ON RECOGNITION BY PRIME GRAPH
OF THE PROJECTIVE SPECIAL
LINEAR GROUP OVER GF(3)

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Abstract. Let $G$ be a finite group. The prime graph of $G$ is denoted by $\Gamma(G)$. We prove that the simple group $\text{PSL}_n(3)$, where $n \geq 9$, is quasirecognizable by prime graph; i.e., if $G$ is a finite group such that $\Gamma(G) = \Gamma(\text{PSL}_n(3))$, then $G$ has a unique nonabelian composition factor isomorphic to $\text{PSL}_n(3)$. Darafsheh proved in 2010 that if $p > 3$ is a prime number, then the projective special linear group $\text{PSL}_p(3)$ is at most 2-recognizable by spectrum. As a consequence of our result we prove that if $n \geq 9$, then $\text{PSL}_n(3)$ is at most 2-recognizable by spectrum.

1. Introduction

If $n$ is an integer, then we denote by $\pi(n)$ the set of all prime divisors of $n$. If $G$ is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. The spectrum of a finite group $G$ which is denoted by $\omega(G)$ is the set of its element orders. We construct the prime graph of $G$ which is denoted by $\Gamma(G)$ as follows: the vertex set is $\pi(G)$ and two distinct primes $p$ and $q$ are joined by an edge (we write $p \sim q$) if and only if $G$ contains an element of order $pq$. Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_i(G)$, $i = 1, \ldots, s(G)$, be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, we always suppose that $2 \in \pi_1(G)$. In graph theory a subset of vertices of a graph is called an independent set if its vertices are pairwise nonadjacent. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$. In other words, if $\rho(G)$ is some independent set with the maximal number of vertices in $\Gamma(G)$, then $t(G) = |\rho(G)|$. Similarly if $p \in \pi(G)$, then let $\rho(p, G)$ be some independent set with the maximal number of vertices in $\Gamma(G)$ containing $p$ and let $t(p, G) = |\rho(p, G)|$.

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A finite group $G$ is called **recognizable by prime graph** whenever if for a finite group $H$, we have $\Gamma(H) = \Gamma(G)$, then $H \cong G$. A nonabelian simple group $P$ is called **quasirecognizable by prime graph** if every finite group whose prime graph is $\Gamma(P)$ has a unique nonabelian composition factor which is isomorphic to $P$ (see [16]). Obviously recognition (quasirecognition) by prime graph implies recognition (quasirecognition) by spectrum, but the converse is not true in general. Also some methods of recognition by spectrum cannot be used for recognition by prime graph. If $\Omega$ is a nonempty subset of the set of natural numbers, we denote by $h(\Omega)$ the number of nonisomorphic groups $G$ with $\omega(G) = \Omega$. If $G$ is a finite group, then $h(\omega(G))$ is denoted by $h(G)$. If $h(G) = \infty$, then $G$ is called **nonrecognizable by spectrum**. If $h(G) = r$, then $G$ is called $r$-recognizable by spectrum.

Hagie in [12], determined finite groups $G$ satisfying $\Gamma(G) = \Gamma(S)$, where $S$ is a sporadic simple group. It is proved that if $q = 3^{2n+1}$ $(n > 0)$, then the simple group $^2G_2(q)$ is uniquely determined by its prime graph [16][35]. A group $G$ is called a CIT group if $G$ is of even order and the centralizer in $G$ of any involution is a 2-group. In [18], finite groups with the same prime graph as a CIT simple group are determined. Also in [19], it is proved that if $p > 11$ is a prime number and $p \neq 1$ (mod 12), then $\text{PSL}_2(p)$ is recognizable by prime graph. In [17][23], finite groups with the same prime graph as $\text{PSL}_2(q)$, where $q$ is not prime, are determined. It is proved that the simple group $F_4(q)$, where $q = 2^n > 2$ (see [15]) and $^2F_4(q)$ (see [1]) are quasirecognizable by prime graph. In [14], it is proved that if $p$ is a prime number which is not a Mersenne or Fermat prime and $p \neq 11, 13, 19$ and $\Gamma(G) = \Gamma(\text{PGL}_2(p))$, then $G$ has a unique nonabelian composition factor which is isomorphic to $\text{PSL}_2(p)$ and if $p = 13$, then $G$ has a unique nonabelian composition factor which is isomorphic to $\text{PSL}_2(13)$ or $\text{PSL}_2(27)$. Then it is proved that if $p$ and $k > 1$ are odd and $q = p^k$ is a prime power, then $\text{PGL}_2(q)$ is uniquely determined by its prime graph [2]. In [20][21][22][24][27][28], finite groups with the same prime graph as $\text{PSL}_n(2)$, $\text{U}_n(2)$, $\text{D}_n(2)$, $B_n(3)$ and $^2D_n(2)$ are obtained. In [3][4], it is proved that $^2D_{2n+1}(3)$ is recognizable by prime graph.

The projective special linear groups defined over a finite field of order 3, called the ternary field, are denoted by $\text{PSL}_n(3)$, $\text{PSL}(n, 3)$, $L_n(3)$ or $A_{n-1}(3)$ as a finite group of Lie type. In this paper as the main result we prove that the simple group $\text{PSL}_n(3)$, where $n \geq 9$, is quasirecognizable by prime graph; i.e., if $G$ is a finite group such that $\Gamma(G) = \Gamma(\text{PSL}_n(3))$, then $G$ has a unique nonabelian composition factor isomorphic to $\text{PSL}_n(3)$. In [8], it is proved that the projective special linear group $\text{PSL}_p(3)$, where $p > 3$ is a prime number, is at most 2-recognizable by spectrum, i.e., if $G$ is a finite group such that $\omega(G) = \omega(\text{PSL}_p(3))$, where $p > 3$ is an odd prime, then $G$ is isomorphic to $\text{PSL}_p(3)$ or $\text{PSL}_p(3) \cdot 2$, the extension of $\text{PSL}_p(3)$ by the graph automorphism. As a consequence of our result we prove that if $n \geq 9$, then $\text{PSL}_n(3)$ is at most 2-recognizable by spectrum, i.e., if $G$ is a finite group such that $\omega(G) = \omega(\text{PSL}_n(3))$, then $G$ is isomorphic to $\text{PSL}_n(3)$ or $\text{PSL}_n(3) \cdot 2$, the extension of $\text{PSL}_n(3)$ by the graph automorphism.

