L-PONOMAREV SYSTEM AND IMAGES
OF LOCALLY SEPARABLE METRIC SPACES

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Abstract. We introduce the notion of an $L$-Ponomarev system $(f, M, X, P^*_n)$, and give characterizations of certain mss-image (resp., mssc-images) of locally separable metric spaces. As an application, we get a new characterization of quotient mss-images (mssc-images) of locally separable metric spaces, which is helpful in solving Velichko's question (1987).

1. Introduction

Lin in [15] introduced the concept of mss-maps (resp., mssc-maps) to characterize spaces with certain $\sigma$-locally countable (resp., $\sigma$-locally finite) networks by mss-image (resp., mssc-images) of metric spaces. After that, some characterizations for certain mss-image (resp., mssc-images) of metric (or semi-metric) spaces are obtained by many authors ([10, 13, 14], for example).

Velichko [26] proved that a space $X$ is a pseudo-open $s$-image of a locally separable metric space iff $X$ is a locally separable space which is a pseudo-open $s$-image of a metric space, and posed the following interesting question about quotient and $s$-images of metric spaces.

Question 1.1. Find a $\Phi$-property such that a space $X$ is a quotient and $s$-image of a metric and $\Phi$-space iff $X$ is a $\Phi$-space which is a quotient and $s$-image of a metric space.

Recently, Dung gave some characterizations for certain mss-image (resp., mssc-images) of locally separable metric spaces in the class of regular and $T_1$-spaces (see in [3, 4]). This leads us to consider the following question.

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Question 1.2. Find a $\Phi$-property such that a space $X$ is a quotient and msss-image (mssc-image) of a metric and $\Phi$-space iff $X$ is a $\Phi$-space which is a quotient and msss-image (resp., mssc-image) of a metric space.

In this paper, we introduce the notion of a generalized Ponomarev system $(f, M, X, P_n)$, calling it an $L$-Ponomarev system, and then prove some statements concerning the properties of such systems corresponding to $\sigma$-locally finite and $\sigma$-locally countable Lindelöf networks. As an application, we get a new characterization of quotient msss-images (mssc-images) of locally separable metric spaces, give an affirmative answer to Question 1.2, and we get an affirmative answer to Question 2.17 from [3].

Throughout this paper, all spaces are assumed to be Hausdorff, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $K \subset X$ and $\mathcal{P}$ be a collection of subsets of $X$, we denote $(\mathcal{P})_x = \{ P \in \mathcal{P} : x \in P \}$, $\mathcal{P}_K = \{ P \in \mathcal{P} : P \cap K \neq \emptyset \}$. For a sequence $\{x_n\}$ converging to $x$ and $P \subset X$, we say that $\{x_n\}$ is eventually in $P$ if $\{x\} \cup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}$ is frequently in $P$ if some subsequence of $\{x_n\}$ is eventually in $P$.

Definition 1.1. [2, 17] Let $\mathcal{P} = \bigcup \{\mathcal{P}_x : x \in X\}$ be a cover of a space $X$. Assume that $\mathcal{P}$ satisfies the following (a) and (b) for every $x \in X$.

(a) $\mathcal{P}_x$ is a network at $x$.
(b) If $P_1, P_2 \in \mathcal{P}_x$, then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is a weak base for $X$, if for $G \subset X$, $G$ is open in $X$ iff for every $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$.

(2) $\mathcal{P}$ is an $sn$-network (resp., so-network) for $X$, if every element of $\mathcal{P}_x$ is a sequential neighborhood of $x$ (resp., sequentially open in $X$) for every $x \in X$.

Definition 1.2. Let $X$ be a space and $\mathcal{P}$ be a cover of $X$.

(1) $\mathcal{P}$ is a Lindelöf (resp., compact) cover, if each element of $\mathcal{P}$ is Lindelöf (resp., compact).

(2) $X$ is an $\mathbb{N}_0$-space, if $X$ is a regular space with a countable cs*-network.

(3) $X$ is an $H_\mathbb{N}_0$-space, if $X$ has a countable cs*-network.

Definition 1.3. Let $f : X \to Y$ be a map.

(1) $f$ is weak-open [27], if there exists a weak base $\mathcal{B} = \bigcup \{ \mathcal{B}_y : y \in Y \}$ for $Y$, and for every $y \in Y$, there exists $x \in f^{-1}(y)$ such that for each open neighborhood $U$ of $x$, $B \subset f(U)$ for some $B \in \mathcal{B}_y$.

(2) $f$ is 1-sequence-covering [17], if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that each sequence converging to $y$ is an image of some sequence converging to $x$.

(3) $f$ is 2-sequence-covering [17], if for every $y \in Y$, $x_y \in f^{-1}(y)$, and sequence $\{y_n\}$ converging to $y$ in $Y$, there exists a sequence $\{x_n\}$ converging to $x_y$ in $X$ with each $x_n \in f^{-1}(y_n)$.

(4) $f$ is an msss-map (resp., mss-map) [15], if $X$ is a subspace of the product space $\prod_{i \in \mathbb{N}} X_i$ of a family $\{X_i : i \in \mathbb{N}\}$ of metric spaces and for each $y \in Y$,
there is a sequence \( \{ V_i : i \in \mathbb{N} \} \) of open neighborhood’s of \( y \) such that each \( p_i f^{-1}(V_i) \) is separable in \( X_i \) (resp., each \( \text{cl}(p_i f^{-1}(V_i)) \) is compact in \( X_i \)).

