MAXIMAL TYCHONOFF SPACES
AND NORMAL ISOLATOR COVERS

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Abstract. We introduce a new kind of cover called a normal isolator cover to characterize maximal Tychonoff spaces. Such a study is used to provide an alternative proof of an interesting result of Feng and Garcia-Ferreira in 1999 that every maximal Tychonoff space is extremally disconnected. Maximal tychonoffness of subspaces is also discussed.

1. Introduction

In the poset $\mathcal{A}(X)$, of all topologies on a given set $X$, having the property $P$, a topological space $(X, \tau)$ is maximal $P$ provided that $\tau$ is a maximal element in $\mathcal{A}(X)$. In [6], it had been shown that a topological space $(X, \tau)$ is maximal $P$ if and only if every continuous bijection from a space $(Y, \tau_1)$ with the property $P$ to $(X, \tau)$ is a homeomorphism. In 1943 Hewitt [15] and in 1947 Vaidyanathaswamy [29] had independently proved that every compact Hausdorff space is maximal compact. Vaidyanathaswamy [29] put forward a question if there exists any non-Hausdorff maximal compact space. One year later in 1948 Hing tong [28] answered affirmatively Vaidyanathaswamy’s question. In the same year Ramanathan [21] characterized maximal compact spaces as those whose compact subsets are precisely the closed sets. Levine [17] answered affirmatively the question of Vaidyanathaswamy by establishing that a one point compactification of rationals with the usual topology is a non-Hausdorff maximal compact space. In the same paper he exhibited that maximal compact topologies are not productive. On the other hand, Mioduszewski and Rudolf [18] demonstrated necessary and sufficient conditions for an absolutely closed (or $H$-closed) space to become maximal absolutely closed.

Thron [27] and Aull [1] investigated maximal countably compact spaces. Aull, in fact, strengthened the result: a first countable Hausdorff countably compact space is maximal countably compact and minimal first countable Hausdorff of Thron [27].
In 1971, Cameron [6] and in 1973 Raha [20] investigated exhaustively the various aspects of certain maximal $P$ spaces, where $P = \text{Lindeloff, countably compact, sequential compact, pseudocompact, lightly compact or connected}$. Thomas [25] had also discussed maximal connected topologies. Cameron characterized maximal QHC spaces [7] and maximal pseudocompact spaces [5, 9] and in [8] he showed that the maximal topologies of a class of topologies which include lightly compact and QHC spaces are submaximal and $T_1$ spaces. In 1999, Kennedy and McCartan [16] investigated spaces which are maximal with respect to a semiregular property and showed new characterizations of maximal QHC spaces and maximal pseudocompact spaces. In 1997, Guthrie, Stone and Wage [13] investigated topologies, which are maximal connected Hausdorff and in 1998, Shakhmatov, Tkacenko, Tkachuk, Watson and Wilson [23] showed that neither first countable nor Cech-complete spaces are maximal Tychonoff connected also in 2007, Zelenyuk [31] investigated almost maximal spaces. In addition, interesting behaviors of some of Maximal topologies and their applications are found in the papers [10, 11, 14, 19, 22, 24, 25].

Considering the usefulness and importance of uniformizability (=Tychonoff-ness in Hausdorff spaces) and the above observations about maximal topologies of various kinds of topological properties, this article is devoted to study maximal uniformizable (=Maximal Tychonoff) spaces. Several characterizations of such spaces have been given in terms of refinement of normally open covers as well as newly introduced normal isolator covers. As a consequence, we provide an alternative proof of the already existing interesting result of Feng, Garcia-Ferreira [12] that every maximal Tychonoff space is extremally disconnected. Maximal uniformizability with respect to subspaces has also been discussed.

2. Preliminaries

The symbol $X$ or $(X, \tau)$ denotes a topological space without any isolated points which is $T_2$ and the base set $X$ is infinite, unless explicitly stated. For two covers $\mathcal{U}$ and $\mathcal{V}$ of $X$, $\mathcal{U}$ is called a refinement of $\mathcal{V}$ denoted by $\mathcal{U} < \mathcal{V}$ if for each $U \in \mathcal{U}$, there exists a $V \in \mathcal{V}$ such that $U \subseteq V$ and we call $\mathcal{U}$ star refines $\mathcal{V}$ or $\mathcal{U}$ is a star refinement of $\mathcal{V}$, denoted by $\mathcal{U} \prec \mathcal{V}$, if for each $U \in \mathcal{U}$, there exists a $V \in \mathcal{V}$ such that $\text{St}(U; \mathcal{U}) \subseteq V$, where $\text{St}(U; \mathcal{U}) = \bigcup \{W \in \mathcal{U} : W \cap U \neq \emptyset\}$. When $U = \{x\}$, we denote $\text{St}(U; \mathcal{U})$ as $\text{St}(x; \mathcal{U})$. We note that if $\mathcal{U} \prec \mathcal{V}$, then $\mathcal{U} < \mathcal{V}$.

