4-DIMENSIONAL
(PARA)-KÄHLER–WEYL STRUCTURES

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Abstract. We give an elementary proof of the fact that any 4-dimensional para-Hermitian manifold admits a unique para-Kähler–Weyl structure. We then use analytic continuation to pass from the para-complex to the complex setting and thereby show that any 4-dimensional pseudo-Hermitian manifold also admits a unique Kähler–Weyl structure.

1. Introduction

1.1. Weyl manifolds. Let \((M, g, \nabla)\) be a pseudo-Riemannian manifold of dimension \(m\). A triple \((M, g, \nabla)\) is said to be a Weyl manifold and \(\nabla\) is said to be a Weyl connection if \(\nabla\) is a torsion free connection with \(\nabla g = -2\phi \otimes g\) for some smooth 1-form \(\phi\). This is a conformal theory; if \(\tilde{g} = e^{2f}g\) is a conformally equivalent metric, then \((M, \tilde{g}, \nabla)\) is a Weyl manifold with associated 1-form \(\tilde{\phi} = \phi - df\). If \(\nabla g\) is the Levi-Civita connection, we may then express \(\nabla = \nabla^\phi\) in the form:

\[
\nabla^\phi_x y = \nabla^g_x y + \phi(x)y + \phi(y)x - g(x, y)\phi^#
\]

where \(\phi^#\) is the dual vector field. Thus \(\phi\) determines \(\nabla\). Conversely, if \(\phi\) is given and if we use Equation (1.1) to define \(\nabla\), then \(\nabla\) is a Weyl connection with associated 1-form \(\phi\). We refer to [5] for further details concerning Weyl geometry.

1.2. Para-Hermitian manifolds. Let \(m = 2\bar{m}\). A triple \((M, g, J_+\) is said to be an almost para-Hermitian manifold with an almost para-complex structure \(J_+\) if \(g\) is a pseudo-Riemannian metric on \(M\) of neutral signature \((\bar{m}, \bar{m})\) and if \(J_+\) is an endomorphism of the tangent bundle \(TM\) so that \(J_+^2 = Id\) and so that \(J_+^* g = -g\); \((M, g, J_+)\) is said to be para-Hermitian with an integrable complex structure \(J_+\) if the para-Nijenhuis tensor

\[
N_{J_+}(x, y) := [x, y] - J_+[J_+x, y] - J_+[x, J_+y] + [J_+x, J_+y]
\]

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vanishes or, equivalently, if there are local coordinates \((u^1, \ldots, u^m, v^1, \ldots, v^m)\) centered on an arbitrary point of \(M\) so that \(J_u \partial_{u_i} = \partial_{v_i}\) and \(J_v \partial_{v_i} = \partial_{u_i}\).

1.3. Pseudo-Hermitian manifolds. Let \(m = 2m\). A triple \((M, g, J)\) is said to be an almost pseudo-Hermitian manifold with an almost complex structure \(J_-\) if \((M, g)\) is a pseudo-Riemannian manifold, if \(J_-\) is an endomorphism of the tangent bundle so that \(J_-^2 = -\id\) and so that \(J_-^* g = g\); \((M, g, J_-)\) is said to be a pseudo-Hermitian manifold with an integrable complex structure \(J_-\) if the Nijenhuis tensor

\[
N_{J_-}(x, y) := [x, y] + J_-[J_- x, y] + J_-[x, J_- y] - [J_- x, J_- y]
\]

vanishes or, equivalently, if there are local coordinates \((u^1, \ldots, u^m, v^1, \ldots, v^m)\) centered on an arbitrary point of \(M\) so that \(J_- \partial_{u_i} = \partial_{v_i}\) and \(J_- \partial_{v_i} = -\partial_{u_i}\).

1.4. Para-Kähler and Kähler manifolds. One says that a Weyl connection \(\nabla\) on a para-Hermitian manifold \((M, g, J)\) is a para-Kähler–Weyl connection if \(\nabla J_\pm = 0\). Similarly, one says that a Weyl connection \(\nabla\) on a pseudo-Hermitian manifold \((M, g, J_-)\) is a Kähler–Weyl connection if \(\nabla J_- = 0\).

The following is well known – see, for example, the discussion in \([9]\), \([10]\), \([11]\), and the generalization given in \([3]\) to the more general context.