In this paper, all groups are finite and by simple groups we mean nonabelian simple groups. All further unexplained notations are standard and refer to [7]. Throughout the proof we use the classification of finite simple groups. In [31]...
Tables 2–9], independent sets also independent numbers for all simple groups are listed and we use these results in the proof of the main theorem of this paper.

2. Preliminary results

**Lemma 2.1.** [33 Theorem 1] Let $G$ be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the following hold:

1. there exists a finite nonabelian simple group $S$ such that $S \leq \bar{G} = G/K \leq \text{Aut}(S)$ for the maximal normal soluble subgroup $K$ of $G$;

2. for every independent subset $\rho$ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in $\rho$ divides the product $|K||G/S|$. In particular, $t(S) \geq t(G) - 1$;

3. one of the following holds:
   
   (a) every prime $r \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ does not divide the product $|K||G/S|$; in particular, $t(2, S) \geq t(2, G)$;

   (b) there exists a prime $r \in \pi(K)$ nonadjacent to 2 in $\Gamma(G)$; in which case $t(G) = 3$, $t(2, G) = 2$, and $S \cong \text{Alt}_7$ or $\text{PSL}_2(q)$ for some odd $q$.

**Remark 2.1.** In Lemma 2.1 for every odd prime $p \in \pi(S)$, we have $t(p, S) \geq t(p, G) - 1$.

**Lemma 2.2.** [26] Let $N$ be a normal subgroup of $G$. Assume that $G/N$ is a Frobenius group with Frobenius kernel $F$ and cyclic Frobenius complement $C$. If $([N], [F]) = 1$, and $F$ is not contained in $NC_G(N)/N$, then $p|C| \in \omega(G)$, where $p$ is a prime divisor of $|N|$.

**Lemma 2.3** (Zsigmondy’s Theorem). [36] Let $p$ be a prime and let $n$ be a positive integer. Then one of the following holds:

1. there is a primitive prime $p'$ for $p^n - 1$, that is, $p' \mid (p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$, (usually $p'$ is denoted by $r_n$)

2. $p = 2$, $n = 1$ or 6,

3. $p$ is a Mersenne prime and $n = 2$.

**Lemma 2.4.** [13] Let $G$ be a finite simple group.

1. If $G = C_n(q)$, then $G$ contains a Frobenius subgroup with kernel of order $q^n$ and cyclic complement of order $(q^n - 1)/(2, q - 1)$.

2. If $G = 2D_n(q)$, and there exists a primitive prime divisor $r$ of $q^{n-2} - 1$, then $G$ contains a Frobenius subgroup with kernel of order $q^{n-2}$ and cyclic complement of order $r$.

3. If $G = B_n(q)$ or $D_n(q)$, and there exists a primitive prime divisor $r_m$ of $q^m - 1$ where $m = n$ or $n - 1$ such that $m$ is odd, then $G$ contains a Frobenius subgroup with kernel of order $q^{m(m-1)/2}$ and cyclic complement of order $r_m$.

**Remark 2.2.** [30] Let $p$ be a prime number and $(q, p) = 1$. Let $k \geq 1$ be the smallest positive integer such that $q^k \equiv 1 \pmod{p}$. Then $k$ is called the order of $q$ with respect to $p$ and we denote it by $\text{ord}_p(q)$. Obviously by the Fermat’s little theorem it follows that $\text{ord}_p(q)|(p - 1)$. Also if $q^a \equiv 1 \pmod{p}$, then $\text{ord}_p(q)|n$. Similarly if $m > 1$ is an integer and $(q, m) = 1$, we can define $\text{ord}_m(q)$. If $a$ is an
odd prime, then \( \text{ord}_q(q) \) is denoted by \( e(a,q) \), too. If \( q \) is odd, then \( e(2,q) = 1 \) for \( q \equiv 1 \pmod{4} \) and \( e(2,q) = 2 \) for \( q \equiv -1 \pmod{4} \).

**Lemma 2.5.** [32] Proposition 2.4] Let \( G \) be a simple group of Lie type, \( B_n(q) \) or \( C_n(q) \) over a field of characteristic \( p \). Define

\[
\eta(m) = \begin{cases} m & \text{if } m \text{ is odd,} \\ m/2 & \text{otherwise.} \end{cases}
\]

Let \( r, s \) be odd primes with \( r, s \in \pi(G) \setminus \{p\} \). Put \( k = e(r,q) \) and \( l = e(s,q) \), and suppose that \( 1 \leq \eta(k) \leq \eta(l) \). Then \( r \) and \( s \) are nonadjacent if and only if \( \eta(k) + \eta(l) > n \), and \( l/k \) is not an odd natural number.

**Lemma 2.6.** [31] Proposition 2.1] Let \( G = A_n(q) \) be a finite simple group of Lie type over a field of characteristic \( p \). Let \( r \) and \( s \) be odd primes and \( r, s \in \pi(G) \setminus \{p\} \). Put \( k = e(r,q) \) and \( l = e(s,q) \), and suppose that \( 2 \leq k \leq l \). Then \( r \) and \( s \) are nonadjacent if and only if \( \eta(k) + \eta(l) > n \), and \( l/k \) is not an odd natural number.