A normal sequence of covers of $X$ is a sequence of covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of $X$ such that $\mathcal{U}_{n+1} \prec \mathcal{U}_n$, for $n = 1, 2, \ldots$; and a normal cover is a cover which is $\mathcal{U}_1$ in some normal sequence of covers [30], §36.9, p. 247. An open cover $\mathcal{U}$ of a topological space $X$ is normally open if and only if $\mathcal{U} = \mathcal{U}_1$ in some normal sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ consisting of open covers of $X$ [30], §36.14, p. 248.

A collection $\mu$ of covers of a space $X$ is a base for some covering uniformity on $X$ if and only if it satisfies the condition that for $\mathcal{U}_1, \mathcal{U}_2 \in \mu$ there is a $\mathcal{U}_3 \in \mu$ such that $\mathcal{U}_3 \prec \mathcal{U}_1$ and $\mathcal{U}_3 \prec \mathcal{U}_2$ [30], §36.3, p. 245. It is well known that if $\mu$ is a base for a covering uniformity $\mu$ on $X$, then $\{\text{St}(x; \mathcal{U}) : \mathcal{U} \in \mu\}$ is a local base at $x \in X$ in the uniform topology [30], §36.6, p. 246. Also if $X$ is any uniformizable topological
space, then there is a finest uniformity on X, compatible with the topology of X, called the fine uniformity on X, denoted by \( \mu_F \), which has a base of all normally open covers of X. So a uniformizable space \((X, \tau)\) has at least one normally open cover consisting of proper subsets of X.

**Lemma 2.1.** If \( U_1, U_2, U_3 \) are three covers of X such that \( U_1 \vartriangleleft U_2 \vartriangleleft U_3 \), then \( U_1 \vartriangleleft U_3 \).

**Proof.** The proof is obvious. \( \square \)

**Lemma 2.2.** If \( \{U_1, U_2, U_3, \ldots\} \) is a normal sequence of covers and if \( U_k, U_m \in \{U_1, U_2, U_3, \ldots\} \), then for a positive integer \( t \) greater than both of \( k \) and \( m \), \( U_t \vartriangleleft U_k \) and \( U_t \vartriangleleft U_m \).

**Proof.** The proof is obvious. \( \square \)

**Theorem 2.1 (Hausdorff criterion [30]).** For each \( x \in X \), let \( B^1_x \) be a neighborhood base at \( x \) for the topology \( \tau_1 \) on X and \( B^2_x \) be a neighborhood base at \( x \) for the topology \( \tau_2 \) on X. Then \( \tau_1 \subset \tau_2 \) if and only if for each \( x \in X \) and each \( B^1 \in B^1_x \), there is some \( B^2 \in B^2_x \) such that \( B^2 \subset B^1 \). [30] §4.8. p. 35.

**Definition 2.1.** For two covers \( U \) and \( V \) of X, we denote the intersection of \( U \) and \( V \) as \( U \wedge V \) and define it as \( U \wedge V = \{U \cap V : U \in U, V \in V\} \). [30] §36.3. p. 245.

## 3. Maximal Tychonoff spaces

Recently, when a space (uniformizable or not) possessing a nontrivial proper uniformizable subtopology is investigated in [3] by Basu and Mandal, by the help of normal sequence of covers and star refinement of covers. A useful consequence of that investigation reflects that a sort of converse of A. H. Stone’s famous theorem is true when Basu and Mandal [3] established that a paracompact \( T_2 \) space \((X, \tau)\) is either metrizable or \((X, \tau)\) has a nontrivial proper uniformizable subtopology, which is pseudometrizable. In the course of that study, disconnectedness is seen to play a major role, especially when that is of very strong in nature viz. zero-dimensionality, it is shown there that for a paracompact \( T_2 \) space \((X, \tau)\) containing no isolated points, the cardinality of such nontrivial proper uniformizable subtopologies of \((X, \tau)\) is at least \( \aleph_0 \). In another paper [4], Basu and Mandal characterized minimal Uniformizable spaces in terms of normal sequence of covers and have shown that a minimal uniformizable non-indiscrete space is pseudometrizable. In this section, we investigate maximal uniformizable (=maximal Tychonoff) spaces in terms of a new kind of cover called a normal isolator cover.

**Definition 3.1.** A cover \( U \) of \((X, \tau)\) is called an isolator cover of \((X, \tau)\) if \( \text{St}(x; U \wedge V) \) is infinite for every normally open cover \( V \) of \((X, \tau)\) and \( U \) is called a normal isolator cover if it is the first term of a normal sequence of isolator covers.