**Theorem 1.1.** Let \(m \geq 6\). If \((M, g, J_\pm, \nabla)\) is a (para)-Kähler–Weyl structure, then the associated Weyl structure is trivial, i.e., there is always locally a conformally equivalent metric \(\tilde{g} = e^{2f} g\) so that \((M, \tilde{g}, J_\pm)\) is (para)-Kähler and so that \(\nabla = \nabla^{\tilde{g}}\).

By Theorem 1.1, only the 4-dimensional setting is relevant. The following is the main result of this short note; it plays a central role in the discussion of \([11]\).

**Theorem 1.2.** (1) If \(\mathcal{M} = (M, g, J)\) is a para-Hermitian manifold of signature \((2, 2)\), then there is a unique para-Kähler–Weyl structure on \(\mathcal{M}\) with \(\phi = \frac{1}{2} J_\pm \delta \Omega_\pm\).

(2) If \(\mathcal{M} = (M, g, J_-)\) is a pseudo-Hermitian manifold of signature \((2, 2)\), then there is a unique Kähler–Weyl structure on \(\mathcal{M}\) with \(\phi = -\frac{1}{2} J_- \delta \Omega_-\).

(3) If \(\mathcal{M} = (M, g, J_-)\) is a Hermitian manifold of signature \((0, 4)\), then there is a unique Kähler–Weyl structure on \(\mathcal{M}\) with \(\phi = -\frac{1}{2} J_- \delta \Omega_-\).

Assertion (3) of Theorem 1.2, which deals with the Hermitian setting, is well known – see, for example, the discussion in \([8]\). Subsequently, Theorem 1.2 was established in full generality (see \([3], [4]\)) by extending the Higa curvature decomposition \([6], [7]\) from the real to the Kähler–Weyl and to the para-Kähler Weyl contexts.

Here is a brief outline to this paper. In Section \([2]\), we show that if a (para)-Kähler–Weyl structure exists, then it is unique. In Section \([3]\) we give a direct
proof of Assertion (1) of Theorem 1.2 in the para-Hermitian setting. In Section 4, we use analytic continuation to derive Assertions (2) and (3), which deal with the complex setting, from Assertion (1). This reverses the usual procedure of viewing para-complex geometry setting as an adjunct to complex geometry and is a novel feature of this paper.

2. Uniqueness of the (para)-Kähler–Weyl structure

This section is devoted to the proof of the following uniqueness result.

\textbf{Lemma 2.1.} (1) If \( \nabla^{\phi_1} \) and \( \nabla^{\phi_2} \) are two para-Kähler–Weyl connections on a 4-dimensional para-Hermitian manifold \((M, g, J_+)\), then \( \phi_1 = \phi_2 \).

(2) If \( \nabla^{\phi_1} \) and \( \nabla^{\phi_2} \) are two Kähler–Weyl connections on a 4-dimensional pseudo-Hermitian manifold \((M, g, J_-)\), then \( \phi_1 = \phi_2 \).

\textbf{Proof.} Let \( \phi = \phi_1 - \phi_2 \) and let \( \Theta_X(Y) = \phi(X)Y + \phi(Y)X - g(X, Y)\phi^\# \). By Equation (1.1), \( \nabla_X^{\phi_1} - \nabla_X^{\phi_2} = \Theta_X \in \text{End}(TM) \). Consequently, \( \{\nabla^{\phi_1} - \nabla^{\phi_2}\}J_\pm = 0 \) implies \( [\Theta_X, J_\pm] = 0 \) for all \( X \).

We first deal with the para-Hermitian case. This is a purely algebraic computation. Let \( \{e_1, e_2, e_3, e_4\} \) be a local frame for \( TM \) so that

\begin{align*}
J_+ e_1 &= e_1, \quad J_+ e_2 = e_2, \quad J_+ e_3 = -e_3, \quad J_+ e_4 = -e_4, \\
g(e_1, e_3) &= g(e_2, e_4) = 1.
\end{align*}

(2.1)

