \), and has a kernel \( K \) satisfying (1).

If \( T_b^g (g) = 0 \) for any \( g \in L^u (\mathbb{R}^n) \) \((1 < u < \infty)\), then there exists a constant \( C > 0 \) such that, for any \( f \in C_0^\infty (\mathbb{R}^n) \) and \( \tilde{x} \in \mathbb{R}^n \),

\[
M^\#(T_b^g (f)) (\tilde{x}) \leq C \|b\|_{\text{Lip}_\beta} \sum_{k=1}^m M_{\beta+s,s} (T^{k,2} (f)) (\tilde{x}).
\]

**Proof.** It suffices to prove for \( f \in C_0^\infty (\mathbb{R}^n) \) and some constant \( C_0 \), the following inequality holds:

\[
\frac{1}{|Q|} \int_Q |T_b^g (f) (x) - C_0| \, dx \leq C \|b\|_{\text{Lip}_\beta} \sum_{k=1}^m M_{\beta+s,s} (T^{k,2} (f)) (\tilde{x}).
\]

Without loss of generality, we may assume \( T_b^g \) are \( T_b \) \((k = 1, \ldots, m)\). Fix a cube \( Q = Q(x_0, d) \) and \( \tilde{x} \in Q \). Write, for \( f_1 = f \chi_{2Q} \) and \( f_2 = f \chi_{(2Q)^c} \),

\[ T_b^g (f) (x) = T_b^{b-b_0} (f) (x) = T_{b-b_0}^g (f) (x) + T_{b-b_0}^g (f) (x) = g(x) + h(x). \]

Then

\[
\frac{1}{|Q|} \int_Q |T_b^g (f) (x) - h(x_0)| \, dx \leq \frac{1}{|Q|} \int_Q |g(x)| \, dx + \frac{1}{|Q|} \int_Q |h(x) - h(x_0)| \, dx = I_1 + I_2.
\]

For \( I_1 \), choose \( 1 < r < \infty \) such that \( 1/r = 1/s - \delta/n \), by \((L^s, L^r)\)-boundedness of \( T_b \) and Hölder’s inequality, we obtain

\[
\frac{1}{|Q|} \int_Q |T_b^{k,1} M_{b-b_0}^g (T^{k,2} (f)) (x)| \, dx \leq \frac{1}{|Q|} \int_{R^n} |T_b^{k,1} M_{b-b_0}^g (T^{k,2} (f)) (x)|^r \, dx \leq C |Q|^{-1/r} \left( \int_{R^n} |M_{b-b_0}^g (T^{k,2} (f)) (x)|^s \, dx \right)^{1/s} \leq C |Q|^{-1/r} \left( \int_{2Q} |(b(x) - b_0) \|T^{k,2} (f) (x)||^s \, dx \right)^{1/s} \leq C |Q|^{-1/r} \|b\|_{\text{Lip}_\beta} |2Q|^{\beta/s} \|2Q\|^{1/(s-\beta/n)} \times \left( \frac{1}{|2Q|^{1-s(\beta+n)}} \int_{2Q} |T^{k,2} (f) (x)|^s \, dx \right)^{1/s} \leq C \|b\|_{\text{Lip}_\beta} M_{\beta+s,s} (T^{k,2} (f)) (\tilde{x}).
\]
such that, for any $T_{\delta}$ any $p, r$

This completes the proof of Theorem 4.1. □

For $I_2$, recalling that $s > q'$, we get, for $x \in Q$,

$|T_{\delta}^{k+1} M(b_{bQ}) x_{<2Q} T_{\delta}^{k+2} (f)(x) - T_{\delta}^{k+1} M(b_{bQ}) x_{<2Q} T_{\delta}^{k+2} (f)(x_0)|$

$\leq \int_{(2Q)^c} |b(y) - b_{2Q}||K(x, y) - K(x, y)||T_{\delta}^{k+2} (f)(y)|dy$

$= \sum_{j=1}^{\infty} \int_{2^j d \leq |y - x| < 2^{j+1} d} |b(y) - b_{2Q}||K(x, y) - K(x, y)||T_{\delta}^{k+2} (f)(y)|dy$