**Lemma 2.7.** [31] Proposition 2.2] Let \( G = 2A_n(q) \) be a finite simple group of Lie type over a field of characteristic \( p \). Define

\[
\nu(m) = \begin{cases} m & \text{if } m \equiv 0 \pmod{4}; \\ m/2 & \text{if } m \equiv 2 \pmod{4}; \\ 2m & \text{if } m \equiv 1 \pmod{2}. \end{cases}
\]

Let \( r \) and \( s \) be odd primes and \( r, s \in \pi(G) \setminus \{p\} \). Put \( k = e(r,q) \) and \( l = e(s,q) \), and suppose that \( 2 \leq \nu(k) \leq \nu(l) \). Then \( r \) and \( s \) are nonadjacent if and only if \( \nu(k) + \nu(l) > n \), and \( \nu(k) \) does not divide \( \nu(l) \).

Let \( q \) be a prime. We denote by \( D_n^+(q) \) the simple group \( D_n(q) \), and by \( D_n^-(q) \) the simple group \( 2D_n(q) \).

**Lemma 2.8.** [32] Proposition 2.5] Let \( G = D_n^+(q) \) be a finite simple group of Lie type over a field of characteristic \( p \) and let function \( \eta(m) \) be defined as in Lemma 2.5 Let \( r \) and \( s \) be odd primes and \( r, s \in \pi(G) \setminus \{p\} \). Put \( k = e(r,q) \) and \( l = e(s,q) \), and \( 1 \leq \eta(k) \leq \eta(l) \). Then \( r \) and \( s \) are nonadjacent if and only if \( 2\eta(k) + 2\eta(l) > 2n - (1 - \varepsilon(-1)^{k+l}), \) \( l/k \) is not an odd natural number, and if \( \varepsilon = + \), then the equality chain \( n = l = 2\eta(l) = 2\eta(k) = 2k \) is not true.

**Lemma 2.9.** [5] Lemma 3.1] Let \( G \) be a finite group satisfying the conditions of Lemma 2.1 and let the groups \( K \) and \( S \) be as in the claim of Lemma 2.1 Let there exist \( p \in \pi(K) \) and \( p' \in \pi(S) \) such that \( p \sim p' \) in \( \Gamma(G) \), and \( S \) contains a Frobenius subgroup with kernel \( F \) and cyclic complement \( C \) such that \( (|F|,|K|) = 1 \). Then \( p|C| \in \omega(G) \).

**Lemma 2.10.** [34] Theorem 1] Let \( L = SL_n(q) \), where \( n \geq 5 \) and \( q = p^a \). If \( L \) acts on a vector space \( W \) over a field of characteristic \( p \), then \( \omega(L) \neq \omega(W \times L) \).

3. Main Results

**Theorem 3.1.** The simple group \( SL_n(3) \), where \( n \geq 9 \), is quasirecognizable by prime graph; i.e., if \( G \) is a finite group such that \( \Gamma(G) = \Gamma(PGL_n(3)) \), then \( G \) has a unique nonabelian composition factor which is isomorphic to \( SL_n(3) \).
Proof. Let $D = \text{PSL}_n(3)$, where $n \geq 9$, and $G$ be a finite group such that $\Gamma(G) = \Gamma(D)$. Using \cite[Tables 4-8]{32}, we conclude that $t(D) = \lceil \frac{n+1}{2} \rceil \geq 5$ and $t(2, D) = 2$. Therefore $t(G) \geq 5$ and $t(2, G) = 2$. Also $\rho(D) = \{r_i \mid \frac{3}{2} < i \leq n\}$, where $r_i$ is a primitive prime divisor of $3^i - 1$. Also using \cite[Table 6]{32}, it follows that $\rho(2, \text{PSL}_n(3)) = \{2, r_{n-1}\}$ if $n$ is even and $\rho(2, \text{PSL}_n(3)) = \{2, r_n\}$ if $n$ is odd. Using Lemma 2.1 we conclude that there exists a finite nonabelian simple group $S$ such that $S \leq G = G/N \leq \text{Aut}(S)$, where $N$ is the maximal normal soluble subgroup of $G$. Also $t(S) \geq t(G) - 1 \geq 4$ and $t(2, S) \geq t(2, G) \geq 2$, by Lemma 2.1.

Now we consider each possibility for $S$, by the tables in \cite{32}.

Step 1. Let $S \cong A_m$, where $m \geq 5$.

We know that $t(S) \geq 4$. Thus $m \geq 17$. So 17 | $|A_m|$. Since $e(17, 3) = 16$, we conclude that $n \geq 16$. Therefore $t(G) = \lceil (n + 1)/2 \rceil \geq 8$, and so $m > 19$. If $p$ is a prime number such that $p \leq m - 17$, then $p \sim 17$ in $\Gamma(S)$. Let $A := [m - 17, m] \cap \mathbb{Z}$. Then $t(S) = 17 - [18/2] + [18/3] - [18/6]$ of $A$ are divisible by 2 or 3. Therefore $t(17, S) \leq 7$.

On the other hand, we know that $e(17, 3) = 16$. Let $k = e(p, 3)$. Then $p$ is not adjacent to 17 in $\Gamma(G)$ if and only if 16 + $k > n$ and 16 | $k$ if 16 $\leq k$, and $k | 16$ if $k \leq 16$. There are 16 consecutive numbers in $(n - 16, n] \cap \mathbb{Z}$. So 16 can divide exactly one of them. Also at most 5 of them divide 16. Hence $t(17, G) \geq 16 - 1 - 5 = 10$. Therefore $7 \geq t(17, S) \geq t(17, G) - 1 \geq 10 - 1 = 9$, which is a contradiction.

Step 2. In this step, we prove that the simple group $S$ is not isomorphic to a simple group of Lie type over $\text{GF}(p^a)$, where $p \neq 3$. Using Table 8 in \cite{32}, we consider the independent set $B = \{r_i \mid n - 4 < i \leq n\}$ in $\Gamma(G)$, since $n \geq 9$ and so $[\frac{n}{4}] < n - 4$. By Lemma 2.1 $|B \cap \pi(S)| \geq 4$.

Case 1. Let $S \cong \text{PSL}_n(q)$, where $q = p^a$, $p \neq 3$.

We know that $t(S) \geq t(G) - 1$. Therefore $\lceil \frac{n+1}{2} \rceil \geq t(S) \geq \lceil \frac{n+1}{2} \rceil - 1 \geq 4$, which implies that $m \geq 7$ and $m \geq n - 3$. Also $t(p, S) \leq 3$ by Table 4 in \cite{32}.

