APPROXIMATIONS OF PERIODIC FUNCTIONS
BY ANALOGUE OF ZYGMUND SUMS
IN THE SPACES $L^{p(\cdot)}$

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Abstract. We found order estimates for the upper bounds of the deviations of analogue of Zygmund’s sums on the classes of $(\phi; \beta)$-differentiable functions in the metrics of generalized Lebesgue spaces with variable exponent.

1. Definition and formulation of the problem

Let $p = p(x)$ be a $2\pi$-periodic measurable and essentially bounded function and let $L^{p(\cdot)}$ be space of measurable $2\pi$-periodic functions $f$ such that

$$\int_{-\pi}^{\pi} |f(x)|^{p(x)} \, dx < \infty.$$

If $p := \text{ess inf}_x |p(x)| \geq 1$ and $\bar{p} := \text{ess sup}_x |p(x)| < \infty$, then $L^{p(\cdot)}$ are Banach spaces \cite{16} (see also \cite{8}) with the norm, which can be given by

$$\|f\|_{p(\cdot)} := \inf \left\{ \alpha > 0 : \int_{-\pi}^{\pi} \left| \frac{f(x)}{\alpha} \right|^{p(x)} \, dx \leq 1 \right\}.$$

Here are some definitions which will be used in the statement and proof of the results of this article.

Definition 1.1. It is said that a function $p = p(x)$ satisfies the Dini–Lipschitz condition of order $\gamma$, if $\omega(p; \delta)((\ln \frac{1}{\delta})^{\gamma} \leq K = \text{const}$, $0 < \delta < 1$, where

$$\omega(p; \delta) = \sup_{x_1, x_2 \in [-\pi; \pi]} \left\{ |p(x_1) - p(x_2)| : |x_1 - x_2| \leq \delta \right\}.$$ 

The set of $2\pi$-periodic exponents $p = p(x) > 1$, satisfying the Dini–Lipschitz condition of order $\gamma \geq 1$ in the period, is denoted by $\mathcal{P}^\gamma$. Obviously, if $p \in \mathcal{P}^\gamma$, then $p > 1$ and $\bar{p} < \infty$. In the work \cite{16} shown that when $1 < p, \bar{p} < \infty$, space $L^{p(\cdot)}$

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where \( q(x) = \frac{p(x)}{p(x) - 1} \), is conjugate for \( L^{p(\cdot)} \) and for arbitrary functions \( f \in L^{p(\cdot)} \) and \( g \in L^{q(\cdot)} \) an analogue of the classical Hölder inequality is true:

\[
\int_{-\pi}^{\pi} |f(x)g(x)| \, dx \leq K_{p,q} \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}, \quad (K_{p,q} \leq 1/p + 1/q),
\]

which, in particular, implies embedding: \( L^{p(\cdot)} \subset L \), where \( L \) is space of 2\( \pi \)-periodic Lebesgue integrable on the period functions.

The spaces \( L^{p(\cdot)} \) are called generalized Lebesgue spaces with variable exponent. It is clear, that if \( p = p(x) \equiv \text{const} > 0 \), spaces \( L^{p(\cdot)} \) coincide with the classical Lebesgue spaces \( L^p \). In its turn, if \( \bar{p} < \infty \), spaces \( L^{p(\cdot)} \) are a special case of the so-called Orlicz–Musielak spaces [10].

For the first time, a Lebesgue space with variable exponent appeared in the literature in the article of W. Orlicz [12]. In the work [11] spaces \( L^{p(\cdot)} \) considered as an example of the more general function spaces and, furthermore, have been studied by many authors in different directions. The basic results of the theory of these spaces are available, for example, in [1,2,4,8,10,12,13,16,18,19]. Note also that the generalized Lebesgue spaces with variable exponent used in the theory elastic mechanics, the theory of differential operators, variations calculus [8,13,15].

Next, we need the definitions of the \((\psi; \beta)\)-derivative and the sets \( L^\psi_\beta \), which belongs to A.I. Stepanetz [20, pp. 142–143].