**Definition 1.4.** For a cover \( \mathcal{P} \) of a space \( X \), let \( (P) \) be a (certain) covering-property of \( \mathcal{P} \). Let us say that \( \mathcal{P} \) has property \( \sigma-(P) \), if \( \mathcal{P} \) can be expressed as \( \bigcup \{ P_n : n \in \mathbb{N} \} \), where each \( P_n \) having the property \( (P) \) and \( P_n \subset \mathcal{P}_{n+1} \) for all \( n \in \mathbb{N} \).

For some undefined or related concepts, we refer the reader to [18].

### 2. Main results

From now on, let us restrict the properties \((P)\) and \( \alpha(P) \) to the following.

1. \((P)\) are locally finite, locally countable.
2. \( \alpha(P) \) is mss if \((P)\) is locally finite, and \( \alpha(P) \) is mss if \((P)\) is locally countable.

**Notation 2.1.** Let \( \mathcal{P} = \bigcup \{ P_n : n \in \mathbb{N} \} \) be a Lindelöf network having property \( \sigma-(P) \) for a space \( X \). For each \( n \in \mathbb{N} \), we put \( P^*_n = \{ X \} \cup P_n = \{ P_\alpha : \alpha \in \Lambda_n \} \) and endow \( \Lambda_n \) with the discrete topology. Assume that for each \( x \in X \), there exists a network \( \{ P_{\alpha_n} : n \in \mathbb{N} \} \) at \( x \) with \( \alpha_n \in \Lambda_n \). Then, \( M = \left\{ \alpha = (\alpha_n) \in \prod_{n \in \mathbb{N}} \Lambda_n : \{ P_{\alpha_n} \} \text{ forms a network at some point } x_\alpha \in X \right\} \) is a metric space and the point \( x_\alpha \) is unique in \( X \) for every \( \alpha \in M \). Define \( f : M \to X \) by \( f(\alpha) = x_\alpha \). Let us call \((f, M, X, P^*_n)\) an L-Ponomarev system.

**Remark 2.1.** (1) Let \( \mathcal{P} = \bigcup \{ P_n : n \in \mathbb{N} \} \) be a Lindelöf network of \( X \), where each \( P_n \) having property \( (P) \). Then, \( \mathcal{P} \) is a Lindelöf network has property \( \sigma-(P) \).

(2) If \((f, M, X, P^*_n)\) an L-Ponomarev system, then \( f \) is an \( s \)-map.

**Lemma 2.1.** If \( \mathcal{P} \) is a cs-network having property \( \sigma-(P) \), then \( \mathcal{P} \) is a cfp-network.

**Proof.** Let \( \mathcal{P} = \bigcup \{ P_n : n \in \mathbb{N} \} \) be a cs-network having property \( \sigma-(P) \) for \( X \), and \( K \subset V \) with \( K \) is compact and \( V \) is open in \( X \). Since \( \mathcal{P} \) is a cs-network having property \( \sigma-(P) \), \( K \) has a countable cs-network. Thus, \( K \) is metrizable. By [19], Lemma 1.2], for each \( x \in K \), there exists \( P_x \in \mathcal{P} \) such that \( x \in \text{int}_{K}(P_x \cap K) \subset P_x \subset V \). By the regularity of \( K \), for each \( x \in K \), there exists an open neighborhood \( V_x \) in \( K \) such that \( x \in V_x \subset \text{cl}_{K}(V_x) \subset \text{int}_{K}(P_x \cap K) \). Since \( K \) is compact, there exists a finite subset \( F \) of \( K \) such that \( K \subset \bigcup_{x \in F} V_x \). Thus, \( \{ P_x : x \in F \} \) is a cfp-cover of \( K \) and \( \bigcup_{x \in F} P_x \subset U \). Therefore, \( \mathcal{P} \) is a cfp-network.

**Lemma 2.2.** If \( X \) has a Lindelöf cs*-network with property \( \sigma-(P) \), then \( X \) has a Lindelöf cs-network with property \( \sigma-(P) \).