Thus $t(p, G) \leq 4$. So we conclude that $p \notin B$. Therefore $p$ is joined to at least two elements of $B$ in $\Gamma(G)$. In the sequel we consider one case and other cases are similar to it. We assume that $p$ is joined to $r_{n-4}$ and $r_{n-3}$ in $\Gamma(G)$. Let $t = e(p, 3)$ and note that $e(r_{n-4}, 3) = n - 4$ and $e(r_{n-3}, 3) = n - 3$. By Lemma 2.0 one of the following subcases occurs:

1. $t + n - 3 \leq n$ and $t + n - 4 \leq n$; 2. $t + n - 3 \leq n$ and $t \mid (n - 3)$;
3. $t + n - 4 \leq n$ and $t \mid (n - 3)$; 4. $t \mid (n - 4)$ and $t \mid (n - 3)$.

Therefore in each case we conclude that $t \leq 4$. Therefore $p \in \{2, 5, 13\}$.

If $p = 5$, then $S \cong \text{PSL}_m(5^a)$. We note that $e(71, 5) = 5$ and $e(71, 3) = 35$. We know that $m \geq 6$ and $e(71, 5^a)$ divides $e(71, 5) = 5$. Therefore $71 \in \pi(S) \subseteq \pi(G)$. Now by Lemma 2.6 if $e(x, 5^a) \leq m - 5$, then $x$ is joined 71 in $\Gamma(S)$. Therefore $t(71, S) \leq 6$. On the other hand, by Lemma 2.6 we conclude that if $e(y, 3) \leq n - 35$, then $y \sim 71$ in $\Gamma(G)$. Therefore $\rho(71, G) \subseteq \{r_i \mid n - 35 < i \leq n\}$. Also we know that $r_i \sim 71$ if and only if $n - 35 < i \leq n$ and $i/35$, $35/i$ are not integers. Let $C = \{n - 34, \ldots, n\}$. Thus there is only one $i \in C$ such that $i/35$ is an integer. Also since $35 = 5 \times 7$, there are at most 4 elements $i \in C$ such that $35/i$ is an integer. Thus $t(71, G) \geq 35 - 5 = 30$, which is a contradiction since $29 \leq t(71, G) - 1 \leq t(71, S) \leq 6$. 


• If $p = 2$, then $S \cong \text{PSL}(2^n)$. Since $e(31, 2) = 5$ and $e(31, 3) = 30$, similarly we get a contradiction.
• If $p = 13$, then $S \cong \text{PSL}(13^n)$. We know that $e(30941, 13) = 5$ and $e(30941, 3) = 30940$. Now similarly to the above we get a contradiction.

Case 2. Let $S \cong U_m(q)$, where $q = p^n$, $p \neq 3$. Since $t(S) = \lfloor (m + 1)/2 \rfloor$, similarly to Case 1 we have $m \geq 7$ and $m \geq n - 3$. By Table 4, we have $t(p, S) \leq 3$, and similarly to the last case, we conclude that $p = 2, 5, 13$.

• If $p = 2$, then $S \cong U_m(2^n)$. If $m = 7$, then $t(S) = 4$, which implies that $t(G) = 5$ and so $n = 9, 10$. Therefore $757 \in \rho(2, G) \subseteq \pi(S)$. On the other hand $e(757, 2) = 756 = 2^2 \times 3^3 \times 7$ and since $m = 7$, the order of $U_7(2^a)$ shows that $9 \times 7 \mid \alpha$. Now $\pi(2^{3^3} - 1) \subseteq \pi(G)$, which is a contradiction. Therefore $m \geq 8$, and so $\pi(2^{3^3} - 1) \subseteq \pi(S)$, which implies that $17 \in \pi(S)$ and since $e(17, 3) = 16$, we get that $n \geq 16$. This implies that $m \geq 13$ and so $31 \in \pi(2^{10a} - 1) \subseteq \pi(S)$. We note that $e(31, 3) = 30$ and $e(31, 2) = 5$. Hence $e(31, 2^n) \mid 5$. We know that if $\nu(e(x, 2^n)) \leq m - 10$, then $31 \sim x$ in $\Gamma(S)$, by Lemma 2.5. Therefore $t(31, S) \leq 10$. Now we determine $t(31, G)$. As $e(31, 3) = 30$ similarly to the above, we conclude that if $e(x, 3) \leq m - 30$, then $31 \sim x$ in $\Gamma(G)$. Let $C = \{n - 29, \ldots, n\}$. So $30$ divides exactly one element of $C$. Also since $30 = 2 \times 3 \times 5$, there are at most $8$ elements in $C$ such that $30/i$ is an integer. Thus $22 \leq t(31, G)$. Therefore $21 \leq t(31, G) - 1 \leq t(31, S) \leq 10$, which is a contradiction.

• If $p = 5$, then $S \cong U_m(5^n)$. Since $m \geq 7$, we have $449 \in \pi(S)$. Also $e(449, 3) = 448$ and $e(449, 5) = 14$. Now similarly to the above we get a contradiction.

• If $p = 13$, then similarly to the above by using $e(157, 3) = 78$ and $e(157, 13) = 6$, we get a contradiction.

Case 3. Let $S \cong D_m(q)$, where $q = p^n$, $p \neq 3$. Then $t(S) \geq t(G) - 1$ implies that $m \geq 5$. Similarly to the last cases, we conclude that $p = 2, 5$ or $13$.

If $p = 2$, then we note that $e(31, 2) = 5$. Therefore $e(31, 2^n) \mid 5$. Thus for every $x \in \pi(S)$, such that $2\eta(e(x, 2^n)) \leq 2n - 10 - 2$, we have $x \sim 31$ in $\Gamma(S)$. Therefore $t(31, S) \leq 12$. On the other hand, as we mentioned above $t(31, G) \geq 22$, which is a contradiction. Similarly if $p = 5$, then we use $t(31, S)$ and if $p = 13$, then we use $t(157, S)$ and similarly to the above we get a contradiction.