$\leq C\|b\|_{\text{Lip}_s} \sum_{j=1}^{\infty} |2^{j+1} Q|^{2/n} \left( \int_{2^j d \leq |y - x| < 2^{j+1} d} |K(x, y) - K(x, y)|^q dy \right)^{1/q}$

$x \left( \int_{2^{j+1} Q} |T_{\delta}^{k+2} (f)(y)|^{q'} dy \right)^{1/q'}$

$\leq C\|b\|_{\text{Lip}_s} \sum_{j=1}^{\infty} |2^{j+1} Q|^{2/n} C_j (2^j d)^{-n/q'} + |2^{j+1} Q|^{1/q'} + |2^{j+1} Q|^{1/q'} - (2^{-j} d)^{n/q'}$

$x \left( \int_{2^{j+1} Q} |T_{\delta}^{k+2} (f)(y)|^{q'} dy \right)^{1/q'}$

$\leq C\|b\|_{\text{Lip}_s} M_{\beta + \delta, s} (T_{\delta}^{k+2} (f)) (x) \sum_{j=1}^{\infty} C_j \leq C\|b\|_{\text{Lip}_s} M_{\beta + \delta, s} (T_{\delta}^{k+2} (f)) (\tilde{x})$,

thus

$I_2 \leq \frac{1}{|Q|} \int_{Q} \sum_{k=1}^{m} |T_{\delta}^{k+1} M(b_{bQ}) x_{<2Q} T_{\delta}^{k+2} (f)(x) - T_{\delta}^{k+1} M(b_{bQ}) x_{<2Q} T_{\delta}^{k+2} (f)(x_0)|dx$

$\leq C\|b\|_{\text{Lip}_s} \sum_{k=1}^{m} M_{\beta + \delta, s} (T_{\delta}^{k+2} (f)) (\tilde{x})$.

This completes the proof of Theorem 4.1. □

**Theorem 4.2.** Let the sequence $\{2^{i \beta} C_j\} \in l^1$, $0 < \beta < 1$, $q' \leq s < \infty$ and $b \in \text{Lip}_s (\mathbb{R}^n)$. Suppose $T_{\delta}$ is a bounded linear operator from $L^p (\mathbb{R}^n)$ to $L^r (\mathbb{R}^n)$ for any $p, r$ with $1 < p < n/\beta$ and $1/r = 1/p - \delta/n$, and has a kernel $K$ satisfying (1). If $T_{\delta}^0 (g) = 0$ for any $g \in L^u (\mathbb{R}^n)$ ($1 < u < \infty$), then there exists a constant $C$ such that, for any $f \in C^\infty_0 (\mathbb{R}^n)$ and $\tilde{x} \in \mathbb{R}^n$,

$\sup_{Q \supset \tilde{x}} \frac{1}{|Q|^{1+\beta/n}} \int_{Q} |T_{\delta}^0 (f)(x) - C_0| dx \leq C\|b\|_{\text{Lip}_s} \sum_{k=1}^{m} M_{\delta, s} (T_{\delta}^{k+2} (f)) (\tilde{x})$. 
Without loss of generality, we may assume the following inequality holds:
\[
\frac{1}{|Q|^{1+\beta/n}} \int_Q |T^b_\delta(f)(x) - C_0| \, dx \leq C \|b\|\text{Lip}_s \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\bar{x}).
\]

By using the same argument as in the proof of Theorem 4.1, we get, for \(T = \delta, b\), write
\[
T^b_\delta(f)(x) = T^{b-h_\delta}(f)(x) = T^{(b-h_\delta)\chi_{2Q}}(f)(x) + T^{(b-h_\delta)\chi_{(2Q)^c}}(f)(x) = g(x) + h(x)
\]
and
\[
\frac{1}{|Q|^{1+\beta/n}} \int_Q |T^b_\delta(f)(x) - h(x_0)| \, dx \leq \frac{1}{|Q|^{1+\beta/n}} \int_Q |g(x)| \, dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q |h(x) - h(x_0)| \, dx = I_3 + I_4.
\]

