COMPARISON OF RANDOM S-BOX GENERATION METHODS

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Abstract. Random bijective S-box generation methods are considered. An alternative S-box generation method by forming compositions of permutations from some fixed set is proposed. Experiments show that the rate of acceptable S-boxes for all the methods considered is essentially the same. The advantage of the composition method is an obvious parametrization, with the potentially large key space.

1. Introduction

A number of known block ciphers are of substitution-permutation (SP) type. S-boxes are used in such cipher systems as the important nonlinear component. A strong block cipher should be resistant to various attacks, such as linear and differential cryptanalysis. In SP networks this is generally achieved if the S-boxes used satisfy a number of criteria, such as strict avalanche criterion (SAC), bit independence criterion (BIC), nonlinearity, XOR Table Distribution and maximum expected linear probability (MELP, see for example [1,2]).

The S-box used in encryption process could be chosen under the control of key, instead of being fixed. For example in [1,3,4] random key-dependent S-boxes are generated once per encryption without discarding any S-boxes, and in [5] the S-boxes are generated until a good one is found.

Here we consider random bijective S-box generation methods satisfying chosen selected criteria. Jakimoski and Kocarev [6] generated the chaotic S-boxes by discretizing the exponential and logistic maps; the numbers of iterations of the discretized exponential map were considered as the keys, making the S-boxes key-dependent. The S-box generation method based on a 2D discretized chaotic Baker map was proposed in [7]. Afterwards, the 2D was further extended to a 3D one [8].

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Yin et al. [9] presented S-box generation method based on the iteration of continuous chaotic maps, with the starting points depending on the key. In [10], a generation method is proposed, which uses the Lorenz system and special shifting method to generate the S-box. The S-boxes obtained by various methods are checked afterwards, and discarded if they do not satisfy common set of criteria (see for example [2, 11, 13]).

We propose another simple method to obtain random S-boxes. After choosing some fixed set of starting S-boxes, output S-boxes are obtained by making various compositions of the starting S-boxes. The sequence of the indices of starting S-boxes used is key-controlled. These methods are experimentally compared, by generating a number of \( n \times n \) S-boxes, \( n = 8, 10, 12 \). It turns out that the rate of S-boxes satisfying all the criteria does not depend substantially on a generation method. The advantage of the proposed generation method is the size of the key space.

2. Notation

Let \( B = \{0, 1\} \). S-box of the type \( m \times n \) is a function \( f : B^m \rightarrow B^n \). S-boxes appearing inside block ciphers are expected to satisfy some standard criteria. Here we consider only bijective S-boxes, where \( m = n \). The vector \( x = (x_1, x_2, \ldots, x_n) \in B^n \) naturally corresponds to the integer \( \sum_{i=1}^{n} x_i 2^{n-i} \), which is also denoted by \( x \), without causing misunderstanding. Bijective S-box \( f : B^n \rightarrow B^n \) is a permutation of the set \( \{0, 1, \ldots, N-1\} \), where \( N = 2^n \). It is represented by the integer vector \( \{f(0), f(1), \ldots, f(N-1)\} \) which is also denoted by \( f \), because there is no danger of confusion. A cycle is a permutation \( (c_1, c_2, \ldots, c_k) \) mapping \( c_i \) into \( c_{i+1} \), \( 1 \leq i < k \), and mapping \( c_k \) into \( c_1 \), leaving the other elements in place. Transposition is a cycle of length two. The composition \( h = fg \) of two permutations \( f \) and \( g \) of the same set \( A \), is the permutation mapping each \( x \in A \) into \( h(x) = f(g(x)) \).

Let \( e_i = [\delta_{i,1}, \delta_{i,2}, \ldots, \delta_{i,n}]^T \), where
\[
\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases}
\]
and let \((\cdot)^T\) denote a matrix transpose. Let \( \oplus \) denote exclusive or operation.

The nonlinearity of a function \( \phi : B^n \rightarrow B \) is defined by
\[
N(\phi) = 2^n - \frac{1}{2} \max_{a \in B^n} \left| \sum_{x \in B^n} (-1)^{f(x)+a \cdot x} \right|
\]
(see [14] for example). The complexity of computation of \( N(\phi) \) using Walsh transform is \( O(n 2^n) \). Let \( P_{i,j}(f) = 2^{-n} \sum_{x \in B^n} f_{i}(x) \oplus f_{j}(x \oplus e_{i}) \).