Clearly every open cover of \((X, \tau)\) is an isolator cover of \((X, \tau)\) and every normally open cover of \((X, \tau)\) is also obviously a normal isolator cover of \((X, \tau)\).
We further note that if \( \tau_1 \) is a topology on \( X \) such that \( \tau_1 \supset \tau \) and if \( \mathcal{U} \) is a normal isolator cover of \( (X, \tau_1) \), then \( \mathcal{U} \) is a normal isolator cover of \( (X, \tau) \).

**Definition 3.2.** A Tyconoff (or uniformizable) space \((X, \tau)\) is called maximal Tyconoff \( \mathbf{[11]} \) (or maximal uniformizable) if no topology without any isolated points stronger than \( \tau \) is Tyconoff (or uniformizable).

**Lemma 3.1.** If \( \mathcal{U}_2 \not< \mathcal{U}_1 \) and \( \mathcal{V}_2 \not< \mathcal{V}_1 \), then

\[
(\text{i}) \quad \mathcal{U}_2 \land \mathcal{V}_2 \not< \mathcal{U}_1 \land \mathcal{V}_1; \quad (\text{ii}) \quad \mathcal{U}_2 \land \mathcal{V}_2 \not< \mathcal{U}_1 \land \mathcal{V}_2, \quad \text{and} \quad \mathcal{U}_2 \land \mathcal{V}_2 \not< \mathcal{V}_1.
\]

**Proof.** (i) Let \( \mathcal{U}_2 \cap \mathcal{V}_2 \in \mathcal{U}_2 \land \mathcal{V}_2 \), where \( \mathcal{U}_2 \in \mathcal{U}_2, \mathcal{V}_2 \in \mathcal{V}_2 \). Then

\[
\text{St}(\mathcal{U}_2 \cap \mathcal{V}_2; \mathcal{U}_2 \land \mathcal{V}_2) \subseteq \bigcup \{\{U_\alpha \cap V_\beta \} : U_\alpha \in \mathcal{U}_2 \text{ with } U_\alpha \cap \mathcal{V}_2 \not= \emptyset \text{ and } \}
\]

\[
\bigcup \{\{U_\alpha \in \mathcal{U}_2 \cap \mathcal{V}_2 \not= \emptyset \} \cap \left( \bigcup \{V_\beta \in \mathcal{V}_2 : V_\beta \cap \mathcal{V}_2 \not= \emptyset \} \right) \}
\]

\[
= \text{St}(\mathcal{U}_2; \mathcal{U}_2) \cap \text{St}(\mathcal{V}_2; \mathcal{V}_2) \subseteq \bigcup U_1 \cap \bigcup V_1 \subseteq \mathcal{U}_1 \land \mathcal{V}_1.
\]

[as \( \mathcal{U}_2 < \mathcal{U}_1 \) and \( \mathcal{V}_2 < \mathcal{V}_1 \), so for \( \mathcal{U}_2 \in \mathcal{U}_2, \mathcal{V}_2 \in \mathcal{V}_2 \) there exist some \( \mathcal{U}_1 \in \mathcal{U}_1 \) and some \( \mathcal{V}_1 \in \mathcal{V}_1 \) such that \( \text{St}(\mathcal{U}_2; \mathcal{U}_2) \subset \mathcal{U}_1 \) and \( \text{St}(\mathcal{V}_2; \mathcal{V}_2) \subset \mathcal{V}_1 \). So for \( \mathcal{U}_2 \cap \mathcal{V}_2 \in \mathcal{U}_2 \land \mathcal{V}_2 \) there exists \( \mathcal{U}_1 \cap \mathcal{V}_1 \in \mathcal{U}_1 \cap \mathcal{V}_1 \) such that \( \text{St}(\mathcal{U}_2 \cap \mathcal{V}_2; \mathcal{U}_2 \land \mathcal{V}_2) \subset \mathcal{U}_1 \cap \mathcal{V}_1 \). Hence \( \mathcal{U}_2 \land \mathcal{V}_2 < \mathcal{U}_1 \land \mathcal{V}_1 \).