By using the same argument as in the proof of Theorem 4.1, we get, for \(1 < r < \infty\) with \(1/r = 1/s - \delta/n\),
\[
I_3 \leq \sum_{k=1}^m \frac{C}{|Q|^{1+\beta/n} |Q|^{1-1/r}} \left( \int_{2Q} |T^{k,1}_{\delta}(M_{b-h_\delta}\chi_{2Q}) T^{k,2}(f)(x)'| \, dx \right)^{1/r}
\leq \sum_{k=1}^m \frac{C}{|Q|^{1+\beta/n} |Q|^{1-1/r}} \left( \int_{2Q} |M_{b-h_\delta}\chi_{2Q} T^{k,2}(f)(x)'| \, dx \right)^{1/s}
\leq \sum_{k=1}^m \frac{C}{|Q|^{1+\beta/n} |Q|^{1-1/r} \|b\|\text{Lip}_s |2Q|^{\beta/n} \left( \int_{2Q} |T^{k,2}(f)(x)'| \, dx \right)^{1/s}
\leq C \|b\|\text{Lip}_s \sum_{k=1}^m \frac{1}{|Q|^{1-\delta/n}} \left( \int_{2Q} |T^{k,2}(f)(x)'| \, dx \right)^{1/s}
\leq C \|b\|\text{Lip}_s \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\bar{x}),
\]
\[
I_4 \leq \sum_{k=1}^m \frac{1}{|Q|^{1+\beta/n}} \int_Q \int_{2^{j+1}d \leq |y-x_0| < 2^{j+1}d} |K(x,y) - K(x_0,y)| T^{k,2}(f)(y) \, dy \, dx
\leq \sum_{k=1}^m \frac{C}{|Q|^{1+\beta/n}} \int_Q \int_{2^{j+1}d \leq |y-x_0| < 2^{j+1}d} |b_{\text{Lip}_s}| 2^{j+1} |Q|^{\beta/n} \left( \int_{2^{j+1}Q} |T^{k,2}(f)(y)'| \, dy \right)^{1/q'}
\times \left( \int_{2^{j+1}d \leq |y-x_0| < 2^{j+1}d} |K(x,y) - K(x_0,y)|^q \, dy \right)^{1/q}
\leq C \|b\|\text{Lip}_s \sum_{k=1}^m \frac{1}{|Q|^{\beta/n}} \sum_{j=1}^\infty |2^{j+1} |Q|^{\beta/n} C_j (2^{j+1} d)^{-n/q' + \delta} |2^{j+1} Q|^{1/q' - \delta/n}
\]
Suppose $\frac{1}{p} < \frac{n}{\delta} < 1$.

Theorem 4.1, we have

Without loss of generality, we may assume

This completes the proof of Theorem 4.2. □

Theorem 4.3. Let the sequence $\{jC_j\} \in l^1$, $q' \leq s < \infty$ and $b \in BMO(R^n)$. Suppose $T_\delta$ is a bounded linear operator from $L^p(R^n)$ to $L^q(R^n)$ for any $p, r$ with $1 < p < n/\delta$ and $1/r = 1/p - \delta/n$, and has a kernel $K$ satisfying (1). If $T_\delta^b(g) = 0$ for any $g \in L^q(R^n)$ ($1 < u < \infty$), then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\hat{x} \in R^n$,

\[ M^b(T_\delta^b(f))(\hat{x}) \leq C\|b\|_{BMO} \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\hat{x}). \]

Proof. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant $C_0$, the following inequality holds:

\[ \frac{1}{|Q|} \int_Q |T_\delta^b(f)(x) - C_0| \, dx \leq C\|b\|_{BMO} \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\hat{x}). \]

Without loss of generality, we may assume $T_\delta^b = T_\delta$ ($k = 1, \ldots, m$). Fix a cube $Q = Q(x_0, d)$ and $\hat{x} \in Q$. For $f_1 = f\chi_Q$ and $f_2 = f\chi_{(2Q)^c}$, similar to the proof of Theorem 4.1, we have

\[ T_\delta^b(f)(x) = T_\delta^{b-b_Q}(f)(x) = T_\delta^{(b-b_Q)\chi_{2Q}}(f)(x) + T_\delta^{(b-b_Q)\chi_{2Q^c}}(f)(x) = g(x) + h(x), \]

\[ \frac{1}{|Q|} \int_Q |T_\delta^b(f)(x) - h(x_0)| \, dx \leq \frac{1}{|Q|} \int_Q |g(x)| \, dx + \frac{1}{|Q|} \int_Q |h(x) - h(x_0)| \, dx = I_5 + I_6. \]