**Proof.** Let \( \mathcal{P} = \bigcup \{ P_i : i \in \mathbb{N} \} \) be a Lindelöf cs*-network having property \( \sigma-(P) \) for \( X \). Since each element of \( \mathcal{P}_i \) is Lindelöf, each \( \mathcal{P}_i \) is star-countable. It follows from [22, Lemma 2.1] that for each \( i \in \mathbb{N} \), \( \mathcal{P}_i = \bigcup \{ Q_{\alpha}^{(i)} : \alpha \in \Lambda_i \} \), where
\( \mathcal{Q}^{(i)}_{\alpha} \) is a countable subfamily of \( \mathcal{P}_i \) for all \( \alpha \in \Lambda_i \) and \( \bigcup \mathcal{Q}^{(i)}_{\alpha} \cap \bigcup \mathcal{Q}^{(i)}_{\beta} = \emptyset \) for all \( \alpha \neq \beta \). For each \( i \in \mathbb{N} \) and \( \alpha \in \Lambda_i \), we put
\[
\mathcal{R}^{(i)}_{\alpha} = \left\{ \bigcup \mathcal{F} : \mathcal{F} \text{ is a finite subfamily of } \mathcal{Q}^{(i)}_{\alpha} \right\}.
\]
Since each \( \mathcal{R}^{(i)}_{\alpha} \) is countable, we can write \( \mathcal{R}^{(i)}_{\alpha} = \{ R^{(i)}_{\alpha,j} : j \in \mathbb{N} \} \). Now, for each \( i, j \in \mathbb{N} \), put \( \mathcal{F}^{(i)}_j = \{ R^{(i)}_{\alpha,j} : \alpha \in \Lambda_i \} \), and denote \( \mathcal{G} = \bigcup \{ \mathcal{F}^{(i)}_j : i, j \in \mathbb{N} \} \). Then, each \( R^{(i)}_{\alpha,j} \) is Lindelöf and each family \( \mathcal{F}^{(i)}_j \) has property \( (P) \). Now, we shall show that \( \mathcal{G} \) is a cs-network. In fact, let \( \{ x_n \} \) be a sequence converging to \( x \in U \) with \( U \) is open in \( X \). Since \( \mathcal{P} \) is a point-countable cs*-network, it follows from \cite{25} Lemma 3 that there exists a finite family \( \mathcal{A} \subset (\mathcal{P})_x \) such that \( \{ x_n \} \) is eventually in \( \bigcup \mathcal{A} \subset U \). Furthermore, since \( \mathcal{A} \) is finite and \( \mathcal{P}_i \subset \mathcal{P}_{i+1} \) for all \( i \in \mathbb{N} \), there exists \( i \in \mathbb{N} \) such that \( \mathcal{A} \subset \mathcal{P}_i \). So, there exists unique \( \alpha \in \Lambda_i \) such that \( \mathcal{A} \subset \mathcal{Q}^{(i)}_{\alpha} \), and \( \bigcup \mathcal{A} \in \mathcal{R}^{(i)}_{\alpha} \). Thus, \( \bigcup \mathcal{A} = R^{(i)}_{\alpha,j} \) for some \( j \in \mathbb{N} \). Hence, \( \bigcup \mathcal{A} \in \mathcal{G} \), and \( \mathcal{G} \) is a cs-network. It follows from Remark \( 2.3(1) \) \( \mathcal{G} \) is a Lindelöf cs-network having property \( \sigma-(P) \). □

**Lemma 2.3.** Let \( f : M \to X \) be a \( \alpha(P) \)-map, and \( M \) be a locally separable metric space. Then,

(1) \( X \) has a Lindelöf cs* network with property \( \sigma-(P) \), if \( f \) is sequentially-quotient.

(2) \( X \) has a Lindelöf sn-network with property \( \sigma-(P) \), if \( f \) is 1-sequence-covering.

(3) \( X \) has a Lindelöf so-network with property \( \sigma-(P) \), if \( f \) is 2-sequence-covering.

**Proof.** By \cite{15} Lemma 1.2 and by the proof of (3) \( \Rightarrow \) (1) in \cite{12} Theorem 4, there exists a base \( \mathcal{B} \) of \( M \) such that \( \mathcal{F} = f(\mathcal{B}) \) is a network for \( X \), and \( \mathcal{F} \) can be expressed as \( \bigcup \{ \mathcal{F} : n \in \mathbb{N} \} \), where each \( \mathcal{F}_n \) has property \( (P) \). Since \( M \) is locally separable, for each \( a \in M \), there exists a separable open neighborhood \( U_a \). Denote
\[
\mathcal{C} = \{ B \in \mathcal{B} : B \subset U_a, a \in M \}.
\]

Then, \( \mathcal{C} \subset \mathcal{B} \) and \( \mathcal{C} \) is a separable base for \( M \). If put \( \mathcal{P} = f(\mathcal{C}) \), then \( \mathcal{P} \subset \mathcal{F} \), and it follows from Remark \( 2.1(1) \) that \( \mathcal{P} \) is a Lindelöf network having property \( \sigma-(P) \). Thus, \( \mathcal{P} \) can be expressed as \( \bigcup \{ \mathcal{P}_n : n \in \mathbb{N} \} \), where each \( \mathcal{P}_n \) having the property \( (P) \) and \( \mathcal{P}_n \subset \mathcal{P}_{n+1} \) for all \( n \in \mathbb{N} \). Furthermore, we have

(1) If \( f \) is sequentially-quotient, then since \( \mathcal{C} \) is a base for \( M \), \( \mathcal{P} \) is a cs*-network. Therefore, \( X \) has a Lindelöf cs*-network with property \( \sigma-(P) \).

(2) If \( f \) is 1-sequence-covering, then for each \( x \in X \), there exists \( a_x \in f^{-1}(x) \) such that each sequence converging to \( x \) is an image of a sequence converging to \( a_x \). Now, for each \( x \in X \), we put \( \mathcal{G}_x = \{ f(B) : a_x \in B \in \mathcal{C} \} \), \( \mathcal{G} = \bigcup \{ \mathcal{G}_x : x \in X \} \). Then, \( \mathcal{G} \subset \mathcal{P} \) and \( \mathcal{G} \) is an sn-network. For each \( n \in \mathbb{N} \), we put \( \mathcal{G}_n = \mathcal{G} \cap \mathcal{P}_n \). Then, \( \bigcup \{ \mathcal{G}_n : n \in \mathbb{N} \} \) is a Lindelöf sn-network having property \( \sigma-(P) \) for \( X \).