Linear probability is defined by \( LP(a, b) = (2^{-n} \sum_{x \in B^n} (-1)^{a \cdot x + b \cdot f(x)})^2 \).

Given a function \( f : B^n \rightarrow B^n \), \( f = (f_1, f_2, \ldots, f_n) \) (where \( f_i : B^n \rightarrow B \)), let
\[
N(f) = \min \{N(f_i) \mid 1 \leq i \leq n\}, \\
B(f) = \min \{N(f_i \oplus f_j) \mid 1 \leq i < j \leq n\}, \\
S(f) = \frac{1}{n^2} \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} \left| \frac{1}{2} - P_{i,j}(f) \right|
\]
The value \( N(f) \) is a measure of the nonlinearity of \( f \). Furthermore, \( S(f), B(f), X(f) \) and \( L(f) \) measure degree to which \( f \) satisfy strict avalanche criterion (SAC), output bits independence criterion (BIC), equiprobable input/output XOR distribution criterion (XOR) and maximum expected linear probability (MELP), respectively. Equiprobable input/output XOR distribution criterion (XOR) is also known as maximum expected differential probability (MEDP).

Let \( a, c, d \) be integers, \( 0 < a, c, d < 2^{n-1} \), and let \( 0 < b, e < 1 \). We say that the S-box \( f : B^n \to B^n \) is \((a, b, c, d, e)\) acceptable if

\[
N(f) \geq a, \quad S(f) \leq b, \quad B(f) \geq c, \quad X(f) \leq d, \quad L(f) < e.
\]

These five criteria will be denoted by \( C_1, C_2, C_3, C_4, C_5 \), respectively. The values \( N(f), S(f), B(f), X(f), L(f) \) can be computed in time \( O(n2^n), O(n^22^n), O(n^32^n), O(n4^n), O(n^4n) \), respectively. Therefore, it is most efficient to check if \( f \) satisfy the conditions \( \mathbb{I} \) following the order \( C_1, i = 1, 2, 3, 4, 5; \) if \( f \) fails to satisfy \( C_i \) for some \( i, 1 \leq i \leq 5 \), the computation is stopped, and \( f \) is discarded.

3. Random S-box generation methods

For given \( n \), after choosing appropriate values of \( a, b, c, d, e \), all \( n \times n \) S-box generation algorithms that will be considered, generate a sequence of permutations of \( \{0, 1, \ldots, N-1\}, N = 2^n \), leaving only \((a, b, c, d, e)\) acceptable ones. There have been recently proposed a number of random bijective S-box generation algorithms, differing mainly in a random permutation generation method used. Each permutation is then checked to see if it satisfies conditions \( \mathbb{I} \).

For the sake of completeness, the two “classical” algorithms to obtain random permutation of \( \{0, 1, \ldots, N-1\} \) are included here. They both use a random number generator \( \text{rnd}() \), giving the pseudorandom numbers uniformly distributed on \([0, 1]\).

The algorithm \texttt{perm1} obtains the \( i \)-th element \( f[i] \) of a permutation \( f \) repeatedly computing \( a = \lfloor N \cdot \text{rnd()} \rfloor \) until \( a \) is different from all \( f[j], 0 \leq j < i \). The average number of calls to \( \text{rnd}() \) is \( \sum_{i=0}^{N-1} \frac{N}{N-i} = O(N \log N) \).

The algorithm \texttt{perm2} (Knuth shuffle, see [15]) gives the permutation equal to the product of \( N-1 \) transpositions \( (N-i, r[i]), i = 0, 1, \ldots, N-1, \) where \( r[i] = \lfloor (N-i) \cdot \text{rnd}() \rfloor \). The number of calls to \( \text{rnd}() \) in the worst case is \( O(N) \), explaining why \texttt{perm2} is much more efficient than \texttt{perm1}.

Denote by \( S0 \) the \( n \times n \) S-box generating algorithm using \texttt{perm2}, and the standard RNG (random number generator) \( \text{rnd}() \), the part of C language.

