ON A CONVERGENT PROCESS OF BERNSTEIN

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Abstract. Bernstein in 1930 defined a convergent interpolation process based on the roots of the Chebyshev polynomials. We prove a similar statement for certain Jacobi roots.

1. Introduction. Preliminary results

1.1. In 1930, Bernstein [1] (cf. [2], too) defined the following convergent interpolatory process on the roots of

\[ T_n(x) = \cos(n \arccos x) = \cos n \vartheta, \quad -1 \leq x \leq 1, \quad 0 \leq \vartheta \leq \pi, \quad n = 1, 2, \ldots \]

(Chebyshev polynomials); the roots are

\[ x_{kn} = \cos \vartheta_{kn} = \cos \frac{2k - 1}{2n} \pi, \quad k = 1, 2, \ldots, n; \quad n = 1, 2, \ldots. \]  

Let \( l, q \) be natural numbers; for simplicity we suppose that \( n = 2lq \). We divide the nodes into \( q \) rows as follows.

\[
\begin{align*}
&x_{1n} & &x_{2n} & &\cdots & &x_{2l,n} \\
x_{2l+1,n} & &x_{2l+2,n} & &\cdots & &x_{4l,n} \\
&\vdots & &\vdots & &\vdots & &\vdots \\
x_{2l(q-1)+1,n} & &x_{2l(q-1)+2,n} & &\cdots & &x_{2lq,n}
\end{align*}
\]

If \( f \in C \) (the set of continuous functions on \([-1, 1])\) and

\[ \ell_{kn}(T, x) = \frac{T_n(x)}{T_n(x_{kn})(x - x_{kn})}, \quad k = 1, 2, \ldots, n; \quad n = 1, 2, \ldots \]

are the Lagrange fundamental polynomials based on (1.1) we define the following interpolatory polynomials \( Q_{nl} \) if \( l = 1, 2 \) and 3.

\[ Q_{n1}(f, x) = Q_{n1}(f) = \left\{ f_1(\ell_1 + \ell_2) \right\} + \left\{ f_2(\ell_3 + \ell_4) \right\} + \left\{ f_3(\ell_5 + \ell_6) \right\} + \cdots + \left\{ f_{n-1}(\ell_{n-1} + \ell_n) \right\}, \]

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Dedicated to Professor Giuseppe Mastroianni on the occasion of his retirement.
(1.3) \[ Q_{n2}(f, x) \equiv Q_{n2}(f) \]
\[ = \{ f_1(t_1 + \ell_4) + f_2(\ell_2 - \ell_4) + f_3(\ell_3 + \ell_4) \} \]
\[ + \{ f_5(\ell_5 + \ell_8) + f_6(\ell_6 - \ell_8) + f_7(\ell_7 + \ell_8) \} \]
\[ + \{ f_9(\ell_9 + \ell_{12}) + f_{10}(\ell_{10} - \ell_{12}) + f_{11}(\ell_{11} + \ell_{12}) \} + \ldots \]
\[ + \{ f_{n-3}(\ell_{n-3} + \ell_n) + f_{n-2}(\ell_{n-2} - \ell_n) + f_{n-1}(\ell_{n-1} + \ell_n) \} \],

(1.4) \[ Q_{n3}(f, x) \equiv Q_{n3}(f) \]
\[ = \{ f_1(t_1 + \ell_6) + f_2(\ell_2 - \ell_6) + f_3(\ell_3 + \ell_6) + f_4(\ell_4 - \ell_6) + f_5(\ell_5 + \ell_6) \} \]
\[ + \{ f_7(\ell_7 + \ell_6) + f_8(\ell_8 - \ell_12) + f_9(\ell_9 + \ell_12) \}
\[ + f_{10}(\ell_{10} - \ell_12) + f_{11}(\ell_{11} + \ell_12) \} + \ldots \]
\[ + \{ f_{n-5}(\ell_{n-5} + \ell_n) + f_{n-4}(\ell_{n-4} - \ell_n) + f_{n-3}(\ell_{n-3} + \ell_n) \}
\[ + f_{n-2}(\ell_{n-2} - \ell_n) + f_{n-1}(\ell_{n-1} + \ell_n) \} \].

The definitions for \( l \geq 4 \) are analogous:

(1.5) \[ Q_{nl}(f, x) \equiv Q_{nl}(f) \]
\[ = \{ f_1(t_1 + \ell_2l) + f_2(\ell_2 - \ell_2l) + \ldots + f_{2l-1}(\ell_{2l-1} + \ell_2l) \} + \]
\[ + \{ f_{2l+1}(\ell_{2l+1} + \ell_{2l}) + f_{2l+2}(\ell_{2l+2} - \ell_{2l}) + \ldots + f_{4l-1}(\ell_{4l-1} + \ell_4l) \} + \ldots \]
\[ + \{ f_{n-(2l-1)}(\ell_{n-(2l-1)} + \ell_n) + \ldots + f_{n-1}(\ell_{n-1} + \ell_n) \} . \]

You may consult with \cite{1} or \cite{2} (above \( f_k = f(x_{kn}) \) and \( \ell_k \equiv \ell_{kn}(T, x) \); moreover \( q \) is large enough).