**Definition 1.2.** Let \( f \in L \) and

\[
S[f] = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \left( a_k(f) \cos kx + b_k(f) \sin kx \right) = \sum_{k=0}^{\infty} A_k(f, x)
\]

be its Fourier series. Let, further, \( \psi(k) \) be arbitrary function of natural argument and \( \beta \in \mathbb{R} \). Assume that the series

\[
\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left( a_k(f) \cos \left( kx + \frac{\beta \pi}{2} \right) + b_k(f) \sin \left( kx + \frac{\beta \pi}{2} \right) \right)
\]

is the Fourier series of some function from \( L \). This function is denoted by \( f^\psi_\beta(\cdot) \) (or \( (D^\psi_\beta f)(\cdot) \)) and called \((\psi; \beta)\)-derivative of a function \( f(\cdot) \). The set of functions \( f(\cdot) \), satisfying this condition is denoted by \( L^\psi_\beta \).

Denote by \( \hat{z}_n(f; x) \) (see [5]) the trigonometric polynomials of the form

\[
(1.1) \quad \hat{z}_n(f; x) := \frac{a_0(f)}{2} + \sum_{k=1}^{n-1} \left( 1 - \frac{\psi(n)}{\psi(k)} \right) A_k(f; x).
\]

Note that in the case \( \psi(k) = 1/k^r \), \( r > 0 \), the sums (1.1) are Zygmund’s well known sums

\[
Z_n(f; x) := \frac{a_0(f)}{2} + \sum_{k=1}^{n-1} \left( 1 - \left( \frac{k}{n} \right)^r \right) A_k(f; x).
\]
In this paper, we study the value
\[ E(L_\beta^{\psi}; \hat{Z}_n)x(:) := \sup_{f \in L_\beta^{\psi}} \|f - \hat{Z}_n(f)\|_{p(x)} \]
upper bounds of deviations analogues of Zygmund’s sums \( \hat{Z}_n(f; x) \) on the classes 
\[ L_\beta^{\psi}(x) := \{ f \in L_\beta^{\psi} : f_\beta^{\psi} \in U_{p(x)} \} , \]
where \( U_{p(x)} := \{ \varphi \in L^{p(x)} : \|\varphi\|_{p(x)} \leq 1 \} \) is the unit ball of \( L^{p(x)} \).

2. Auxiliary results

In the proof of the main assertions of this work we use the following well-known results.

**Theorem 2.1.** [19] If \( p \in P^\gamma \), then for an arbitrary function \( f \in L^{p(x)} \) the inequalities hold
\[
\begin{align*}
(2.1) & \quad \|S_n(f)\|_{p(x)} \leq C_p\|f\|_{p(x)}, \\
(2.2) & \quad \|\hat{f}\|_{p(x)} \leq K_p\|f\|_{p(x)},
\end{align*}
\]
where
\[
S_n(f; x) = \frac{a_0(f)}{2} + \sum_{k=1}^{n-1} (a_k(f) \cos kx + b_k(f) \sin kx), \quad n = 0, 1, \ldots,
\]
is the Fourier partial sums of the order \( n \) of function \( f \), \( \hat{f}(\cdot) \) is trigonometric conjugate to \( f(\cdot) \) functions, and \( C_p, K_p \) are positive constants which don’t depend on \( n \) and \( f \).

From inequality (2.1), it follows that for an arbitrary function \( f \in L^{p(x)} \), on condition \( p \in P^\gamma \), its Fourier series converges to \( f \) in the metric of the spaces \( L^{p(x)} \), that is \( \|f - S_n(f)\|_{p(x)} \to 0 \), as \( n \to \infty \), and the relation holds
\[
E_n(f)_{p(x)} \leq \|f - S_{n-1}(f)\|_{p(x)} \leq K_p E_n(f)_{p(x)},
\]
where
\[
E_n(\varphi)_{p(x)} := \inf_{t_{n-1} \in T_{n-1}} \|\varphi - t_{n-1}\|_{p(x)}, \quad \varphi \in L^{p(x)},
\]
is the best approximation of function \( \varphi \) by subspace \( T_{2n-1} \) of trigonometric polynomials of order, not higher than \( n - 1 \), and \( K_p \) is the value which depends only on \( p = p(x) \).