Case 4. Let $S \cong B_m(q)$ or $S \cong C_m(q)$, where $q = p^n$ and $p \neq 3$.

So $(3m + 5)/4 \geq \lfloor (3m + 5)/4 \rfloor = t(S) \geq t(G) - 1 = \lfloor (n + 1)/2 \rfloor - 1 \geq 4$. Thus similarly $m \geq 4$ and $p = 2, 5$ or $13$. For $p = 5$, we note that $e(31, 3) = 30$ and $e(31, 5) = 3$. By Lemma 2.5 we know that if $\eta(e(x, 5^n)) \leq m - 3$, then $x \sim 31$ in $\Gamma(S)$. Therefore $t(31, S) \leq 6$, which is a contradiction since $t(31, G) \geq 22$.

If $p = 13$, then using $e(157, 3) = 78$ and $e(157, 13) = 6$, we get a contradiction.

If $p = 2$, then $S \cong B_m(2^n)$. Now we note that $e(17, 3) = 16, e(17, 2) = 8$. Then $e(17, 2^n) \mid 8$ and so $\eta(e(17, 2^n)) \leq 4$. If $\eta(e(x, 2^n)) \leq m - 4$, then $x \sim 17$ in $\Gamma(S)$ by Lemma 2.5. So only $8$ elements, where $\eta(e(x, 2^n)) > m - 4$ may not be joined $17$ in $\Gamma(S)$. Hence the independent set which contains $17$ has at most $9$ elements in $\Gamma(S)$. On the other hand, if $r_i \sim 17$ in $\Gamma(G)$, then $m - 16 < i \leq m$; and $i/16, 16/i$ are not integers. Then $16$ divides one of the numbers in $m \in m, m] \cap Z$. Also at most $10$ elements are not adjacent to $17$ in $\Gamma(G)$. Therefore $\rho(17, G)$ has at least $11$ elements and we get a
contradiction since $10 \leq t(17, G) - 1 \leq t(17, S) \leq 9$.

**Case 5.** Let $S \cong D_m(q)$, where $q = p^n$, $p \neq 3$. Therefore $(3m + 4)/4 \geq t(S) \geq 4$ implies that $3m \geq 2n - 6$ and $m \geq 4$. Now we consider the following cases.

- Let $n \geq 11$. Then $B' = \{ r'_i \mid n - 5 \leq i \leq n \}$ is an independent set in $\Gamma(G)$. Since $t(p, S) \leq 4$, we conclude that $p$ is joined to at least two elements of $B$ in $\Gamma(G)$. In each case similarly to the previous cases we conclude that $p = 2, 5, 11$ or 13.

If $p = 2$, then since $e(31, 3) = 30$ and $e(31, 2) = 5$, similarly to the last cases we get a contradiction. Also we know that $e(31, 3) = 30, e(31, 5) = 3; e(7321, 3) = 1830, e(7321, 11) = 8$ and $e(157, 3) = 78, e(157, 13) = 6$. Hence for $p = 5, 11$ and 13 we get a contradiction.

- Let $n = 9$ and $S \cong D_m(q)$, where $p \in \pi(PSL_3(3)) \sim \{3\}$.

  If $p \in \{2, 13, 41, 757, 1093\}$, then $\pi(p^8 - 1) \not\in \pi(G)$. Also for $p \in \{5, 7, 11\}$, we see that $\pi(p^8 - 1) \not\in \pi(G)$.

- If $n = 10$, then similarly we get a contradiction.

**Case 6.** In this case we prove that $S$ is not isomorphic to an exceptional simple group. Let $S \cong F_4(q)$, $E_6(q)$ or $^2E_6(q)$, where $q = p^n$ and $p \in \pi(G)$. Then $t(S) \leq 5$ and so $9 \leq n \leq 12$. Easily we can see that for each $3 \neq p \in \pi(G)$, $\pi((p^8 - 1)(p^{12} - 1)) \not\subseteq \pi(G)$, which is a contradiction since $\pi((p^8 - 1)(p^{12} - 1)) \subseteq \pi(S)$.

If $S \cong E_7(q)$, where $q = p^n$, then $t(S) = 8$ and so $t(G) \leq 9$. Therefore $9 \leq n \leq 18$. Similarly to the last case for each $3 \neq p \in \pi(G)$, we can get a contradiction.

If $S \cong E_8(q)$, where $q = p^n$, then $9 \leq n \leq 24$ and for each $p \in \pi(PSL_3(3))$ we have $\pi(p^{10} - p^8 + 1) \not\subseteq \pi(G)$, which is a contradiction.

If $S \cong F_4(2^{2n+1})$, then $9 \leq n \leq 12$. If $n = 9$ or $n = 10$, then 757 $\in \pi(2, G)$ and so 757 $\in \pi(F_4(2^{2n+1}))$. We know that $e(757, 2) = 756$ and so 756 $\mid 12(2n + 1)$. Therefore $7 \mid (2n + 1)$, and so $\pi(2^7 - 1) \subseteq \pi(S) \subseteq \pi(G)$, which is a contradiction.

If $n = 11, 12$, then 3851 $\in \rho(2, G)$ and similarly we get a contradiction, since $e(3851, 2) = 3850$.

If $S \cong B_2(2^{2n+1})$, then similarly we get a contradiction.

**Step 3.** Now we consider the simple groups of Lie type over $GF(3^n)$. In the sequel, we use $r'_k$ for a primitive prime divisor of $(3^n)^k - 1$.

**Case 1.** Let $S \cong PSL_m(q)$, where $q = 3^n$.

By Table 6], $r_{n-1} \in \pi(S)$ or $r_n \in \pi(S)$. Also $m \geq n - 3$.

(I) Let $n$ be odd and so $r_n \in \rho(2, S) = \{ 2, r'_m, r''_m \}$.