(3) If \( f \) is 2-sequence-covering, then for each \( x \in X \), we put
\[
\mathcal{C}_x = \{ B \in \mathcal{C} : B \cap f^{-1}(x) \neq \emptyset \},
\]
and let $\mathcal{G}_x$ be the family of all finite intersections of members of $f(\mathcal{C}_x)$, and $\mathcal{G} = \bigcup \{ \mathcal{G}_x : x \in X \}$. Then, $\mathcal{G} \subseteq \mathcal{P}$ and $\mathcal{G}$ is an so-network. For each $n \in \mathbb{N}$, we put $\mathcal{G}_n = \mathcal{G} \cap \mathcal{P}_n$. Then, $\bigcup \{ \mathcal{G}_n : n \in \mathbb{N} \}$ is a Lindelöf so-network having property $\sigma$-(P) for $X$.

**Lemma 2.4.** Let $\mathcal{P} = \bigcup \{ \mathcal{P}_n : n \in \mathbb{N} \}$ be a Lindelöf network having property $\sigma$-(P) and $(f, M, X, \mathcal{P}_n')$ be an L-Ponomarev system. Then, the following statements hold.

1. $f$ is a $\alpha(P)$-map.
2. $M$ is locally separable.
3. $f$ is sequence-covering compact-covering, if $\mathcal{P}$ is a cs-network.
4. $f$ is 1-sequence-covering compact-covering, if $\mathcal{P}$ is an sn-network.
5. $f$ is 2-sequence-covering compact-covering, if $\mathcal{P}$ is an so-network.

**Proof.**

2. Let $a = (\alpha_i) \in M$. Then, $\{ \mathcal{P}_n \}$ is a network at some point $x_a \in X$. Thus, there exists $i_0 \in \mathbb{N}$ such that $\mathcal{P}_{\alpha_{i_0}}$ is Lindelöf. Put

$$U_a = M \cap \left\{ (\beta_i) \in \prod_{i \in \mathbb{N}} \Lambda_i : \beta_i = \alpha_i, i \leq i_0 \right\}.$$ 

Then, $U_a$ is an open neighborhood of $a$ in $M$. Now, for each $i \leq i_0$, put $\Delta_i = \{ \alpha_i \}$, and for each $i > i_0$, we put $\Delta_i = \{ \alpha \in \Lambda_i : \mathcal{P}_\alpha \cap \mathcal{P}_{\alpha_{i_0}} \neq \emptyset \}$. Then, $U_a \subseteq \prod_{i \in \mathbb{N}} \Delta_i$. Furthermore, since each $\mathcal{P}_i$ having property (P) and $\mathcal{P}_{\alpha_{i_0}}$ is Lindelöf, $\Delta_i$ is countable for every $i > i_0$. Thus, $U_a$ is separable, and $M$ is locally separable.

3. Let $\mathcal{P}$ be a cs-network. Then,

(3.1) $f$ is sequence-covering. Let $S = \{ x_n : n \in \mathbb{N} \}$ be a sequence converging to $x$ in $X$. Since $\mathcal{P}$ is a point-countable cs-network, we can write

$$\{ P \in \mathcal{P} : S \text{ is eventually in } P \} = \{ P_i : i \in \mathbb{N} \}.$$ 

On the other hand, since $\mathcal{P}_i \subseteq \mathcal{P}_{i+1}$ for all $i \in \mathbb{N}$, we can choose sequence $\{i_n\} \subseteq \mathbb{N}$ such that $i_n < i_{n+1}$, and $P_n \in \mathcal{P}_{i_n}$ for every $n \in \mathbb{N}$. Now, for each $j \in \mathbb{N}$, we take

$$F_{a_j} = \begin{cases} P_n, & \text{if } j = i_n, \\ X, & \text{if } j \neq i_n, \end{cases}$$

and $a = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Lambda_i$. Then $f(a) = x$ and $S$ is eventually in each $F_{a_j}$. Now, for each $n \in \mathbb{N}$, put $B_n = \{ (\gamma_i) \in M : \gamma_i = \alpha_i \text{ for each } i \leq n \}$. It is easy to check that $\{ B_n \}$ is a decreasing neighborhood base at $a$ in $M$ and $f(B_n) = \bigcap_{i \leq n} P_{\alpha_i}$ for all $n \in \mathbb{N}$. Because $S$ is eventually in each $f(B_n)$, it follows from [8] Lemma 6] that for each $n \in \mathbb{N}$, there exists $a_n \in f^{-1}(x_n)$ such that the sequence $\{ a_n \}$ converging to $a$ in $M$. Therefore, $f$ is sequence-covering.

(3.2) $f$ is compact-covering. Let $K$ be a compact subset of $X$. Since $\mathcal{P}$ is a Lindelöf cs-network having property $\sigma$-(P), it follows from Lemma 2.1] that $\mathcal{P}$ is a cfp-network for $X$. Furthermore, since $\mathcal{P}_K$ is countable, we can put

$$\{ Q \subseteq \mathcal{P}_K : Q \text{ is a finite cfp-cover of } K \} = \{ Q_i : i \in \mathbb{N} \}. $$
Since \( Q_n \subset P \) and \( P_n \subset P_{n+1} \) for all \( n \in \mathbb{N} \), then we can choose a sequence \( \{i_n\} \subset \mathbb{N} \) such that \( i_n < i_{n+1} \), and \( Q_n \subset P_{i_n} \) for every \( n \in \mathbb{N} \). Now, we choose a sequence \( \{A_i\} \) as follows

\[
A_j = \begin{cases} 
Q_n, & \text{if } j = i_n, \\
X, & \text{if } j \neq i_n.
\end{cases}
\]