We expand \( \phi = a_1 e^1 + a_2 e^2 + a_3 e^3 + a_4 e^4 \) and compute:

\begin{align*}
\Theta_{e_1} e_4 &= a_1 e_4 + a_4 e_1, \quad J_+ \Theta_{e_1} e_4 = -a_1 e_4 + a_4 e_1, \quad \Theta_{e_1} J_+ e_4 = -a_1 e_4 - a_4 e_1, \\
\Theta_{e_2} e_3 &= a_2 e_3 + a_3 e_2, \quad J_+ \Theta_{e_2} e_3 = -a_2 e_3 + a_3 e_2, \quad \Theta_{e_2} J_+ e_3 = -a_2 e_3 - a_3 e_2, \\
\Theta_{e_3} e_1 &= a_3 e_1 + a_1 e_3, \quad J_+ \Theta_{e_3} e_1 = a_3 e_1 - a_1 e_3, \quad \Theta_{e_3} J_+ e_1 = a_3 e_1 + a_1 e_3, \\
\Theta_{e_4} e_2 &= a_4 e_2 + a_2 e_4, \quad J_+ \Theta_{e_4} e_2 = a_4 e_2 - a_2 e_4, \quad \Theta_{e_4} J_+ e_2 = a_4 e_2 + a_2 e_4.
\end{align*}

Equating \( \Theta_{e_j} J_+ e_j \) with \( J_+ \Theta_{e_j} e_j \) then implies \( a_1 = a_2 = a_3 = a_4 = 0 \) so \( \phi = 0 \) and \( \phi_1 = \phi_2 \). This establishes Assertion (1).

Next assume we are in the pseudo-Hermitian setting. Complexify and extend \( g \) to be complex bilinear. Choose a local frame \( \{Z_1, Z_2, \bar{Z}_1, \bar{Z}_2\} \) for \( TM \otimes_{\mathbb{R}} \mathbb{C} \) so

\begin{align*}
J_- Z_1 &= \sqrt{-1} Z_1, \quad J_- Z_2 = \sqrt{-1} Z_2, \\
J_- \bar{Z}_1 &= -\sqrt{-1} \bar{Z}_1, \quad J_- \bar{Z}_2 = -\sqrt{-1} \bar{Z}_2, \\
g(Z_1, Z_1) &= 1, \quad g(\bar{Z}_2, \bar{Z}_2) = \varepsilon_2
\end{align*}

where we take \( \varepsilon_2 = +1 \) in signature \((0, 4)\) and \( \varepsilon_2 = -1 \) in signature \((2, 2)\). We set \( J_+ := -\sqrt{-1} J_- \), \( e_1 := Z_1 \), \( e_2 := Z_2 \), \( e_3 := \bar{Z}_1 \), and \( e_4 := \varepsilon_2 \bar{Z}_2 \) and apply the argument given to prove Assertion (1) (where the coefficients \( a_i \) are now complex) to derive Assertion (2). \( \square \)
3. Para-Hermitian geometry

3.1. The algebraic context. Let \((V, \langle \cdot, \cdot \rangle, J_+)\) be a para-Hermitian vector space of dimension 4. Here \(\langle \cdot, \cdot \rangle\) is an inner product on \(V\) of signature \((2,2)\) and \(J_+\) is an endomorphism of \(V\) satisfying \(J_+^2 = \text{Id}\) and \(J_+^\ast \langle \cdot, \cdot \rangle = -\langle \cdot, \cdot \rangle\). We may then choose a basis \(\{e_1, e_2, e_3, e_4\}\) for \(V = \mathbb{R}^4\) so that the relations of Equation (2.1) are satisfied. The Kähler form and orientation \(\mu\) are then given by

\[
\Omega_+ = -e^1 \wedge e^3 - e^2 \wedge e^4 \quad \text{and} \quad \mu = 2 \Omega_+ \wedge \Omega_+ = e^1 \wedge e^3 \wedge e^2 \wedge e^4.
\]

Let \(\star\) be the Hodge operator, characterized by

\[
\omega_1 \wedge \star \omega_2 = \langle \omega_1, \omega_2 \rangle e^1 \wedge e^3 \wedge e^2 \wedge e^4 \quad \text{for all } \omega_i.
\]

Consequently:

\[
\begin{align*}
\star e^1 \wedge e^3 &= -e^2 \wedge e^4, \\
\star e^2 \wedge e^4 &= -e^1 \wedge e^3, \\
\star e^1 \wedge e^3 &\neq e^2 \wedge e^4 = e^3, \\
\star e^3 \wedge e^4 &= -e^3 \wedge e^4.
\end{align*}
\]