- If $r_n = r'_m$, then $n \mid \alpha m$ and so $n \leq \alpha m$. On the other hand, using Zsigmondy’s Theorem, we conclude that $am \leq n$, since $\pi(S) \subseteq \pi(G)$. Therefore $am = n$.

  Also we know that $m \geq n - 3$. If $\alpha \geq 2$, then $n = \alpha m \geq 2m \geq 2n - 6$, which implies that $6 \leq n$ and this is a contradiction. Thus $\alpha = 1$, and so $m = n$. Therefore $S = PSL_n(3)$.

- If $r_n = r''_m$, then $n \mid \alpha(m - 1)$ and so $n \leq \alpha(m - 1)$. Also by Zsigmondy’s Theorem, $\alpha(m - 1) \leq n$. Hence $\alpha(m - 1) = n$. On the other hand, $t = (3^m - 1) \mid |S|$ and since $\pi(S) \subseteq \pi(G)$, we conclude that $\alpha m \leq n$, which is a contradiction.

(II) Let $n$ be even and so $r_{n-1} \in \pi(S)$. Therefore $r_{n-1} \in \rho(2, S) = \{ 2, r'_m, r''_m \}$.

- If $r_{n-1} = r'_m$, then $(n - 1) \mid \alpha(m - 1)$ and $\alpha(m - 1) \leq n$, since $\pi(S) \subseteq \pi(G)$. 


Hence \( \alpha(m - 1) = n - 1 \). We know that \( m \geq n - 3 \). If \( \alpha \geq 2 \), then 
\[
\begin{align*}
\n - 1 = \alpha(m - 1) &\geq 2(m - 1) \\
&\geq 2(n - 3) - 2 \geq 2n - 8.
\end{align*}
\]
Hence \( n \leq 7 \), which is a contradiction. Thus \( \alpha = 1 \) and so \( m = n \). Therefore \( S \cong \text{PSL}_n(3) \).

- If \( r_{n-1} = \alpha_m' \), then \( (n - 1) \mid \alpha m \). Also \( \alpha m \leq n \), since \( \pi(S) \subseteq \pi(G) \). Therefore \( \alpha m = n - 1 \). If \( \alpha \geq 2 \), we get that \( n - 1 = \alpha m \geq 2m \geq 2n - 6 \). Thus \( n \leq 5 \), which is a contradiction. Thus \( \alpha = 1 \) and so \( m = n - 1 \). Therefore \( S \cong \text{PSL}_{n-1}(3) \).

- So \( r_n \in \pi(K) \cup \pi(G/S) \). Also we note that \( \pi(G/S) \subseteq \pi(\text{Out}(S)) = \{2\} \). So \( r_n \in \pi(K) \). We note that there exists a Frobenius subgroup of \( \text{PSL}_{n-1}(3) \) of the form \( 3^{n-2} : (3^{n-2} - 1)/d \), where \( d = (n - 1, 2) \). On the other hand, \( r_n \sim r_{n-1} \) in \( \Gamma(G) \). So by Lemma 2.4, we conclude that \( r_n \) is joined to \( r_{n-2} \) in \( \Gamma(G) \), which is a contradiction.

**Case 2.** Let \( S \cong U_m(q) \), where \( q = 3^\alpha \). Then \( m \geq n - 3 \) and \( r_n \in \pi(S) \) or \( r_{n-1} \in \pi(S) \).

(I) Let \( n \) be odd and so \( r_n \in \pi(S) \). Using [32, Table 4] we must consider four cases, since \( r_n \in \{r'_{2m}, r'_{2m-2}, r_m, r_{m/2}\} \).

- If \( r_n = r'_{2m} \), then \( m \) is odd by [32, Table 4]. Also similar to the previous cases, \( n \mid 2\alpha m \). On the other hand, since \( \pi(S) \subseteq \pi(G) \), we conclude that \( 2\alpha m \leq n \). Therefore \( 2\alpha m = n \), which is a contradiction since \( n \) is odd.

- If \( r_n = r'_{2m-2} \), then \( m \) is even and \( n \mid 2\alpha(m - 1) \). Also since \( \pi(S) \subseteq \pi(G) \), we get that \( 2\alpha(m - 1) \leq n \). Thus \( n = 2\alpha(m - 1) \), which is a contradiction since \( n \) is odd.

- If \( r_n = r_m \), then \( 4 \mid m \), by [32, Table 4]. Also \( n \mid \alpha m \). Thus \( n = \alpha m \), a contradiction since \( n \) is odd.

- Let \( r_n = r'_{m/2} \). Thus \( n \mid \alpha m/2 \), which implies that \( n \leq \alpha m/2 \leq n \), since \( \pi(S) \subseteq \pi(G) \). Therefore \( n = \alpha m/2 \). Hence \( \alpha m = 2n \) and \( r_{am} \in \pi(S) \subseteq \pi(G) \), which is a contradiction.

(II) Let \( n \) be even and so \( r_{n-1} \in \pi(S) \).

- Let \( r_{n-1} = r'_{2m} \). Thus \( (n-1) \mid 2\alpha m \). So \( n-1 \leq 2\alpha m \leq n \). Therefore \( 2\alpha m = n - 1 \), which is a contradiction, since \( n \) is even.

- Let \( r_{n-1} = r'_{2m-2} \). So \( (n-1) \mid 2\alpha(m-1) \). So similarly to the above, we conclude that \( 2\alpha(m-1) = n - 1 \), which is a contradiction since \( n \) is even.

- Let \( r_{n-1} = r_m' \). So \( n-1 = \alpha m \). If \( \alpha \geq 2 \), then \( n - 1 = \alpha m \geq 2m \geq 2n - 6 \). Therefore \( 5 \geq n \), which is a contradiction. If \( \alpha = 1 \), then \( m = n - 1 \). Therefore \( S = U_{n-1}(3) \). Since \( n - 1 \) is odd, we conclude that \( (3^{n-1} + 1) \mid |S| \). Hence \( r_{2(n-1)} \in \pi(G) \), which is a contradiction.