**Lemma 2.1.** [7] Let the sequence \( \mu(k), \ k = 0, 1, 2, \ldots, \) satisfies the conditions
\[
\begin{align*}
\nu_0 = \nu_0(\mu) &= \sup_k |\mu(k)| \leq C, \\
\sigma_0 = \sigma_0(\mu) &= \sup_{m \in \mathbb{N}} \sum_{k=2^m}^{2^{m+1}} |\mu(k+1) - \mu(k)| \leq C,
\end{align*}
\]
where \( C \) is the value which does not depend on \( k \) and \( m \). Then, if \( p \in P^\gamma \), for a given function \( f \in L^{p(x)} \there exists a function \( F \in L^{p(x)} \) such that the series
\[
\frac{\mu(0)a_0(f)}{2} + \sum_{k=1}^{\infty} \mu(k)(a_k(f) \cos kx + b_k(f) \sin kx)
\]
is the Fourier series of $F$ and the inequality is true
\begin{equation}
\|F\|_{p(\cdot)} \leq K\lambda \|f\|_{p(\cdot)}, \quad \lambda = \max\{\nu_0, \sigma_0\},
\end{equation}
where the value $K$ does not depend on the function $f$.

In the case $p = p(x) \equiv \text{const}$ this statement is a well-known lemma of Marcinkiewicz for multipliers [9].

We will also use the following theorem of Hardy–Littlewood.

**Theorem 2.2.** [6] Let $1 < p < s < \infty$, $p, s = \text{const}$, and $D_n(t) := \sum_{k=1}^{\infty} k^{-\alpha} \cos kt$. Then, for an arbitrary function $\varphi \in L_p$ the convolution
\begin{equation}
\Phi_{\alpha}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x + t) D_n(t) \, dt
\end{equation}
belongs to $L_s$, and $\|\Phi_{\alpha}\|_s \leq C_{s,p} \|\varphi\|_p$, where $C_{s,p}$ depends on $s$ and $p$ only.

Note that if $\varphi \in L_p$ and $S[\varphi] = \sum_{k=0}^{\infty} A_k(\varphi; x)$, then
\begin{equation}
S[\Phi_{\alpha}] = \sum_{k=0}^{\infty} k^{-\alpha} A_k(\varphi; x),
\end{equation}
that is $\Phi_{\alpha} = M_{\alpha}(\varphi)$, where $M_{\alpha}$ is an operator-multiplier, which is determined by the sequence of $\mu_{\alpha}(k) = k^{-\alpha}$, $k = 0, 1, 2, \ldots$, and it acts from $L_p$ to $L_s$, where indicators $1 < p < s < \infty$, $p, s = \text{const}$, are related by the equation $p^{-1} - s^{-1} = \alpha$.

3. Approximation by analogue of Zygmund’s sums

We define the sequences $\mu(k)$ and $\tilde{\mu}(k)$, $k \in \mathbb{N}$, as follows:
\begin{align}
\mu(k) &= \mu_{n,\alpha}(k) := \begin{cases} 
k^{-\alpha} \psi(n) \cos \frac{\beta x}{2}, & 1 \leq k \leq n - 1, \\
k^{-\alpha} \psi(k) \cos \frac{\beta t}{2}, & n \leq k,
\end{cases} \\
\tilde{\mu}(k) &= \tilde{\mu}_{n,\alpha}(k) := \begin{cases} 
k^{-\alpha} \psi(n) \sin \frac{\beta x}{2}, & 1 \leq k \leq n - 1, \\
k^{-\alpha} \psi(k) \sin \frac{\beta t}{2}, & n \leq k,
\end{cases}
\end{align}
where $n \in \mathbb{N}$, $\alpha \geq 0$, $\beta \in \mathbb{R}$. For each fixed $\alpha \geq 0$ we denote by $\Upsilon_{\alpha,n}$ the set of pairs $(\psi; \beta)$, such that for any positive number $n$ the conditions
\begin{align}
\nu_{\alpha}(\psi; \beta; n) := \sup_k |\mu_{n,\alpha}(k)| &\leq C \nu(n)n^\alpha < K, \\
\sigma_{\alpha}(\psi; \beta; n) := \sup_{m \in \mathbb{N}} \sum_{k=2m}^{2m+1} |\mu_{n,\alpha}(k + 1) - \mu_{n,\alpha}(k)| &\leq C \nu(n)n^\alpha < K,
\end{align}
and similar conditions for the function $\tilde{\mu}_{n,\alpha}(k)$ hold, where $\nu(n) = \sup_{k \geq n} |\psi(k)|$, $C$ and $K$ are positive constants uniformly bounded on $n$.