The method proposed in [7] (which will be denoted by \( S1 \)) uses the random number generator based on recurrent sequence \( x_m = \tau(x_{m-1}), m \geq 1, x_0 \in [0, 1] \), where \( \tau(x) = \mu(1 - x), \mu \in [4 \cdot \frac{2^n - 1}{2^n}, 4] \). Here \( \mu \) is chosen so that the length of the interval \( [0, 1] \setminus \{\tau(x) \mid x \in [0, 1]\} \) is at most \( 1/(2N) \). Given \( x \), the state of the RNG, the next random number generated \( \text{rnd}() \) is \( \tau(x) \), which also replaces the state \( x \). After obtaining the permutation \( \pi \) by \texttt{perm1}, the permutation \( \pi \beta^0 \)
is returned, where $\beta$ is the Baker 2D-map [7]. Experiments show that the rate of $(a, b, c, d, e)$-acceptable S-boxes is not changed if the permutations are generated by more efficient algorithm perm2.

The method proposed in [8] (which will be called $S_2$) uses similar RNG, based on the recurrent sequence, where $\tau(x) = \cos(k \arccos(x))$ ($k \in R$); instead of Baker 2D map $\beta$, the Baker 3D-map [8] is used.

The method from [9] (which will be called $S_3$) gives random permutations of $B^n$ depending on the key — the sequence $o$ of $N = 2^n$ integers from the interval $[0, N-1]$. The permutation returned is the product of $N-1$ transpositions $(i, r[i])$, $i = 0, 1, \ldots, N-1$, where

$$r[i] = \left\lfloor N \tau^m \left( \frac{i}{2^n} + \frac{a(i)}{4^n} \right) \right\rfloor,$$

and $m = 9$, for example.

4. Proposed S-box generation method

We now describe the proposed simple algorithm, which will be called $S_4$. The set of $k > 1$ fixed bijective starting S-boxes $f_1, f_2, \ldots, f_k$ is used. For an arbitrary sequence of $m > 1$ indexes $i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, k\}$, $S_4$ returns the S-box $\prod_{j=1}^m f_{i_j}$. For example, if $n = 8$, $k = 5$, and $f_1, f_2, f_3, f_4$ are the bijective chaotic S-boxes from [7], [8], [16], [17] respectively, and if $f_5$ is the AES S-box, then the S-box $f_5 f_1 f_4 f_3 f_5 f_1 f_4 f_3 f_5$ is good enough — it is $(106, 0.02856, 100, 10, 0.071)$-acceptable.

The S-box generation method could be incorporated in practical system as follows. Communicating parties A and B share the set of starting S-boxes, the parameter $(a, b, c, d, e)$ values and the key for chosen PRNG (pseudo random number generator). The PRNG is used to generate index sequences, until the first $(a, b, c, d, e)$-acceptable S-box is found. Let $K$ be the total number of S-boxes to be generated by the system, i.e., the number of different keys; we suppose that keys are not to be repeated. The key space size $\log_2 K$ could be estimated as follows. Denote by $\delta$ the probability that the randomly chosen S-box is $(a, b, c, d, e)$-acceptable. Denote by $\epsilon$ the probability that all S-boxes generated are different. Starting from the approximate expression (birthday problem, see for example [18], $\epsilon$ small)

$$\epsilon \simeq \frac{K^2}{2 \delta (2^n)!},$$

the size of key space is approximately $\frac{1}{4} \log_2 N! + \frac{1}{4} \log_2 (2\epsilon \delta)$. If, for example, $\epsilon = \delta = 2^{-20} \simeq 10^{-6}$, then the size of key space is approximately $\frac{1}{4} \log_2 N! - 9.5 \simeq 832, 4375, 21615$ bits for $n = 8, 10, 12$ respectively. The speed of the S-box generation is proportional to the selection rate $\delta$. It is determined mostly by the efficiency of checking the conditions $C_4$ and $C_5$. A small change in key bits causes substantially different sequence of index sequences, and therefore substantially different output S-boxes.

5. Experimental results

In order to compare the rates of $(a, b, c, d, e)$-acceptable S-boxes that could be obtained by algorithms $S_0$, $S_1$, $S_2$, $S_3$ and $S_4$, each of these algorithms is used
Comparison of Random S-Box Generation Methods

Table 1. The bounds \((a, b, c, d, e)\) used for corresponding characteristics of S-boxes.