- Let \( r_{n-1} = r'_{m/2} \). Thus \( m \) is even and \( n - 1 = \alpha m/2 \). We know that \( n - 1 \) is odd and so \( \alpha \) is odd. If \( \alpha \geq 3 \), then \( n - 1 \geq 3m/2 \geq 3(n-3)/2 \). Therefore \( 7 \geq n \), which is a contradiction. If \( \alpha = 1 \), then \( m = 2n - 2 \). So \( r_{2n-2} \in \pi(S) \subseteq \pi(G) \), which is a contradiction.

**Case 3.** Let \( S \cong B_m(q) \), where \( q = 3^\alpha \). Since \( t(S) \geq t(G) - 1 \). We have \( 3m > 2n - 11 \). Also \( \rho(2, S) = \{2, r_m', r'_{2m}\} \).

(I) Let \( n \) be odd and so \( r_n \in \rho(2, G) \). Therefore \( r_n = r_m' \) or \( r_n = r'_{2m} \).

- Let \( r_n = r_m' \). Then \( n \) is odd by [32, Table 6]. Also \( n \mid \alpha m \). Hence \( n \leq \alpha m \leq n \), since \( \pi(S) \subseteq \pi(G) \). Thus \( n = \alpha m \). Obviously \( \alpha \) is odd. If \( \alpha \geq 5 \), then \( n = \alpha m \geq
5m ≥ \frac{10}{3}n - \frac{55}{3}. So 55 ≥ 7n, which is a contradiction. If α = 1, then n = m. So S ∼ B_n(3^\alpha). Hence r_{2m} ∈ \pi(S) ⊆ \pi(G), which is a contradiction. If α = 3, then n = 3m. Hence S ∼ B_{3n}(27). Thus \pi(27^{2n/3} - 1) = \pi(3^{2n} - 1) ⊆ \pi(S) ⊆ \pi(G), which is a contradiction.

- Let r_n = r'_{2m}. So m is even. Therefore similarly to the above, we conclude that n = 2αm, which is a contradiction since n is odd.

**Case 4.** Let S = D_m(q), where q = 3^\alpha. Similarly, we conclude that if m ≠ 3 (mod 4), then 3m ≥ 2n - 2 and if m = 3 (mod 4), then 3m > 2n - 4, since t(S) ≥ t(G) - 1. Therefore in each case we have 3m > 2n - 4. We know that \rho(2, S) = 2, r'_{m-1}, r'_{m}, r'_{2m-2}. Also if r'_{m-1} ∈ \rho(2, S), then m is even and if r'_{m} ∈ \rho(2, S), then m is odd.

**I** If n is odd, then r_n ∈ \pi(S).

- Let r_n = r'_{m-1}. So n is odd and m is odd. Similarly to the above, we conclude that n = α(m - 1). Now since \pi((3^\alpha)^{2(m-1)} - 1) ⊆ \pi(S) ⊆ \pi(G), we get a contradiction.

- Let r_n = r'_{2m-2}. Then n | 2α(m - 1), and so n = 2α(m - 1), which is a contradiction since n is odd.

**II** If n is even, then r_{n-1} ∈ \pi(S).

- Let r_{n-1} = r'_{m-1}. Hence (n - 1) | α(m - 1) and α(m - 1) ≤ n. Hence n - 1 = α(m - 1). Then r_{2α(m-1)} ∈ \pi(S) ⊆ \pi(G), which is a contradiction.

- Let r_{n-1} = r'_{m}. So m is even and m is odd. Similarly to the above, we conclude that (n - 1) | αm. Thus αm = n - 1 and so α is odd. Also we know that 3m > 2n - 4. If α ≥ 3, then n - 1 = αm ≥ 3m ≥ 2n - 4. Therefore n ≤ 3, which is a contradiction.

If α = 1, then n - 1 = m and so S ∼ D_{n-1}(3). Hence r_{2m-2} ∈ \pi(S) ⊆ \pi(G), which is a contradiction.

- If r_{n-1} = r'_{2m-2}, then n - 1 = 2(m - 1)α, which is a contradiction since n is even.

**Case 5.** Let S ≡ 2D_m(q), where q = 3^\alpha.

Similarly to the above, we conclude that 3m > 2n - 10. By [32, Table 6] it follows that if r'_{2m-2} ∈ \rho(2, S), then m is odd.

**I** If n is odd, then r_n ∈ \pi(S). Hence r_n = r'_{2m} or r_n = r'_{2m-2}. If r_n = r'_{2m} or r'_{2m-2}, then similarly to the above, we conclude that n = 2αm or 2α(m - 1), which is a contradiction since n is odd.

**II** If n is even, then r_{n-1} ∈ \pi(S). If r_{n-1} = r'_{m}, or r'_{2m-2}, then n - 1 = 2αm or 2α(m - 1), which is a contradiction since n - 1 is odd.

**Case 6.** Let S ∼ F_4(q), where q = 3^\alpha.

Since t(S) = 5, t(G) ≤ 6 and so 9 ≤ n ≤ 12. We know that \pi(3^{12α} - 1) ⊆ \pi(F_4(q)). Therefore α = 1 and so q = 3, n = 12. Now r_{11} ∈ \rho(2, G) ⊆ \pi(S), which is a contradiction.
Similarly, we conclude that $S$ can not be isomorphic to $E_6(q)$ and $^2E_6(q)$. 

**Case 7.** Let $S \cong 2G_2(3^{2m+1})$, where $m \geq 1$. Since $t(S) = 5$, we get that $9 \leq n \leq 12$. Similarly to the previous case if $n \geq 11$, then we get a contradiction since $3 \in \rho(S)$, for each independent set $\rho(S)$. Therefore $t(G) = 5$ and so $n = 9$ or $n = 10$.

Now Zsigmondy’s Theorem implies that $6(2m + 1) \leq 10$, which is a contradiction.