(3.1)

3.2. Example. We begin the proof of Theorem 1.2 by considering a very specific example. Let \((x^1, x^2, x^3, x^4)\) be the usual coordinates on \(\mathbb{R}^4\), let \(\partial_i := \partial_{x_i}\), and let \(J_+\) be the standard para-complex structure:

\[
J_+ \partial_1 = \partial_1, \quad J_+ \partial_2 = \partial_2, \quad J_+ \partial_3 = -\partial_3, \quad J_+ \partial_4 = -\partial_4.
\]

Let \(f(0) = 0\). We take the metric to have non-zero components determined by \(g(\partial_1, \partial_3) = 1\) and \(g(\partial_2, \partial_4) = e^{2f}\). Let \(f_i := \{\partial_i f\}(0)\). The (possibly) non-zero Christoffel symbols of \(g^\circ\) at the origin are given by:

\[
\begin{align*}
g(\nabla^g_{\partial_1} \partial_2, \partial_4) &= g(\nabla^g_{\partial_3} \partial_1, \partial_4) = g(\nabla^g_{\partial_4} \partial_1, \partial_2) = f_1, \\
g(\nabla^g_{\partial_2} \partial_4, \partial_1) &= g(\nabla^g_{\partial_3} \partial_4, \partial_2) = g(\nabla^g_{\partial_4} \partial_3, \partial_2) = f_3.
\end{align*}
\]

Consequently the (possibly) non-zero covariant derivatives at the origin are:

\[
\begin{align*}
\nabla^g_{\partial_1} \partial_2 &= \nabla^g_{\partial_3} \partial_1 = f_1 \partial_2, \\
\nabla^g_{\partial_2} \partial_3 &= \nabla^g_{\partial_4} \partial_3 = f_3 \partial_2, \\
\nabla^g_{\partial_3} \partial_4 &= \nabla^g_{\partial_1} \partial_3 = f_1 \partial_3.
\end{align*}
\]

Since \(\nabla^g_{\partial_1}\) and \(\nabla^g_{\partial_3}\) are diagonal, they commute with \(J_+\) so \(\nabla^g_{\partial_1}(J_+) = \nabla^g_{\partial_3}(J_+) = 0\).

We compute

\[
\begin{align*}
(\nabla^g_{\partial_2} J_+) \partial_1 &= (1 - J_+) \nabla^g_{\partial_2} \partial_1 = (1 - J_+) f_1 \partial_2 = 0, \\
(\nabla^g_{\partial_2} J_+) \partial_2 &= (1 - J_+) \nabla^g_{\partial_2} \partial_2 = (1 - J_+) 2f_2 \partial_2 = 0, \\
(\nabla^g_{\partial_2} J_+) \partial_3 &= (-1 - J_+) \nabla^g_{\partial_2} \partial_3 = (-1 - J_+) f_3 \partial_2 = -2f_3 \partial_2, \\
(\nabla^g_{\partial_2} J_+) \partial_4 &= (-1 - J_+) \nabla^g_{\partial_2} \partial_4 = (-1 - J_+) (-f_1 \partial_3 - f_3 \partial_1) = 2f_3 \partial_1.
\end{align*}
\]
symmetric 2-cotensors
\[ \nabla \] is non-degenerate near the origin. Since only the 1-jets of $W$ we now observe that $\nabla W$ we use $\epsilon$ we apply Equation (3.1). We have Since $\Theta(\partial_{i}) = \Theta(\partial_{j}) = \Theta(\partial_{k}) \neq 0$ for this metric and Assertion (1) of Theorem 1.2 holds in this special case.

**Proof of Theorem 1.2 (1).** Let $V = \mathbb{R}^4$, let $S^2_\ast$ be the vector space of symmetric 2-cotensors $\omega$ so that $J_\ast^\rho \omega = -\omega$, and let $\epsilon \in C^\infty(S^2)$ satisfy $\epsilon(0) = 0$. We use $\epsilon$ to define a perturbation of the flat metric by setting:

\[ g = dx^1 \circ dx^3 + dx^2 \circ dx^4 + \epsilon. \]