At first we consider the case when the functions $p = p(x)$ and $s = s(x)$ on the period satisfy the inequality $s(x) \leq p(x)$. In our notation, the following assertion is true.
Theorem 3.1. Let \((\psi; \beta) \in \mathcal{Y}_{0,n}, p, s \in \mathcal{P}^\gamma, s(x) \leq p(x), x \in [0; 2\pi]\). Then, for all \(n \in \mathbb{N}\) the inequality
\[
C_{p,s}(n) \leq \mathcal{E}(L^p_\beta, \hat{Z}_n)(x) \leq K_{p,s}(n),
\]
holds, where \(C_{p,s}\) and \(K_{p,s}\) are some constants depending on \(p = p(x)\) and \(s = s(x)\) only.

**Proof.** Suppose first that \(p(x) \equiv s(x), x \in [0; 2\pi]\). For an arbitrary function \(f \in L^p_\beta, p(x)\), the equality is true
\[
f(x) - \hat{Z}_n(f; x) = \sum_{k=1}^{n-1} \frac{\psi(n)}{\psi(k)} A_k(f; x) + \sum_{k=n}^{\infty} A_k(f; x)
\]
\[
= \sum_{k=1}^{\infty} \mu_{n,0} A_k(f^n_\beta; x) + \sum_{k=1}^{\infty} \tilde{\mu}_{n,0} \tilde{A}_k(f^n_\beta; x) := M_0(f^n_\beta) + \tilde{M}_0(f^n_\beta),
\]
where \(M_0\) and \(\tilde{M}_0\) are operators-multipliers, which are defined by the sequences (3.1) and (3.2), respectively, \(\alpha = 0\). According to the conditions of the theorem, the couples \((\psi; \beta)\) belong to the set \(\mathcal{Y}_{0,n}\), therefore, sequence (3.1) and (3.2) satisfy the conditions of Lemma 2.1. Applying this lemma, given inequalities (2.2), (3.3) and (3.4), for an arbitrary function \(f \in L^p_\beta, p(x)\), on the basis of the equality (3.5) we find
\[
\|f(\cdot) - \hat{Z}_n(f; \cdot)\|_{\mathcal{P}(x)} = \|M_0(f^n_\beta) + \tilde{M}_0(f^n_\beta)\|_{\mathcal{P}(x)}
\]
\[
\leq K\nu(n)(\|f^n_\beta\|_{\mathcal{P}(x)} + \|\tilde{f}^n_\beta\|_{\mathcal{P}(x)}) \leq C_p\nu(n),
\]
where \(C_p\) is a positive constant, which depends only on the function \(p = p(x)\). In the article [17] it was shown that if \(1 \leq s(x) \leq p(x) \leq \tilde{p} < \infty\), then for an arbitrary function \(f \in L^\tilde{p}(\cdot)\) the inequality holds
\[
\|f\|_{\mathcal{P}(x)} \leq K_{s, p}\|f\|_{\mathcal{P}(x)}.
\]

From relations (3.6) and (3.7) we obtain the estimate
\[
\mathcal{E}(L^p_\beta, \hat{Z}_n)(x) \leq \mathcal{E}(L^p_\beta, \hat{Z}_n)(x) \leq C_{p,s}\nu(n),
\]
where \(C_{p,s}\) is a positive constant, which depends only on the functions \(p = p(x)\) and \(s = s(x)\). We now obtain the lower estimate for the value of \(\mathcal{E}(L^p_\beta, \hat{Z}_n)(x)\). If for given sequence \(\psi(k)\) and the number \(n \in \mathbb{N}\), there exists the natural number \(k_n\), for which the equality
\[
\nu(n) = \sup_{k \geq n} |\psi(k)| = \psi(k_n),
\]
is true, then the corresponding lower estimate can be obtained with help of the function
\[
f_n(x) = \frac{\psi(k_n)}{\|\cos k_n x\|_{\mathcal{P}(x)}} \cos \left( k_n x - \frac{\beta \pi}{2} \right).
\]
Indeed, since
\[ \| (f_n(x))^\psi \|_{p(\cdot)} = \left\| \frac{\cos k_n x}{\cos k_n x} \right\|_{p(\cdot)} = 1, \]
then \( f_n \in L^\psi_{p(\cdot)} \) and
\[ \mathcal{E}(L^\psi_{p(\cdot)}; \hat{Z}_n)_{s(\cdot)} = \| f_n - \hat{Z}(f_n) \|_{s(\cdot)} = \frac{\psi(k_n)}{\| \cos k_n x \|_{p(\cdot)}} \| \cos k_n x \|_{s(\cdot)} = C_{p,s} \nu(n). \]