Since each \( A_i \) is a cfp-cover for \( K \), there exists a finite subfamily \( \mathcal{H}_i = \{P_a\}_{a \in \Gamma_i} \) of \( A_i \) and a cover \( \{F_{a}\}_{a \in \Gamma_i} \) of \( K \) consisting of closed subset of \( K \) satisfying that each \( F_a \subset P_a \). Put \( L = \{(\alpha_i) \in \prod_{i \in \mathbb{N}} \Gamma_i : \bigcap_{i \in \mathbb{N}} F_{\alpha_i} \neq \emptyset \} \). Then, we have

\[
(3.2.1) \quad L \subset M, \text{ and } f(L) \subset K. \quad \text{Suppose } a = (\alpha_i) \in L, \text{ then } \bigcap_{i \in \mathbb{N}} F_{\alpha_i} \neq \emptyset. \quad \text{Pick } x_a \in \bigcap_{i \in \mathbb{N}} F_{\alpha_i}. \quad \text{Now we will show that } \{P_{\alpha_i}\} \text{ is a network at } x_a \text{ in } X. \quad \text{Then, } a \in M \text{ and } f(a) = x_a \in K, \text{ so } L \subset M \text{ and } f(L) \subset K. \quad \text{Indeed, let } V \text{ be a neighborhood of } x_a \text{ in } X. \quad \text{Since } K \text{ is a regular subspace of } X, \text{ there exists an open neighborhood } W \text{ of } x_a \text{ in } K \text{ such that } \text{cl}_K(W) \subset V. \quad \text{Since } \text{cl}_K(W) \text{ is a compact subset of } K, \text{ there exists a finite collection } Q' \text{ of } P_K \text{ such that } Q' \text{ is a cfp-cover of } \text{cl}_K(W) \text{ and } \bigcup Q' \subset V. \quad \text{On the other hand, since } K - W \text{ is a compact subset of } K \text{ satisfying } K - W \subset X - \{x_a\}, \text{ there exists a finite collection } Q'' \text{ of } P_K \text{ such that } Q'' \text{ is a cfp-cover for } K - W \text{ and } \bigcup Q'' \subset X - \{x_a\}. \quad \text{Pick } Q = Q' \cup Q''. \quad \text{Then, } Q \text{ is a cfp-cover for } K, \text{ and so } Q = Q_k \text{ for some } k \in \mathbb{N}. \quad \text{But } x_a \in F_{\alpha_k} \subset P_{\alpha_k} \subset Q_k, \text{ thus } P_{\alpha_k} \subset Q' \text{ and } P_{\alpha_k} \subset V. \quad \text{Hence, } \{P_{\alpha_i}\} \text{ is a network at } x_a \text{ in } X.
\]

\[
(3.2.2) \quad K \subset f(L). \quad \text{Assume that } x \in K. \quad \text{For each } i \in \mathbb{N}, \text{ pick } \alpha_i \in \Gamma_i \text{ such that } x \in F_{\alpha_i}. \quad \text{Put } a = (\alpha_i), \text{ it follows that } a \in L. \quad \text{By the proof of } (3.2.1), f(a) = x. \quad \text{So, } K \subset f(L).
\]

\[
(3.2.3) \quad L \text{ is compact}. \quad \text{Because each } \Gamma_i \text{ is finite, } \prod_{i \in \mathbb{N}} \Gamma_i \text{ is compact. Note that } L \subset \prod_{i \in \mathbb{N}} \Gamma_i, \text{ we only need to prove that } L \text{ is closed in } \prod_{i \in \mathbb{N}} \Gamma_i. \quad \text{In fact, let } a = (\alpha_i) \in \prod_{i \in \mathbb{N}} \Gamma_i - L. \quad \text{Then, } \bigcap_{i \in \mathbb{N}} F_{\alpha_i} = \emptyset. \quad \text{From the compactness of } K, \text{ there exists } i_0 \in \mathbb{N} \text{ such that } \bigcap_{i \leq i_0} F_{\alpha_i} = \emptyset. \quad \text{Put } W = \{(\beta_i) \in \prod_{i \in \mathbb{N}} \Gamma_i : \beta_i = \alpha_i \text{ for each } i \leq i_0\}. \quad \text{Then, } W \text{ is an open subset of } \prod_{i \in \mathbb{N}} \Gamma_i \text{ satisfying } a \in W \text{ and } W \cap L = \emptyset. \quad \text{This implies that } L \text{ is a closed subset of } \prod_{i \in \mathbb{N}} \Gamma_i. \quad \text{Therefore, } L \text{ is a compact subset of } M.
\]

\[1\] Let \( P \) be an sn-network. Then, \( X \) is sn-first countable. Since every sn-network is cs-network, it follows from \[9\] that \( f \) is a sequence-covering, compact-covering map. By Remark \[2.1(2)\] and \[1\] Proposition 2.2(1), \( f \) is 1-sequence-covering.

\[5\] Let \( P \) be an so-network. Since each so-network is a cs-network, by \[3\], it suffices to prove that \( f \) is 2-sequence-covering.