This is non-degenerate near the origin. Since only the 1-jets of $\epsilon$ are relevant in examining $\nabla^\theta(J_\ast)(0)$, this is a linear problem and we may take $\epsilon \in S^2_\ast \otimes V^\ast$ so:

\[ g = g_0 + \sum_i x^i \epsilon(e_i). \]
Then $\varepsilon \to (\nabla^g J_+)(0)$ defines a linear map
\[ \mathcal{E} : S_- (V) \otimes V^* \to \text{End}(V) \otimes V^* \] or equivalently
\[ \mathcal{E} : S_- (V) \to \text{Hom}(V^*, \text{End}(V) \otimes V^*). \]
The analysis of Section 3.2 shows that $\mathcal{E}$ is non-degenerate at 0 and defines a complex metric on some neighborhood of 0 so
\[ \mathcal{E} \] defines a complex Weyl connection $T(e^1) = e^1 + ae^2$, $T(e^2) = e^2$, $T(e^3) = e^3$, $T(e^4) = e^4 - ae^3$.

Then
\[ T(e^1 \circ e^3) = e^1 \circ e^3 + ae^2 \circ e^3. \]
Consequently, $\mathcal{E}(e^2 \circ e^3) = 0$. Permuting the indices $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ then yields $\mathcal{E}(e^1 \circ e^4) = 0$. Since $S_- = \text{Span}\{e_1 \circ e^3, e_1 \circ e^4, e_2 \circ e^3, e_2 \circ e^4\}$, we see that $\mathcal{E} = 0$ in general; this completes the proof of Assertion (1) of Theorem 1.2.

\section{4. Hermitian and pseudo-Hermitian manifolds}

In this section, we will use analytic continuation to derive Theorem 1.2 in the complex setting from Theorem 1.2 in the para-complex setting. Let $V = \mathbb{R}^4$ with the usual basis \{e_1, e_2, e_3, e_4\} and coordinates \{x^1, x^2, x^3, x^4\}, where we expand $v = x^1 e_1 + x^2 e_2 + x^3 e_3 + x^4 e_4$. Let $S^2$ denote the space of symmetric 2-tensors. We complexify and consider
\[ S := \{ S^2 \otimes \mathbb{C} \} \oplus \{ (V^* \otimes S^2) \otimes \mathbb{C} \}. \]
Let $J_+ \in M_2(\mathbb{C})$ be a complex $2 \times 2$ matrix with $J_+^2 = \text{Id}$ and $\text{Tr}(J_+) = 0$. Let
\[ S(J_+) := \{(g_0, g_1) \in S : \det(g_0 - J_+^* g_0) \neq 0\}. \]
For $(g_0, g_1) \in S(J_+)$, define:
\[ g(x)(X, Y) := \frac{1}{2} \{ g_0(X, Y) - g_0(J_+ X, J_+ Y) \}
+ \sum_{i=1}^4 x^i \cdot \frac{1}{2} \{ g_1(e_i, X, Y) - g_1(e_i, J_+ X, J_+ Y) \}. \]
By Equation (4.1), this is non-degenerate at 0 and defines a complex metric on some neighborhood of 0 so $J_+^* g = -g$. Let $\nabla^g$ be the complex Levi-Civita connection:
\[ \nabla^g \partial_i \partial_j = \frac{1}{2} g^{kl} \{ \partial_i g_{jk} + \partial_j g_{ik} - \partial_{x^k} g_{ij} \} \partial_k. \]
Then $\nabla^g$ is a torsion free connection on $T_C := T_M \otimes \mathbb{C}$. The para-Kähler form is defined by setting $\Omega_+(x, y) = g(x, J_+ y)$ and we have
\[ \delta \Omega_+ = *d \Omega_+ \text{ and } \phi := \frac{1}{2} J_+ \delta \Omega. \]
We then use $\phi$ to define a complex Weyl connection $\nabla^g$ on $T_C$ and define a holomorphic map from $S(J_+)$ to $\mathcal{V} := V^* \otimes M_4(\mathbb{C})$ by setting
\[ \mathcal{E}(g_0, g_1; J_+) := \nabla^g(J_+)_{|x=0}. \]
Lemma 4.1. Let $J_+ \in M_4(\mathbb{C})$ with $J_+^2 = id$ and $\text{Tr}(J_+) = 0$. Suppose that $(g_0, g_1) \in S(J_+)$. 