But if for the sequence \( \psi(k) \) and the number \( n \in \mathbb{N} \), there exists no natural number \( k_n \), for which equality (3.1) holds, due to the limitation of the set \( \{ |\psi(k)| \} \) of values of the function \( \psi(k) \) we will have
\[ \nu(n) = \sup_{k \geq n} |\psi(k)| = \sup_{k \geq n} \{ |\psi(k)| \}. \]

In this case, there exists a sequence \( k_j, j \in \mathbb{N} \) such that \( k_j > n \) and the numbers \( \psi(k_j) \) don’t decrease and converge to \( \nu(n) \). Let \( \Lambda = \bigcup_j f_j(x) \), where the function \( f_j(x) \) is defined by equality (3.2). Since \( f_j \in L^\psi_{p(\cdot)} \) for any \( j \in \mathbb{N} \), then
\[ \mathcal{E}(L^\psi_{p(\cdot)}; \hat{Z}_n)_{s(\cdot)} = \sup_{f \in L^\psi_{p(\cdot)}} \| f - \hat{Z}_n(f) \|_{s(\cdot)} \geq \sup_{f \in \Lambda} \| f - \hat{Z}_n(f) \|_{s(\cdot)} \]
\[ = \sup_{j \in \mathbb{N}} \frac{\psi(k_j)}{\| \cos k_j x \|_{p(\cdot)}} \| \cos k_j x \|_{s(\cdot)} = C_{p,s} \nu(n). \]

We now obtain an estimate of the sequence \( \mathcal{E}(L^\psi_{p(\cdot)}; \hat{Z}_n)_{s(\cdot)} \) in the case when the function \( p = p(x) \) and \( s = s(x) \) on the period satisfies the inequality \( p(x) < s(x) \). The following result gives the upper estimate.

**Theorem 3.2.** Let \( p, s \in \mathcal{P}^\gamma \), \( p(x) < s(x) \), \( x \in [0, 2\pi] \) and \( (\psi; \beta) \in \Upsilon_{\alpha,n} \), \( \alpha = 1/p - 1/\gamma \). Then, for all \( n \in \mathbb{N} \) the following inequality
\[ \mathcal{E}(L^\psi_{p(\cdot)}; \hat{Z}_n)_{s(\cdot)} \leq C_{p,s} n^\alpha \nu(n), \]
holds, where \( C_{p,s} \) is a positive constant depending on \( p = p(x) \) and \( s = s(x) \) only.

**Proof.** For an arbitrary function \( f \in L^\psi_{p(\cdot)} \), the equality
\[ (3.10) \quad f(x) - \hat{Z}_n(f; x) = \sum_{k=1}^{n-1} \frac{\psi(n)}{\psi(k)} A_k(f; x) + \sum_{k=n}^{\infty} A_k(f; x) \]
\[ = \sum_{k=1}^{\infty} \mu_{n,k} k^{-\alpha} A_k(f^\psi_\beta; x) + \sum_{k=1}^{\infty} \tilde{\mu}_{n,k} k^{-\alpha} \tilde{A}_k(f^\psi_\beta; x) \]
\[ = : M_n(g_\alpha) + \tilde{M}_n(\tilde{g}_\alpha), \]
holds, where \( M_\alpha \) and \( \tilde{M}_\alpha \) are operators-multipliers, which are defined by the sequences (3.1) and (3.2) respectively, \( \alpha = 1/p - 1/\gamma \) and
\[ g_\alpha(x) := \sum_{k=1}^{\infty} k^{-\alpha} A_k(f^\psi_\beta; x) = \frac{1}{\pi} \int_0^{2\pi} \left[ f^\psi_\beta(x + t)D_\alpha(t)dt, \right. \]
\[ \tilde{g}_n(x) := \sum_{k=1}^{\infty} k^{-\alpha} \hat{A}_k(f^\psi_t; x) = \frac{1}{\pi} \int_0^{2\pi} f^\psi_t(x + t) D_n(t) dt, \]

\( D_n(t) \) is function defined in Theorem 2.2.