**Step 4.** In this step we prove that $S$ is not isomorphic to a sporadic simple group. If $S \cong J_4$, then $43 \mid |S|$ and since $e(43, 3) = 42$ we have $n \geq 42$. So $t(G) \geq \lfloor \frac{42 + 1}{42} \rfloor = 21$, which is a contradiction since $t(J_4) = 7$. For the rest of sporadic simple groups $t(S) \leq 5$ and so $9 \leq n \leq 12$. Hence $\{757, 1093\} \cap \pi(S) \neq \emptyset$, which is a contradiction.

Therefore the quasirecognition of $\text{PSL}_n(3)$, where $n \geq 9$, is proved. \hfill \square

**Theorem 3.2.** If $\Gamma(G) = \Gamma(\text{PSL}_n(3))$, where $n \geq 9$, then $\text{PSL}_n(3) \leq G/N \leq \text{Aut}(\text{PSL}_n(3))$, where $N$ is a 3-group for even $n$ and $N$ is a $\{2, 3\}$-group for odd $n$.

**Proof.** By Theorem 3.1 we know that $\text{PSL}_n(3) \leq G/N \leq \text{Aut}(\text{PSL}_n(3))$. Similarly to [20], we can assume that $N$ is an elementary abelian $p$-group for some $p \in \pi(G)$. Now we prove that $\text{PSL}_n(3)$ acts faithfully on $N$. For this reason, we prove that $C = C_G(N) \leq N$. Since $C$ is a normal subgroup of $G$, if $C \not\leq N$, then $CN/N$ is a nontrivial normal subgroup of $G/N$. As the proof of the main theorem in [20] shows that socle($G/N$) $\cong$ $\text{PSL}_n(3)$ and so $CN/N$ has a subgroup isomorphic to $\text{PSL}_n(3)$. Therefore $r_{n-1}, r_n \in \pi(\text{PSL}_n(3))$ implies that $r_{n-1}, r_n$ divide the order of $CN/N \cong C/(C \cap N)$. Hence $p \sim r_n$ and $p \sim r_{n-1}$ in $\Gamma(G)$, which implies that $p = 1$, by Lemma 2.3. Therefore $C \leq N$ and $\text{PSL}_n(3)$ acts faithfully on $N$. Also $\text{PSL}_n(3)$ contains Frobenius subgroups of the form $3^{n-1} : (3^{n-1} - 1)/(n, 2)$ and $3^{n-2} : (3^{n-2} - 1)/(n - 1, 2)$. Hence if $p \neq 3$, then using Lemma 2.2 it follows that $p \sim r_{n-1}$ and $p \sim r_{n-2}$ in $\Gamma(G)$. Therefore $p = 2$ or $p = 3$, using Lemma 2.4. Now if $n$ is even, then $2 \sim r_{n-1}$, which is a contradiction. Therefore if $n$ is odd, then $N$ is a $\{2, 3\}$-group and if $n$ is even, then $N$ is a 3-group. \hfill \square

**Corollary 3.1.** Let $\Gamma(G) = \Gamma(\text{PSL}_n(3))$, where $n \geq 9$. Then $G/N \cong \text{PSL}_n(3)$ or $\text{PSL}_n(3) \cdot 2$, the extension of $\text{PSL}_n(3)$ by the graph automorphism, where $N$ is a $\Delta$-group, if $n$ is even and $N$ is a $\{2, 3\}$-group, if $n$ is odd.

**Proof.** We know that using the notations of [7], $f = 1$, $g = 2$ and $d = (n, 2)$. By the assumption, we know that $\text{PSL}_n(3) \leq G := G/N \leq \text{Aut}(\text{PSL}_n(3))$. Let $S = \text{PSL}_n(3)$. Then $G/S \leq \text{Aut}(S)$. Now if $\phi$ is a diagonal automorphism of $S$, and $\psi$ is a graph automorphism of $S$, then $S \cdot \phi$ and $S \cdot (\phi\psi)$ have elements of orders $2r_{n-1}$ and $2r_n$, which is a contradiction, since in the prime graph of $G$ we have $2 \sim r_{n-1}$ if $n$ is even and $2 \sim r_n$ if $n$ is odd. Therefore $G \cong S$ or $S \cdot \psi$, the extension of $\text{PSL}_n(3)$ by the graph automorphism. \hfill \square

**Theorem 3.3.** Let $\omega(G) = \omega(\text{PSL}_n(3))$, where $n \geq 9$. Then $G \cong \text{PSL}_n(3)$ or $\text{PSL}_n(3) \cdot 2$, the extension of $\text{PSL}_n(3)$ by the graph automorphism.
Proof. Using Corollary 3.1 we know that if $n$ is even, then $G/N \cong PSL_n(3)$ or $PSL_n(3) \cdot 2$, where $N$ is a 3-group. Now using Lemma 2.10 it follows that $N = 1$. Therefore $G \cong PSL_n(3)$ or $PSL_n(3) \cdot 2$.

Similarly if $n$ is odd, then $N$ can not be a 3-group. Also as we stated in the proof of Theorem 3.2, $PSL_n(3)$ has a Frobenius subgroup of the form $3^{n-1} : (3^{n-1} - 1)$. Now if $2$ $| |N|$, then using Lemma 2.9 we get that $2(3^{n-1} - 1) \in \omega(PSL_n(3))$, which is a contradiction by [6]. Therefore in each case we have $G \cong PSL_n(3)$ or $PSL_n(3) \cdot 2$.

Remark 3.1. In [29], it is proved that $h(PSL_3(3)) = \infty$. Also $PSL_4(3)$ and $PSL_5(3)$ are recognizable by spectrum (see [10][25]). In [11], it is shown that $h(PSL_6(3)) = 2$. In [9], it is proved that $h(PSL_7(3)) = 2$ and $h(PSL_8(3)) = 1$. Also in [8] for each prime number $p > 3$, the following conjectures arise.

Conjecture 1. If $p \equiv 1$ (mod 3), then $PSL_p(3)$ is 2-recognizable by spectrum.

Conjecture 2. If $p \equiv 2$ (mod 3), then $PSL_p(3)$ is recognizable by spectrum.

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