1. If $J_+$ is real and if $(g_0, g_1)$ is real, then $\mathcal{E}(g_0, g_1; J_+) = 0$.
2. If $J_+$ is real and if $(g_0, g_1)$ is complex, then $\mathcal{E}(g_0, g_1; J_+) = 0$.
3. If $J_+$ is complex and if $(g_0, g_1)$ is complex, then $\mathcal{E}(g_0, g_1; J_+) = 0$.

Proof. Assertion (1) follows from Theorem 1.2 (1). We argue as follows to prove Assertion (2). $S(J_+)$ is an open dense subset of $S$ and inherits a natural holomorphic structure thereby. Assume that $J_+$ is real. The map $\mathcal{E}$ is a holomorphic map from $S(J_+)$ to $\mathfrak{M}$. By Assertion (1), $\mathcal{E}(g_0, g_1; J_+)$ vanishes if $(g_0, g_1)$ is real. Thus, by the identity theorem, $\mathcal{E}(g_0, g_1; J_+)$ vanishes for all $(g_0, g_1) \in S(J_+)$. This establishes Assertion (2) by removing the assumption that $(g_0, g_1)$ is real.

We complete the proof by removing the assumption that $J_+$ is real. The general linear group $GL_4(\mathbb{C})$ acts on the structures involved by change of basis (i.e., conjugation). Let $(g_0, g_1) \in S(J_+)$ where $J_+$ is real and $\text{Tr}(J_+) = 0$. We consider the real and complex orbits

$$\mathcal{O}_R(g_0, g_1; J_+) := GL_4(\mathbb{R}) \cdot (g_0, g_1; J_+),$$
$$\mathcal{O}_C(g_0, g_1; J_+) := GL_4(\mathbb{C}) \cdot (g_0, g_1; J_+).$$

Let $\mathcal{F}(A) := \mathcal{E}(A \cdot (g_0, g_1; J_+))$ define a holomorphic map from $GL_4(\mathbb{C})$ to $\mathfrak{M}$. By Assertion (2), $\mathcal{F}$ vanishes on $GL_4(\mathbb{R})$. Thus by the identity theorem, $\mathcal{F}$ vanishes on $GL_4(\mathbb{C})$ or, equivalently, $\mathcal{E}$ vanishes on the orbit space $\mathcal{O}_C(g_0, g_1; J_+)$. Given any $J_+ \in M_4(\mathbb{C})$ with $J_+^2 = id$ and $\text{Tr}(J_+) = 0$, we can choose $A \in GL_4(\mathbb{C})$ so that $A \cdot J_+$ is real. The general case now follows from Assertion (2).

Proof of Theorem 1.2 (2,3). Let $(M, g, J_-)$ be a 4-dimensional pseudo-Hermitian manifold of dimension 4. Fix a point $P$ of $M$. Since $J_-$ is integrable, we may choose local coordinates $(x^1, x^2, x^3, x^4)$ so the matrix of $J_-$ relative to the coordinate frame $\{\partial_i\}$ is constant. Define a Weyl connection with associated 1-form given by $\phi = -\frac{1}{2} J_- \delta \Omega_-$. Only the 0 and the 1-jets of the metric play a role in the computation of $(\nabla^g J_-)(P)$. So we may assume $g = g(g_0, g_1)$. We set $J_+ = \sqrt{-1} J_-$. We have that

$$J_+^2 = \sqrt{-1} J_- \sqrt{-1} J_- = -J_-^2 = id, \quad \text{Tr}(J_+) = \sqrt{-1} \text{Tr}(J_-) = 0, \quad J_+(g)(X, Y) = g(\sqrt{-1} J_- X, \sqrt{-1} J_- Y) = -g(J_- X, J_- Y) = -g(X, Y)$$

so $J_+(g) = -g$ and $(g_0, g_1) \in S(J_+)$. Finally, since $J_- = -\sqrt{-1} J_+$, we have

$$\Omega_- = -\sqrt{-1} \Omega_+, \quad \phi_{J_-} = -\frac{1}{2} J_- \delta_g \Omega_- = -\frac{1}{2} \left( -\sqrt{-1} J_+ \right) \delta_g \left( -\sqrt{-1} \Omega_+ \right) = \frac{1}{2} J_+ \delta_g \Omega_+ = \phi_{J_+}.$$ 

We apply Lemma 4.1 to complete the proof.

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