Let \( x \in X \) and \( a = (\alpha_i) \in f^{-1}(x) \). It is obvious that each \( P_{\alpha_i} \) is a sequential neighborhood of \( x \) in \( X \). For each \( n \in \mathbb{N} \), put \( B_n = \{\gamma_i \in M : \gamma_i = \alpha_i \text{ for each } i \leq n\}. \quad \text{Then, } \{B_n\} \text{ is a decreasing neighborhood base of } a \text{ in } M, \text{ and } f(B_n) = \bigcap_{i \leq n} P_{\alpha_i} \text{ for all } n \in \mathbb{N}. \quad \text{Now, let } \{x_n\} \text{ be a sequence converging to } x \text{ in } X. \quad \text{Since each } f(B_n) \text{ is a sequential neighborhood at } x \text{ in } X, \text{ it follows from } \[10\] Lemma 3.2 \text{ that for each } n \in \mathbb{N}, \text{ there exists } a_n \in f^{-1}(x_n) \text{ such that the sequence } \{a_n\} \text{ converging to } a \text{ in } M. \quad \text{Therefore, } f \text{ is 2-sequence-covering.} \quad \square
Theorem 2.1. The following are equivalent for a space $X$.

1. $X$ has a Lindelöf $c^*$-network with property $\sigma(P)$;
2. $X$ has a Lindelöf $c_0$-network with property $\sigma(P)$;
3. $X$ has a Lindelöf $c$-network with property $\sigma(P)$;
4. $X$ is a sequence-covering, compact-covering $\alpha(P)$-image of a locally separable metric space;
5. $X$ is a sequentially-quotient $\alpha(P)$-image of a locally separable metric space;
6. $X$ is a sequentially-quotient $\alpha(P)$-image of a metric space, and has an $H_0$-subspace.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). By Lemma 2.1 and Lemma 2.2.

(3) $\Rightarrow$ (4). By Lemma 2.4.

(4) $\Rightarrow$ (5). Assume that (5) holds. It suffices to prove that $X$ has an $H_0$-subspace of $H_0$-subspaces. In fact, by Lemma 2.3, $X$ has a Lindelöf $c$-network $P$ having property $\sigma(P)$. Then, each element of $P$ is an $H_0$-subspace. By the proof of (2) $\Rightarrow$ (3) in [20] Theorem 3.4, $X$ has an $H_0$-subspace.

(5) $\Rightarrow$ (6). Let $O$ be an $H_0$-subspace of $X$ and $f : M \to X$ be a sequentially-quotient $\alpha(P)$-map, where $M$ is a metric space. Similar to the proof of Lemma 2.3, there exists a base $B$ of $M$ such that $\mathcal{P} = f(B)$ with property $\sigma(P)$. Since $f$ is sequentially-quotient, $\mathcal{P}$ is a $c^*$-network for $X$.

We can assume that $\mathcal{P}$ is closed under finite intersections. Let $\mathcal{G} = \{P \in \mathcal{P} : P \subset O, O \in O\}$. Then, each element of $\mathcal{G}$ is an $H_0$-subspace. Hence, each element of $\mathcal{G}$ is Lindelöf. We now proved that $\mathcal{G}$ is a $c^*$-network. In fact, let $L$ be a sequence converging to $x \in U$ with $U$ open in $X$. Since $O$ is an $H_0$-subspace of $X$, there exists $O \in O$ such that $x \in O$. On the other hand, since $\mathcal{P}$ is a point-countable $c^*$-network, it follows from [25] Lemma 3 that there exists a finite subfamily $\mathcal{H} \subset (\mathcal{P})_x$ such that $L$ is eventually in $\bigcup \mathcal{H} \subset U$. So, the family $\{H \in (\mathcal{P})_x : \mathcal{H}$ is finite and $L$ is eventually in $\bigcup \mathcal{H} \subset U\}$ is non-empty. Furthermore, since $(\mathcal{P})_x$ is countable, we can write $\{H \in (\mathcal{P})_x : \mathcal{H}$ is finite and $L$ is eventually in $\bigcup \mathcal{H} \subset U\} = \{H_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $H_n = \bigcap_{i \leq n} \bigcup \mathcal{H}_i$. It is obvious that $L$ is eventually in each $H_n$. Now, we shall show that $H_n \subset O$ for some $n \in \mathbb{N}$. If not, for each $n \in \mathbb{N}$, there exists $x_n \in H_n - O$. Then, $\{x_n\}$ converges to $x$. Indeed, let $x \in W$ with $W$ is open in $X$. Then, $U \cap W$ is an open neighborhood of $x$. By [25] Lemma 3, there exists a finite subfamily $\mathcal{Q} \subset (\mathcal{P})_x$ such that $L$ is eventually in $\bigcup \mathcal{Q}$ and $\bigcup \mathcal{Q} \subset U \cap W$. Since $\mathcal{Q}$ is a finite subfamily of $(\mathcal{P})_x$ and $L$ is eventually in $\bigcup \mathcal{Q} \subset U$, $\mathcal{Q} = H_n$ for some $n \in \mathbb{N}$. Furthermore, since $x_i \in H_i$ for all $i \in \mathbb{N}$ and

$$H_i = \bigcap_{j \leq i} \left( \bigcup \mathcal{H}_j \right) \subset \bigcap_{j \leq n} \left( \bigcup \mathcal{H}_j \right) \subset \bigcup \mathcal{H}_n \subset W,$$

for all $i \geq n$, we get $x_i \in W$ for all $i \geq n$. Therefore, $\{x_i\}$ converges to $x$. Since $O$ is a sequential neighborhood of $x$, this implies that there exists $n \in \mathbb{N}$ such that $x_i \in O$ for all $i \geq n$. This is a contradiction to $x_i \notin O$ for all $i \in \mathbb{N}$. Thus, $H_n \subset O$ for some $n \in \mathbb{N}$. 