For $I_5$, choose $1 < t < s$, by Hölder’s inequality and the boundedness of $T_\delta$ with $1 < r < \infty$ and $1/r = 1/t - \delta/n$, we obtain

\[ \frac{1}{|Q|} \int_Q |T_\delta^{b,1}(M_{b-b_Q}\chi_{2Q}T^{k,2}(f)(x))| \, dx \]

\[ \leq \left( \frac{1}{|Q|} \int_{R^n} |T_\delta^{b,1}(M_{b-b_Q}\chi_{2Q}T^{k,2}(f))(x)|^r \, dx \right)^{1/r} \]

\[ \leq C|Q|^{-1/r} \left( \int_{R^n} |M_{b-b_Q}\chi_{2Q}T^{k,2}(f)|^r \, dx \right)^{1/r} \]

\[ \leq C|Q|^{-1/r} \left( \int_{2Q} |T^{k,2}(f)(x)|^s \, dx \right)^{1/s} \left( \int_{2Q} |b(x) - b_Q|^s(t/(s-t)dx \right)^{(s-t)/st} \]

\[ \leq C\|b\|_{BMO} \left( \frac{1}{|2Q|^{1-s/\delta}} \int_{2Q} |T^{k,2}(f)(x)|^s \, dx \right)^{1/s}. \]
\[ \leq C \| b \|_{\text{BMO}} M_{\delta,s}(T^{k,2}(f))(\tilde{x}), \]

thus
\[
I_5 \leq \sum_{k=1}^{l} \frac{1}{|Q|} \int_{Q} |T_{\delta}^{k,1} M_{(b-b_0)\times \Omega} T^{k,2}(f)(x)| \, dx \leq C \| b \|_{\text{BMO}} \sum_{k=1}^{m} M_{\delta,s}(T^{k,2}(f))(\tilde{x}).
\]

For \( I_6 \), recalling that \( s > q' \), taking \( 1 < p < \infty \) with \( 1/p + 1/q + 1/s = 1 \), we get, for \( x \in Q \),
\[
|T_{\delta}^{k,1} M_{(b-b_0)\times \Omega} T^{k,2}(f)(x) - T_{\delta}^{k,1} M_{(b-b_0)\times \Omega} T^{k,2}(f)(x_0)|
\leq \sum_{j=1}^{\infty} \int_{2^j \delta \leq \| y - x_0 \| < 2^{j+1} \delta} |K(x, y) - K(x_0, y)| |b(y) - b_{2Q}| T^{k,2}(f)(y) \, dy
\leq \sum_{j=1}^{\infty} \left( \int_{2^j \delta \leq \| y - x_0 \| < 2^{j+1} \delta} |K(x, y) - K(x_0, y)|^q \, dy \right)^{1/q}
\times \left( \int_{2^{j+1}Q} |b(y) - b_Q|^p \, dy \right)^{1/p} \left( \int_{2^{j+1}Q} T^{k,2}(f)(y)^s \, dy \right)^{1/s}
\leq C \| b \|_{\text{BMO}} \sum_{j=1}^{\infty} C_j (2^j \delta)^{-n/q + s/j} (2^j \delta)^{n/p} (2^j \delta)^{n/s - s}
\times \left( \frac{1}{|2^{j+1}Q|^{1 - s/\rho}} \int_{2^{j+1}Q} T^{k,2}(f)(y)^s \, dy \right)^{1/s}
\leq C \| b \|_{\text{BMO}} M_{\delta,s}(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} j C_j \leq C \| b \|_{\text{BMO}} M_{\delta,s}(T^{k,2}(f))(\tilde{x}),
\]

thus
\[
I_6 \leq \frac{1}{|Q|} \int_{Q} \sum_{k=1}^{l} |T_{\delta}^{k,1} M_{(b-b_0)\times \Omega} T^{k,2}(f)(x) - T_{\delta}^{k,1} M_{(b-b_0)\times \Omega} T^{k,2}(f)(x_0)| \, dx
\leq C \| b \|_{\text{BMO}} \sum_{k=1}^{m} M_{\delta,s}(T^{k,2}(f))(\tilde{x}).
\]