Since \( f \in L^\psi_{\beta,p'(\cdot)} \), then \( f^\psi_t \in L^p(\cdot) \), and moreover \( f^\psi_{\beta} \in L^\infty \). By Theorem 2.2, we conclude that the convolution \( g_n(x) \) belongs to \( L^\infty \), and moreover \( g_n \in L^\alpha(\cdot) \). From the condition \((\psi; \beta) \in \mathcal{T}_{n,\alpha}\) by Lemma 2.1, we conclude that the operator-multiplier \( M_n \) acts from \( L^\alpha(\cdot) \) to \( L^\infty(\cdot) \) for any \( s \in \mathcal{P}^\tau \). Using analogous arguments for the function \( \tilde{g}_n(x) \), taking into account inequalities (3.10), (3.11), and (3.12), for an arbitrary function \( f \in L^\psi_{\beta,p'(\cdot)} \) on the basis of equality (3.11), we find

\[
\| f - \tilde{Z}_n(f) \|_{\tau(s)} \leq \| M_n(g_n) \|_{\tau(s)} + \| M_n(\tilde{g}_n) \|_{\tau(s)} \leq K n^\alpha \nu(n)(\| g_n \|_{\tau(s)} + \| \tilde{g}_n \|_{\tau(s)})
\]

\[
\leq C_{p,s} n^\alpha \nu(n)(\| f^\psi_{\beta} \|_{p'(\cdot)} + \| f^\psi_{\beta} \|_{p(\cdot)}) \leq C_{p,s} n^\alpha \nu(n). \quad \square
\]

To make formulate the following assertion, which gives a lower estimate for the quantity \( \mathcal{E}(L^\psi_{\beta,p'(\cdot)}; \tilde{Z}_n)_{s(\cdot)} \) in the case, if the function \( p = p(x) \) and \( s = s(x) \) satisfies the inequality \( p(x) < s(x) \) on the period, we need the following definition.

Denoting by \( \mathfrak{B} \) the set of pairs \((\psi; \beta)\), such that for any \( n \in \mathbb{N} \) the relations are true:

\[
\sup_{n \leq k \leq 2n} \left| \frac{\nu(n)}{\psi(k)} \right| \leq C, \quad \sup_{m \in \mathbb{N}} \sum_{k=2m}^{2m+1} |\tau(k + 1) - \tau(k)| \leq C,
\]

where \( C \) is a positive constant, which is independent of \( n, \nu(n) = \sup_{k \geq n} \psi(k) \) and \( \tau(k) := \begin{cases} 0, & 1 \leq k \leq n - 1, \\ \frac{\nu(n)}{\psi(k)}, & n \leq k \leq 2n. \end{cases} \)

Theorem 3.3. Let \( p, s \in \mathcal{P}^\tau, p(x) < s(x), x \in [0; 2\pi] \) and \((\psi; \beta) \in \mathfrak{B}\). Then, for all \( n \in \mathbb{N} \) we have \( \mathcal{E}_n(L^\psi_{\beta,p'(\cdot)}; \tilde{Z}_n)_{s(\cdot)} \geq C_{p,s} n^{1/\mathfrak{P} - 1/2} \), where \( C_{p,s} \) is a positive constant depending on \( p = p(x) \) and \( s = s(x) \) only.

Proof. For obtaining a lower estimate, let us show that for any positive integer \( n \) in class \( L^\psi_{\beta,p'(\cdot)} \) there exists a function \( f^*_n \), for which the inequality is true

\[
\| f^*_n - \tilde{Z}_n \|_{\tau(s)} \geq C_{p,s} n^{1/\mathfrak{P} - 1/2}.
\]

For this, we fix \( n \in \mathbb{N} \) and consider the function \( f^*_n(x) = \sum_{k=n}^{2n} \psi(k) \cos \left( kx - \frac{2\pi}{\mathfrak{P}} \right). \)

Since

\[
(f^*_n(x))^\psi_{\beta} = \sum_{k=n}^{2n} \cos kx = \frac{\sin nx/2 \cos(3n+1)x/2}{\sin x/2},
\]

then using relation (3.7) and also the well-known inequality

\[
x/\pi \leq \sin x/2, \quad \sin x \leq x, \quad x \in [0; \pi],
\]