If \( N = n + r, n = 2lq, 0 < r < 2l \), the definition of \( Q_{Nl} \) is as follows (cf. \cite{1} or \cite{2})
\[ Q_{Nl}(f) := Q_{nl}(f) + \sum_{k=n+1}^{N} f_k \ell_k. \]

1.2. By the above definitions we have with \( e_0(x) \equiv 1 \)

(1.6) \[ Q_{nl}(e_0, x) \equiv \sum_{k=1}^{n} \ell_{kn}(T, x) \equiv 1, \]

(1.7) \[ Q_{nl}(f, x_{kn}) = f(x_{kn}) \quad \text{if} \quad k \neq 2l, 4l, \ldots, 2lq, \]

i.e. \( Q_{nl} \) interpolates at \( n - q = 2lq - q \) nodes. This number is "very close" to \( n \) if the (fixed) \( l \) is large enough while \( q \) (and \( n \), too) tends to infinity, i.e., for large \( l \) our \( Q_{nl} \) is "very close" to the Lagrange interpolation \( L_{n} \). However, \( Q_{nl} \) converges for every \( f \in C \), when \( n \to \infty \) (cf. Proposition 1.1 and Theorem 2.1), which generally does not hold for \( L_{n} \).

Later we use that (1.6) and (1.7) hold true for arbitrary point system.
1.3. In [1] Bernstein proved

**Proposition 1.1.** Let \( l \) be a fixed positive integer and \( f \in C \). Then

\[
\lim_{n \to \infty} \|f(x) - Q_{nl}(f, x)\| = 0.
\]

Above, \( \|g(x)\| = \max_{|x| \leq 1} |g(x)|, \) \( g \in C \). Actually, he proved for \( N = n + r \), too; the case when \( N = n + r \) demands only small technical changes in the proof.

1.4. The Bernstein process and its generalizations were exhaustively investigated by Kis (sometimes with coauthors). For more details we suggest the papers [6, 7, 8] and references therein.

2. The Bernstein process for Jacobi abscissas

2.1. The aim of this note is to prove a statement similar to Proposition 1.1 for Jacobi roots. Let the Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \) be defined by

\[
(1 - x)^{\alpha} (1 + x)^{\beta} P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \left[ (1 - x)^{\alpha + n} (1 + x)^{\beta + n} \right] \quad (\alpha, \beta > -1).
\]

For the roots \( x_{k_1}^{(\alpha, \beta)} = \cos \vartheta_{k_1}^{(\alpha, \beta)}, 0 < \vartheta_{k_1}^{(\alpha, \beta)} < \pi, \) of \( P_n^{(\alpha, \beta)}(x) \) we have

\[
-1 < x_{\alpha, \beta}^{(\alpha, \beta)} < x_{\alpha, \beta}^{(\alpha, \beta)} < \cdots < x_{\alpha, \beta}^{(\alpha, \beta)} < 1.
\]

Let

\[
\ell_{k_1}^{(\alpha, \beta)}(x) = \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(x_{k_1})}(x_{k_1} - x_{k_1}).
\]

For a fixed positive integer \( l \), we define \( Q_{nl}^{(\alpha, \beta)}(f, x) \) according to (1.2)–(1.5); now \( \ell_k \) and \( f_k \) stand for \( \ell_k^{(\alpha, \beta)}(x) \) and \( f(x_{\alpha, \beta}^{(\alpha, \beta)}) \), respectively. As we noticed we have the properties analogous to (1.6) and (1.7) for \( Q_{nl}^{(\alpha, \beta)}(f, x) \), too.

2.2. We prove (compare with Vétesi [3] dealing with Lagrange interpolation)

**Theorem 2.1.** Let \( l \) be a fixed positive integer, \( n = 2\ell q \) \( (q = 1, 2, \ldots) \) and \( f \in C \). Then

\[
\lim_{n \to \infty} \|f(x) - Q_{nl}^{(\alpha, \beta)}(f, x)\| = 0
\]

for any processes \( Q_{nl}^{(\alpha, \beta)} \) supposing \(-1 < \alpha, \beta < 0.5\).

Our statement follows from the next more informative pointwise estimations (compare with the result in Vétesi [4] on Lagrange interpolation).