This completes the proof of Theorem 4.3. \( \square \)

**Theorem 4.4.** Let the sequence \( \{C_j\} \in l^1 \), \( 0 < \beta < \min(1, n - \delta) \), \( q' < p < n/(\beta + \delta) \), \( 1/r = 1/p - (\beta + \delta)/n \) and \( b \in \text{Lip}_\beta(R^n) \). Suppose \( T_\delta \) is a bounded linear operator from \( L^p(R^n) \) to \( L^r(R^n) \) and has a kernel \( K \) satisfying (1). If \( T_\delta^b(g) = 0 \) for any \( g \in L^2(R^n) \) \((1 < u < \infty)\) and \( T^{k,2} \) are the bounded operators on \( L^p(R^n) \) for \( 1 < p < \infty \), \( k = 1, \ldots, m \), then \( T_\delta^b \) is bounded from \( L^p(R^n) \) to \( L^r(R^n) \).

**Proof.** Choose \( q' < s < p \) in Theorem 4.1, we have, by Lemma 3.3 and 3.4,
\[
\| T_\delta^b(f) \|_{L^r} \leq \| M(T_\delta^b(f)) \|_{L^r} \leq C \| M^\#(T_\delta^b(f)) \|_{L^r}
\leq C \| b \|_{\text{Lip}_\beta} \sum_{k=1}^{m} \| M_{\beta + \delta,s}(T^{k,2}(f)) \|_{L^r} \leq C \| b \|_{\text{Lip}_\beta} \sum_{k=1}^{m} \| T^{k,2}(f) \|_{L^p}
\]
This completes the proof. □

**Theorem 4.5.** Let the sequence \( \{C_j\} \in l^1 \), \( 0 < \beta < \min(1, n - \delta) \), \( q' < p < n/(\beta + \delta) \), \( 1/r = 1/p - (\beta + \delta)/n \), \( 0 < D < 2^n \) and \( b \in \text{Lip}_\beta(R^n) \). Suppose \( T_\delta \) is a bounded linear operator from \( L^p(R^n) \) to \( L^r(R^n) \) and has a kernel \( K \) satisfying (1). If \( T_\delta^k(g) = 0 \) for any \( g \in L^u(R^n) \) \((1 < u < \infty)\) and \( T^{k,2} \) are the bounded operators on \( L^{p',\infty}(R^n) \) for \( 1 < p < \infty \), \( k = 1, \ldots, m \), then \( T_\delta^k \) is bounded from \( L^{p',\infty}(R^n) \) to \( L^{r',\infty}(R^n) \).

**Proof.** Choose \( q' < s < p \) in Theorem 4.1, we have, by Lemma 3.5 and 3.6,

\[
\|T^k_\delta(f)\|_{L^{p',\infty}} \leq \|M(T^k_\delta(f))\|_{L^{r',\infty}} \leq C\|M^\#(T^k_\delta(f))\|_{L^{r',\infty}} \\
\leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \|M_{k,s}(T^{k,2}(f))\|_{L^{p',\infty}} \leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^{p',\infty}} \\
\leq C\|b\|_{\text{Lip}_\beta} \|f\|_{L^{p',\infty}}.
\]

This completes the proof. □

**Theorem 4.6.** Let the sequence \( \{2^jC_j\} \in l^1 \), \( 0 < \beta < 1 \), \( q' < p < n/\delta \), \( 1/r = 1/p - \delta/n \) and \( b \in \text{Lip}_\beta(R^n) \). Suppose \( T_\delta \) is a bounded linear operator from \( L^p(R^n) \) to \( L^r(R^n) \) and has a kernel \( K \) satisfying (1). If \( T_\delta^k(g) = 0 \) for any \( g \in L^u(R^n) \) \((1 < u < \infty)\) and \( T^{k,2} \) are the bounded operators on \( L^p(R^n) \) for \( 1 < p < \infty \), \( k = 1, \ldots, m \), then \( T_\delta^k \) is bounded from \( L^p(R^n) \) to \( \mathcal{F}^{k,\infty}(R^n) \).