On the other hand, since $H_n = \bigcap_{i \leq n} (\bigcup H_i) = \bigcup \{ \bigcap_{i \leq n} F_i : F_i \in H_i \}$, and $L$ is eventually in $H_n$, it implies that for each $i \leq n$, there exists $F_i \in H_i$ such that $L$ is frequently in $F = \bigcap_{i \leq n} F_i$. Since $\mathcal{P}$ is closed under finite intersections, $F \in \mathcal{P}$. Then, $L$ is frequently in $F$, $F \subset U$ and $F \in \mathcal{G}$. Thus, $\mathcal{G}$ is a cs*-network for $X$. By Remark 2.1(1), $\mathcal{G}$ is a Lindelöf cs*-network having property $\sigma$-($\mathcal{P}$).

**Remark 2.2.** By Theorem 2.1 in case that the property ($\mathcal{P}$) is locally countable, we get an affirmative answer to Question 2.17 of [3].

By Theorem 2.1 the following corollary holds.

**Corollary 2.1.** The following are equivalent for a space $X$.

1. $X$ is a $k$-space with a Lindelöf cs*-network having property $\sigma$-($\mathcal{P}$);
2. $X$ is a $k$-space with a Lindelöf cfp-network having property $\sigma$-($\mathcal{P}$);
3. $X$ is a $k$-space with a Lindelöf cs-network having property $\sigma$-($\mathcal{P}$);
4. $X$ is a sequence-covering, compact-covering, quotient $\alpha(\mathcal{P})$-image of a locally separable metric space;
5. $X$ is a quotient $\alpha(\mathcal{P})$-image of a locally separable metric space;
6. $X$ is a local $H_\aleph_0$-space and a quotient $\alpha(\mathcal{P})$-image of a metric space.

**Remark 2.3.** By Corollary 2.1 we get an affirmative answer to the Question 1.2.

**Remark 2.4.** Let $\mathcal{P}$ be a network having property $\sigma$-($\mathcal{P}$) for a regular space $X$. Then,

1. If $\mathcal{P}$ is a cs*-network (cfp-network; cs-network), then $\mathcal{P}$ is Lindelöf iff each element of $\mathcal{P}$ is a cosmic subspace, iff each element of $\mathcal{P}$ is an $\aleph_0$-subspace.
2. If $\mathcal{P}$ is an sn-network, then $\mathcal{P}$ is Lindelöf iff each element of $\mathcal{P}$ is a cosmic subspace, iff each element of $\mathcal{P}$ is an sn-second countable subspace.
3. If $\mathcal{P}$ is an so-network, then $\mathcal{P}$ is Lindelöf iff each element of $\mathcal{P}$ is a cosmic subspace, iff each element of $\mathcal{P}$ is an so-second countable subspace.

By Theorem 2.1 and Remark 2.4, we obtain the following results for Nguyen Van Dung in case $X$ is a regular space.

**Corollary 2.2.** [3, Theorem 2.8], The following are equivalent for a regular space $X$.

1. $X$ has a $\sigma$-locally countable cs-network consisting of $\aleph_0$-subspaces;
2. $X$ has a $\sigma$-locally countable cs-network consisting of cosmic subspaces;
3. $X$ is a sequence-covering mssss-image of a locally separable metric space.

**Corollary 2.3.** [4, Theorem 2.1], The following are equivalent for a regular space $X$.

1. $X$ has a $\sigma$-locally finite cs-network consisting of $\aleph_0$-subspaces;
2. $X$ has a $\sigma$-locally finite cs-network consisting of cosmic subspaces;
3. $X$ is a sequence-covering mssss-image of a locally separable metric space.

The following results hold by means of the above results.
Theorem 2.2. The following are equivalent for a space $X$.

1. $X$ has a Lindelöf sn-network with property $\sigma(P)$;
2. $X$ is a 1-sequence-covering, compact-covering $\alpha(P)$-image of a locally separable metric space;
3. $X$ is a 1-sequence-covering $\alpha(P)$-image of a locally separable metric space;
4. $X$ is a 1-sequence-covering $\alpha(P)$-image of a metric, and has an so-cover consisting of $H-\aleph_0$-subspaces.

Corollary 2.4. The following are equivalent for a space $X$.

1. $X$ has a Lindelöf weak base with property $\sigma(P)$;
2. $X$ is a weak-open, compact-covering $\alpha(P)$-image of a locally separable metric space;
3. $X$ is a weak-open $\alpha(P)$-image of a locally separable metric space;
4. $X$ is a local $H-\aleph_0$-space and a weak-open $\alpha(P)$-image of a metric.

By Theorem 2.2 and Remark 2.4, we obtain the following results for Nguyen Van Dung in case $X$ is a regular space.

Corollary 2.5. [3, Theorem 2.11] The following are equivalent for a regular space $X$.

1. $X$ has a $\sigma$-locally countable sn-network consisting of sn-second countable subspaces;
2. $X$ has a $\sigma$-locally countable sn-network consisting of cosmic subspaces;
3. $X$ is a 1-sequence-covering msss-image of a locally separable metric space.

Corollary 2.6. [4, Theorem 2.2] The following are equivalent for a regular space $X$.

1. $X$ has a $\sigma$-locally finite sn-network consisting of sn-second countable subspaces;
2. $X$ has a $\sigma$-locally finite sn-network consisting of cosmic subspaces;
3. $X$ is a 1-sequence-covering mssss-image of a locally separable metric space.

Remark 2.5. By Theorem 2.2, it is possible to add the prefix “compact-covering” before “1-sequence-covering” in Corollary 2.5(3) and Corollary 2.6(3).

Theorem 2.3. The following are equivalent for a space $X$.