**Theorem 2.2.** Let \( l \) be fixed natural number. Then for arbitrary fixed \( \alpha, \beta > -1 \) and \( f \in C \)

\[
\left| Q_{nl}^{(\alpha, \beta)}(f, x) - f(x) \right| = O(1) \sum_{i=1}^{n} \omega \left( f; \frac{\sqrt{1 - x^2}}{n^2} - \frac{x^2}{n^2} \right) \frac{1}{i^\gamma}
\]

uniformly in \( n \) and \( x \in [-1, 1] \), where \( \gamma = \min(2; 1.5 - \alpha; 1.5 - \beta) \). \( (\omega(f; t) \) is the modulus of continuity of \( f(x) \).)
2.3. It is easy to get (2.1) using Theorem 2.2. Indeed, let

\[ \varepsilon_n = \begin{cases} \frac{1}{4} \log n & \text{if } -1 < \alpha, \beta \leq -0.5, \\ n^\delta & \text{if } \max(\alpha, \beta) =: \delta > -0.5. \end{cases} \]

We have by (2.2)

\[ \|Q_{nl}(f, x) - f(x)\| = O(1)\omega(f; \varepsilon_n) \]

if \( f \in C \), whence we obtain (2.1).

2.4. Another consequence of Theorem 2.2 is the following

**Corollary 2.1.** If \(-1 < \alpha, \beta \leq -0.5 \) and \( \omega(f; t) \sim t^\varrho \) \((0 < \varrho < 0.5) \) then for \( f \in C \)

\[ |Q_{nl}(f, x) - f(x)| = O(1)\left[ \left( \frac{1}{n} \sqrt{1 - x^2} \right)^\gamma + \frac{1}{n^\gamma} \right] \]

uniformly for \( n \) and \( |x| \leq 1 \). This formula of Timan type can be obtained by simple calculation.

Other estimations showing the connections between the parameters \( \gamma \in (0, 2] \) and \( \varrho \in (0, 1] \) are as follows

\[ |Q_{nl}(f, x) - f(x)| = O(1) \left\{ \begin{array}{ll} \left( \frac{1}{n} \sqrt{1 - x^2} \right)^\gamma & \text{if } 0 < \varrho < \frac{1}{2}(\gamma - 1), \\ n^{-2\varrho} \log n & \text{if } \varrho = \frac{1}{2}(\gamma - 1), \\ n^{\gamma - 1} & \text{if } \frac{1}{2}(\gamma - 1) < \varrho < \gamma - 1; \end{array} \right. \]

uniformly for \( n \) and \( |x| \leq 1 \). These formulae can be obtained by simple calculation.

2.5. It is interesting to compare (2.2) to

\[ |H_n^{(\alpha, \beta)}(f; x) - f(x)| = O(1) \sum_{i=1}^{n} \omega \left( f; \frac{1}{n} \sqrt{1 - x^2} i + \frac{1}{n^2} i^{2\gamma - 1} \right) (x \in [-1, 1]) \]

where \( H_n^{(\alpha, \beta)}(f; x) \) is the Hermite–Fejér interpolatory polynomial of degree \( \leq 2n - 1 \) defined by \( H_n^{(\alpha, \beta)}(f; x_{kn}^{(\alpha, \beta)}) = f(x_{kn}^{(\alpha, \beta)}) \), \( H_n^{(\alpha, \beta)}(f; x_{kn}^{(\alpha, \beta)}) = 0 \) \((k = 1, 2, \ldots, n), f \in C \) and \( \eta = \max\{-0.5, \alpha, \beta\} \) (see [5, 2.1]).

3. Proof of Theorem 2.2

We apply the main idea from [4]. Let \( x = \cos \vartheta, x \in [-1, 1], \vartheta \in [0, \pi] \) and define the index \( j = j(n) \) by \( \min_{1 \leq k \leq n} |x - x_{kn}^{(\alpha, \beta)}| = |x - x_{jn}^{(\alpha, \beta)}| \).
3.1. First let \( l = 1 \). By (1.5) and (1.6) we can write

\[
Q_{nl}^{(\alpha, \beta)}(f, x) - f(x) = \sum_{k=1}^{q} \left\{ f(x_{2k-1}^{(\alpha, \beta)}) - f(x) \right\} \left( \ell_{2k-1}^{(\alpha, \beta)}(x) + \ell_{2k}^{(\alpha, \beta)}(x) \right)
\]

\[
= \sum_{k=1}^{q} \cdots + \sum_{k>q} \cdots = \sum_{l} + \sum_{II}.
\]

Now we use Lemma 4.1 of [3], which says the following: Let \(-1 < \alpha, \beta \) and \( \varepsilon, \eta > 0 \) be fixed. If \( k \geq M, \) \( \theta_{kn}^{(\alpha, \beta)} \leq \pi - \varepsilon, \) then for any \( x \in [-1 + \eta, 1] \) we have

\[
|\ell_{kn}^{(\alpha, \beta)}(x) + \ell_{k+1,n}^{(\alpha, \beta)}(x)| = O(1) \left| \ell_{kn}^{(\alpha, \beta)}(x) \right| \left[ \frac{1}{k} + \frac{k}{(k+j)(|k-j|+1)} \right]
\]

uniformly in \( x \) and \( k \).