**Proof.** Choose \( q' < s < p \) in Theorem 4.2, we have, by Lemma 3.2 and 3.3,

\[
\|T^k_\delta(f)\|_{\mathcal{F}^{k,\infty}} \leq C\sup_{Q \subset \mathcal{E}} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T^k_\delta(f)(x) - C_0| \, dx \\
\leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \|M_{k,s}(T^{k,2}(f))\|_{L^r} \leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p} \\
\leq C\|b\|_{\text{Lip}_\beta} \|f\|_{L^p}.
\]

This completes the proof. □

**Theorem 4.7.** Let the sequence \( \{JC_j\} \in l^1 \), \( q' < p < n/\delta \), \( 1/r = 1/p - \delta/n \) and \( b \in \text{BMO}(R^n) \). Suppose \( T_\delta \) is a bounded linear operator from \( L^p(R^n) \) to \( L^r(R^n) \) and has a kernel \( K \) satisfying (1). If \( T_\delta^k(g) = 0 \) for any \( g \in L^u(R^n) \) \((1 < u < \infty)\) and \( T^{k,2} \) are the bounded operators on \( L^p(R^n) \) for \( 1 < p < \infty \), \( k = 1, \ldots, m \), then \( T_\delta^k \) is bounded from \( L^p(R^n) \) to \( L^r(R^n) \).

**Proof.** Choose \( q' < s < p \) in Theorem 4.3, we have, by Lemma 3.3 and 3.4,

\[
\|T^k_\delta(f)\|_{L^r} \leq \|M(T^k_\delta(f))\|_{L^p} \leq C\|M^\#(T^k_\delta(f))\|_{L^r} \\
\leq C\|b\|_{\text{BMO}} \sum_{k=1}^m \|M_{k,s}(T^{k,2}(f))\|_{L^r} \leq C\|b\|_{\text{BMO}} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p} \\
\leq C\|b\|_{\text{BMO}} \|f\|_{L^p}.
\]
This completes the proof. □

Theorem 4.8. Let the sequence \( \{ jC_j \} \in l^1 \), \( q' < p < n/\delta \), \( 1/r = 1/p - \delta/n \), \( 0 < D < 2^n \) and \( b \in \text{BMO}(\mathbb{R}^n) \). Suppose \( T_\delta \) is a bounded linear operator from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) and has a kernel \( K \) satisfying (1). If \( T_\delta^1(g) = 0 \) for any \( g \in L^q(\mathbb{R}^n) \) \((1 < u < \infty)\) and \( T^{k,2} \) are the bounded operators on \( L^{p,\varphi}(\mathbb{R}^n) \) for \( 1 < p < \infty \), \( k = 1, \ldots, m \), then \( T_\delta^b \) is bounded from \( L^{p,\varphi}(\mathbb{R}^n) \) to \( L^{q,\varphi}(\mathbb{R}^n) \).

Proof. Choose \( q' < s < p \) in Theorem 4.3, we have, by Lemma 3.5 and 3.6,
\[
\| T_\delta^b(f) \|_{L^{q,\varphi}} \leq \| M(T_\delta^b(f)) \|_{L^{q,\varphi}} \leq C \| M^\#(T_\delta^b(f)) \|_{L^{q,\varphi}} \\
\leq C\|b\|_{\text{BMO}} \sum_{k=1}^m \| M_{\delta,s}(T^{k,2}(f)) \|_{L^{q,\varphi}} \leq C\|b\|_{\text{BMO}} \sum_{k=1}^m \| T^{k,2}(f) \|_{L^{p,\varphi}} \\
\leq C\|b\|_{\text{BMO}} \|f\|_{L^{p,\varphi}}.
\]
This completes the proof. □

Corollary. Let \( T_\delta \in \text{GSIO}(\delta) \), that is \( T_\delta \) is the singular integral operator as Definition 1.2. Then Theorems 4.1–4.8 hold for \( T_\delta^b \).

References