<p>| | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>8</td>
<td>106</td>
<td>0.030</td>
</tr>
<tr>
<td>10</td>
<td>456</td>
<td>0.015</td>
</tr>
<tr>
<td>12</td>
<td>1920</td>
<td>0.008</td>
</tr>
</tbody>
</table>

Table 2. The rates of S-boxes satisfying one of five criteria, among those satisfying previous ones.

\[ M_0 = 10^7 \]

\[
\begin{array}{cccccccc}
\text{S0} & 1455 & 0.015 & 257 & 17.7 & 43 & 16.7 & 14 & 32.6 & 9 & 64.3 \\
\text{S1} & 1452 & 0.015 & 246 & 16.9 & 48 & 19.5 & 15 & 31.2 & 9 & 60.0 \\
\text{S2} & 1431 & 0.014 & 282 & 19.7 & 52 & 18.4 & 21 & 40.4 & 12 & 57.1 \\
\text{S3} & 1514 & 0.015 & 273 & 18.0 & 48 & 17.6 & 10 & 20.8 & 7 & 70.0 \\
\text{S4} & 1340 & 0.013 & 236 & 17.6 & 33 & 14.0 & 16 & 48.5 & 14 & 56.2 \\
\end{array}
\]

\[ M_0 = 10^5 \]

\[
\begin{array}{cccccccc}
\text{S0} & 2336 & 2.3 & 87 & 3.7 & 48 & 55.2 & 17 & 35.4 & 16 & 64.1 \\
\text{S1} & 2273 & 2.3 & 100 & 4.4 & 40 & 40.0 & 16 & 40.0 & 15 & 93.7 \\
\text{S2} & 2265 & 2.3 & 82 & 3.6 & 33 & 40.2 & 11 & 33.3 & 11 & 100.0 \\
\text{S3} & 2282 & 2.3 & 87 & 3.8 & 41 & 47.1 & 14 & 34.1 & 13 & 92.9 \\
\text{S4} & 2267 & 2.3 & 84 & 3.7 & 39 & 46.4 & 14 & 35.9 & 14 & 100.0 \\
\end{array}
\]

\[ M_0 = 10^4 \]

\[
\begin{array}{cccccccc}
\text{S0} & 672 & 6.7 & 52 & 7.7 & 21 & 40.4 & 6 & 28.6 & 6 & 100.0 \\
\text{S1} & 619 & 6.2 & 47 & 7.6 & 21 & 44.7 & 5 & 23.8 & 5 & 100.0 \\
\text{S2} & 663 & 6.6 & 57 & 8.6 & 26 & 45.6 & 10 & 38.5 & 10 & 100.0 \\
\text{S3} & 624 & 6.2 & 52 & 8.3 & 25 & 48.1 & 9 & 36.0 & 9 & 100.0 \\
\text{S4} & 676 & 6.8 & 50 & 7.4 & 18 & 36.0 & 6 & 33.3 & 6 & 100.0 \\
\end{array}
\]

to generate and check \(M = M(n) = 10^7, 10^5, 10^4\) random S-boxes for \(n = 8, 10, 12\) respectively.

Let \(M_0 = M\), and let \(M_i\) denote the number of S-boxes satisfying all the criteria \(C_0, \ldots, C_i, 1 \leq i \leq 5\). The values of bounds \((a, b, c, d, e)\) are chosen so that the ratio \(M_i/M_{i-1}\) is not unreasonably small, \(1 \leq i \leq 5\). The bounds for \(n = 8\) are stronger than or equal to those from \([7, 8, 10, 16, 17, 19]\) except for nonlinearity bound in \([19]\) obtained with the heuristic S-box generation method. No S-box satisfying \(C_1\) with \(a = 108\) was found, so the value \(a = 106\) was chosen. The bounds chosen are shown in Table 1.
The results are given in Table 2. The numbers $M_i$ and the ratios $100M_i/M_{i-1}$, $i = 1, 2, 3, 4, 5$, are shown for $n = 8, 10, 12$. It is seen that the ratios $100M_i/M_{i-1}$ do not vary substantially for different S-box generation methods.

6. Conclusion

Several methods for random S-box generation have been proposed in recent years. The results show that the rates of good S-boxes among those generated by various methods do not depend substantially on the method of generation. The consequence of the large complexity of the S-box testing is that the total generation times per one good S-box obtained do not differ substantially, also. The advantage of the proposed composition method is a possibility to achieve a large key space.

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