1. $X$ has a Lindelöf so-network with property $\sigma(P)$;
2. $X$ is a 2-sequence-covering, compact-covering $\alpha(P)$-image of a locally separable metric space;
3. $X$ is a 2-sequence-covering $\alpha(P)$-image of a locally separable metric space;
4. $X$ is a 2-sequence-covering $\alpha(P)$-image of a metric, and has an so-cover consisting of $H-\aleph_0$-subspaces.

Corollary 2.7. The following are equivalent for a space $X$.

1. $X$ has a Lindelöf base with property $\sigma(P)$;
2. $X$ is an open, compact-covering $\alpha(P)$-image of a locally separable metric space;
(3) $X$ is an open $\alpha(P)$-image of a locally separable metric space;
(4) $X$ is a local $H$-$\aleph_0$-space and an open $\alpha(P)$-image of a metric.

By Theorem 2.3 and Remark 2.4 we obtain the following results for Nguyen Van Dung in case $X$ is a regular space.

Corollary 2.8. \cite[Theorem 2.14]{3} The following are equivalent for a regular space $X$.

1. $X$ has a $\sigma$-locally countable so-network consisting of so-second countable subspaces;
2. $X$ has a $\sigma$-locally countable so-network consisting of cosmic subspaces;
3. $X$ is a 2-sequence-covering msss-image of a locally separable metric space.

Corollary 2.9. \cite[Theorem 2.3]{4}, The following are equivalent for a regular space $X$.

1. $X$ has a $\sigma$-locally finite so-network consisting of so-second countable subspaces;
2. $X$ has a $\sigma$-locally finite so-network consisting of cosmic subspaces;
3. $X$ is a 2-sequence-covering mssc-image of a locally separable metric space.

Remark 2.6. By Theorem 2.3 it is possible to add the prefix “compact-covering” before “2-sequence-covering” in Corollary 2.8\cite{3} and Corollary 2.9\cite{4}.

3. Examples

Example 3.1. A quotient s-image of a locally separable metric space need not be locally separable (see \cite[Example 9.8]{11} or \cite[Example 2.9.27]{16}). Then, Question \cite{17} is not true in the case $\Phi$-property is an $\aleph_0$-space (or locally separable).

Example 3.2. There exists a space $X$ with a $\sigma$-locally finite compact $k$-network (hence, $X$ has a $\sigma$-locally finite Lindelöf cs-network by Theorem \cite{2.1}, but $X$ is not locally Lindelöf (hence, $X$ has no locally countable network) (see \cite[Example 4.1(2)]{24}). Then,

1. A space $X$ has a Lindelöf cs-network with property $\sigma$-(P) need not have a locally countable cs-network.
2. In Theorem \cite[2.4]{24}, $X$ need not be local $\aleph_0$-space.

Example 3.3. $S_\omega$ is a Fréchet and $\aleph_0$-space, but it is not first countable. Then, it has a $\sigma$-locally finite Lindelöf cs-network. Since $S_\omega$ is not first countable, it doesn’t have a $\sigma$-locally countable sn-network (or weak base).

1. A space with a $\sigma$-locally finite (hence, $\sigma$-locally countable) Lindelöf cs-network need not have a $\sigma$-locally finite (or $\sigma$-locally countable) Lindelöf sn-network.
2. A $k$-space with a $\sigma$-locally finite (hence, $\sigma$-locally countable) Lindelöf cs-network need not have a $\sigma$-locally finite (or $\sigma$-locally countable) Lindelöf weak base.
Example 3.4. There exists a $g$-second countable space $X$, but it is not Fréchet (see, [23 Example 2.1]). Then, $X$ has a $\sigma$-locally finite Lindelöf weak base. Since $X$ is sequential and it is not Fréchet, $X$ does not have a $\sigma$-locally countable so-network (or weak base). Therefore,

1. A space with a $\sigma$-locally finite (hence, $\sigma$-locally countable) Lindelöf sn-network need not have a $\sigma$-locally finite (or $\sigma$-locally countable) so-network.
2. A space with a $\sigma$-locally finite (hence, $\sigma$-locally countable) Lindelöf weak base need not have a $\sigma$-locally finite (or $\sigma$-locally countable) base.

Example 3.5. There exists a space $X$ having a locally countable sn-network, which is not an $\aleph$-space (see [5 Example 2.19]). Then, $X$ has a $\sigma$-locally countable Lindelöf sn-network. Therefore,

1. A space with a locally countable sn-network need not have a $\sigma$-locally finite Lindelöf cs-network.
2. A space with a $\sigma$-locally countable Lindelöf sn-network need not have a $\sigma$-locally finite Lindelöf sn-network (or cs-network).
3. A space with a $\sigma$-locally countable Lindelöf cs-network need not have a $\sigma$-locally finite Lindelöf cs-network.

Example 3.6. Using [7 Example 3.1], it is easy to see that $X$ is Hausdorff, non-regular and $X$ has a countable base, but it is not a sequentially-quotient $\pi$-image of a metric space. Then, $X$ is not an $\aleph_0$-space. By Theorem [23], $X$ is a 2-sequence-covering (and open) mssc-image of a locally separable metric space.

1. There exists an $H$-$\aleph_0$-space, but it is not an $\aleph_0$-space.
2. A space with a $\sigma$-locally finite Lindelöf cs-network (or an sn-network, or an so-network) need not be a sequentially-quotient $\pi$, mssc-image (or msss-image) of a metric space.

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