We note that instead of \( \ell_{kn}^{(\alpha, \beta)}(x) \) of [3] one can write \( \ell_{k+1,n}^{(\alpha, \beta)}(x) \). Moreover (3.2) obviously holds true if \( 1 \leq k \leq M \) (maybe with another \( O(1) \)).

From (3.1) with obvious short notations we have

\[
\sum_{l} = O(1) \sum_{k=1}^{q} |f(x_{2k-1}) - f(x)|
\]

\[
\times \left( |\ell_{2k-1}(x)| \left[ \frac{1}{2k-1} + \frac{2k-1}{(2k-1+j)(|2k-1-j|+1)} \right] \right)
\]

if \( \alpha, \beta > -1 \) and \( \varepsilon, \eta > 0 \) are fixed.

By (3.3) we get as in [3]: If \( \gamma = \min(2; 1.5 - \alpha; 1.5 - \beta) \), then

\[
\sum_{k=1}^{n-1} |f(x_{2k-1}) - f(x)| |\ell_{2k-1}(x) + \ell_{2k}(x)| = O(1) \sum_{i=1}^{n} \omega \left( f; \frac{\sin \frac{i}{n}}{n \frac{i^2}{n^2}} \right) \frac{1}{i^{\gamma}}
\]

uniformly in \( x \in [-1, 1]; \) see [3] 4.10, where \( \sum |f - f_k| |\ell_k^{k-1}| \) (which by (3.2), is analogous to \( \sum |f - f_k| |\ell_k + \ell_{k+1}| \)) is estimated.

Let us remark that getting (3.1) we have to define \( J = [\varrho_{j+1,n}^{(\alpha, \beta)}, \varrho_{j+1,n}^{(\alpha, \beta)}] \) and for \( r = 1, 2, \ldots \)

\[
I_r = [\varrho_{j-r,n}^{(\alpha, \beta)}, \varrho_{j-r,n}^{(\alpha, \beta)}], \quad K_r = [\varrho_{j+2r-1,n}^{(\alpha, \beta)}, \varrho_{j+2r,n}^{(\alpha, \beta)}]
\]

instead of the definition (4.2) of [4].

From the above formulas we obtain our theorem for \( l = 1 \).

3.2. Now let \( l = 2 \). By (1.3) and (1.6) we get

\[
Q_{n2}(f) - f = \{(f_1 - f)(\ell_1 + \ell_4) + (f_2 - f)(\ell_2 - \ell_4) + (f_3 - f)(\ell_3 + \ell_4)\}
\]

\[
+ \{(f_5 - f)(\ell_5 + \ell_8) + (f_6 - f)(\ell_6 - \ell_8) + (f_7 - f)(\ell_7 + \ell_8)\}^1_2 + \cdots.
\]
In $\{\cdots\}_1$, 
\[
|\ell_1 + \ell_4| = |(\ell_1 + \ell_2) - (\ell_2 + \ell_3) + (\ell_3 + \ell_4)| \leq |\ell_1 + \ell_2| + |\ell_2 + \ell_3| + |\ell_3 + \ell_4| \\
\leq c \sum_{k=1}^3 |\ell_k(x)| \left( \frac{1}{k} + \frac{k}{|k + j|(|k - j| + 1)} \right) \\
\leq c \left\{ |\ell_s(x)| \left[ \frac{1}{s} + \frac{s}{|s + j|(|s - j| + 1)} \right] \right\}_{s=k=1}.
\]

Here we used \(3.2\) and that $|\ell_k(x) \cdot \ell_{k+1}^{-1}| \sim 1$ for any $k$ whenever $0 \leq m \leq C$.

Similar considerations are valid for the second term in $\{\cdots\}_1$ by $\ell_2 - \ell_4 = (\ell_2 + \ell_3) - (\ell_3 + \ell_4)$.

Taking into account that $|f_k - f| \leq c\omega \left( \frac{\sin \vartheta}{n} i + \frac{i^2}{n^2} \right)$ whenever $i = |k - j| + m$ (see \(4.4\)); $0 \leq m \leq C$), we get that 
\[
|\{\cdots\}_1| \leq c\omega \left( \frac{\sin \vartheta}{n} i + \frac{i^2}{n^2} \right) \left( |\ell_s(x)| \left[ \frac{1}{s} + \frac{s}{|s + j|(|s - j| + 1)} \right] \right)_{s=1}.
\]

Using this last estimation and similar ones for $\{\cdots\}_2, \{\cdots\}_3, \ldots$, we can get \(3.4\).

If $l > 2$, the argument is similar. We may omit the further details. \(\square\)

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