(3.12)
we obtain
\[ \| (f_n^*)_2^\bullet \|_{\mathcal{L}(\mathcal{B})} = \left\| \sum_{k=n}^{2n} \cos kx \right\|_{\mathcal{L}(\mathcal{B})} \leq K_p \left\| \sum_{k=n}^{2n} \cos kx \right\|_{\mathcal{B}} \]
\[ = \left( 2 \int_0^\pi \left| \sum_{k=n}^{2n} \cos kx \right|^2 \frac{dx}{\sin x/2} \right)^{1/2} \leq \left( 2 \int_0^\pi \sin nx/2 \frac{dx}{\sin x/2} \right)^{1/2} \]
\[ \leq C_p n^{1-1/\mathcal{B}}. \]

This implies that the function
\[ g_n^* (x) = \frac{n^{1/\mathcal{B} - 1}}{C_p} f_n^* (x) = \frac{n^{1/\mathcal{B} - 1}}{C_p} \sum_{k=n}^{2n} \psi (k) \cos \left( \frac{kx - \beta \pi}{2} \right) \]

belongs to the class \( L_{\mathcal{B}, p}^\bullet \).

Again using the inequalities (3.11) and (3.12), we find
\[ \left\| \sum_{k=n}^{2n} \cos \left( \frac{kx - \beta \pi}{2} \right) \right\|_{s(\cdot)} \geq \left( 2 \int_0^\pi \left| \frac{\sin nx/2 \cos (3n + 1)x/2}{\sin x/2} \right|^2 \frac{dx}{\sin x/2} \right)^{1/2} \]
\[ \geq C_s n^{1-1/\mathcal{B} / (\cos x \mathcal{B})} \left( \int_0^\pi (\cos x \mathcal{B}) dx \right)^{1/2} \geq K_s n^{1-1/\mathcal{B}}. \]

If now by \( T_\psi \) we denote the operator-multiplier that generates sequence (3.11), then by applying Lemma [2.1] to the condition \( (\psi; \beta) \in \mathcal{B} \) we will have
\[ \left\| \sum_{k=n}^{2n} \psi (k) \cos \left( \frac{kx - \beta \pi}{2} \right) \right\|_{s(\cdot)} \leq C \left\| \sum_{k=n}^{2n} \psi (k) \cos \left( \frac{kx - \beta \pi}{2} \right) \right\|_{s(\cdot)} \]
\[ \leq C \left\| \sum_{k=n}^{2n} \psi (k) \cos \left( \frac{kx - \beta \pi}{2} \right) \right\|_{s(\cdot)}. \]

Hence, considering inequality (3.13) we find
\[ \left\| \sum_{k=n}^{2n} \psi (k) \cos \left( \frac{kx - \beta \pi}{2} \right) \right\|_{s(\cdot)} \geq K \left\| \sum_{k=n}^{2n} \cos \left( \frac{kx - \beta \pi}{2} \right) \right\|_{s(\cdot)} \]
\[ \geq C_s \left\| \sum_{k=n}^{2n} \cos kx \right\|_{\mathcal{B}} \geq K_s n^{1-1/\mathcal{B}}. \]

Using relation (3.14), we obtain
\[ E_n (L_{\beta, \mathcal{B}, p}^\psi; \mathcal{Z}_n)_{s(\cdot)} \geq \| g_n - \mathcal{Z}_n (g_n^*) \|_{s(\cdot)} \]
\[ \geq n^{1/\mathcal{B} - 1} \frac{C_p}{\nu(n)} \left\| \sum_{k=n}^{2n} \psi (k) \cos \left( \frac{kx - \beta \pi}{2} \right) \right\|_{s(\cdot)} \]
\[ \geq n^{1/\mathcal{B} - 1} \frac{C_p \nu(n)}{\nu(n)} \left\| \sum_{k=n}^{2n} \psi (k) \cos \left( \frac{kx - \beta \pi}{2} \right) \right\|_{s(\cdot)} \]
\[ \geq C_{p, s} n^{1/\mathcal{B} - 1} \nu(n) n^{1-1/\mathcal{B}} = C_{p, s} \nu(n) n^{1/\mathcal{B} - 1/\mathcal{B}}. \]
References


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