(ii) We know that \( \mathcal{U}_1 \land \mathcal{V}_1 < \mathcal{U}_1 \) and \( \mathcal{U}_1 \land \mathcal{V}_1 < \mathcal{V}_1 \) and also we know that for three covers \( \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \) of \( X \), if \( \mathcal{W}_3 < \mathcal{W}_2 < \mathcal{W}_1 \), then \( \mathcal{W}_3 < \mathcal{W}_1 \). Hence the result follows. \( \Box \)

**Lemma 3.2.** For a Tychonoff space \((X, \tau)\), if \( \mu \) is the collection of all normally open covers of \((X, \tau)\) and \( \mathcal{U} \) is a normal isolator cover of \((X, \tau)\) with the corresponding normal sequence of isolator covers \( \ldots \mathcal{U}_3 \not< \mathcal{U}_2 \not< \mathcal{U}_1 = \mathcal{U} \), then \( \mu_1 = \mu \cup \{\mathcal{U}_1, \mathcal{U}_2, \ldots \} \cup \{\mathcal{V} \land \mathcal{U}_k : k = 1, 2, \ldots ; \mathcal{V} \in \mu \} \) forms a base for some uniformity on \( X \) which generates a stronger Tychonoff (uniformizable) topology \( \tau_0 \) on \( X \) such that \((X, \tau_0)\) is T2 and contains no isolated points.

**Proof.** Here \( \mu_1 = \mu \cup \{\mathcal{U}_1, \mathcal{U}_2, \ldots \} \cup \{\mathcal{V} \land \mathcal{U}_k : k = 1, 2, \ldots ; \mathcal{V} \in \mu \} \). First we shall prove that for \( \mathcal{W}_1, \mathcal{W}_2 \in \mu_1 \), there exists a \( \mathcal{W}_3 \in \mu_1 \) such that \( \mathcal{W}_3 \not< \mathcal{W}_1 \) and \( \mathcal{W}_3 \not< \mathcal{W}_2 \).

Now \( \mu \) is itself a base for the fine uniformity on \( X \) generating the topology \( \tau \) on \( X \). So for \( \mathcal{W}_1, \mathcal{W}_2 \in \mu \), there obviously exists a \( \mathcal{W}_3 \in \mu \) such that \( \mathcal{W}_3 \not< \mathcal{W}_1 \) and \( \mathcal{W}_3 \not< \mathcal{W}_2 \).

Also by Lemma 2.2, for \( \mathcal{U}_k, \mathcal{U}_m \in \{\mathcal{U}_1, \mathcal{U}_2, \ldots \} \), there exists an \( \mathcal{U}_t \in \{\mathcal{U}_1, \mathcal{U}_2, \ldots \} \) such that \( \mathcal{U}_t \not< \mathcal{U}_k \) and \( \mathcal{U}_t \not< \mathcal{U}_m \).

Now we have to check it for four possible cases:

\( (i) \quad \mathcal{W}_1 \in \mu \) and \( \mathcal{U}_k \in \{\mathcal{U}_1, \mathcal{U}_2, \ldots \} \); \( (ii) \quad \mathcal{W}_1 \in \mu \) and \( \mathcal{W} \in \{\mathcal{V} \land \mathcal{U}_k : k = 1, 2, \ldots ; \mathcal{V} \in \mu \} \);

\( (iii) \quad \mathcal{W}_1, \mathcal{W}_2 \in \{\mathcal{V} \land \mathcal{U}_k : k = 1, 2, \ldots ; \mathcal{V} \in \mu \} \); \( (iv) \quad \text{for } \mathcal{U}_k \in \{\mathcal{U}_1, \mathcal{U}_2, \ldots \} \) and \( \mathcal{W} \in \{\mathcal{V} \land \mathcal{U}_k : k = 1, 2, \ldots ; \mathcal{V} \in \mu \} \).
Case (i): Suppose \( \mathcal{W}_1 \in \mu \) and \( \mathcal{U}_k \in \{ \mathcal{U}_1, \mathcal{U}_2, \ldots \} \). As \( \mu \) is a base for the fine uniformity on \( X \), there exists a \( \mathcal{W}_2 \in \mu \) such that \( \mathcal{W}_2 \prec \mathcal{W}_1 \) and also \( \mathcal{U}_k \prec \mathcal{U}_k \) [where \( t \) is a positive integer such that \( t > k \)]. Hence by Lemma 3.1, \( \mathcal{W}_2 \prec \mathcal{U}_k \prec \mathcal{U}_1 \), \( \mathcal{W}_2 \prec \mathcal{U}_k \prec \mathcal{U}_k \), where \( \mathcal{W}_2 \prec \mathcal{U}_k \prec \mathcal{U}_k \in \mu_1 \).

Case (ii): Let \( \mathcal{U}' \in \mu \), \( \mathcal{U} \prec \mathcal{U}_k \in \{ \mathcal{V} \cap \mathcal{U}_l : l = 1, 2, \ldots ; \mathcal{V} \in \mu \} \).

Since \( \mathcal{U}' \cap \mathcal{U}_k \in \mu \) and \( \mu \) is being a base for the fine uniformity on \( X \), there exists a \( \mathcal{U}'' \in \mu \) such that \( \mathcal{U}'' \prec \mathcal{U}' \) and \( \mathcal{U}'' \prec \mathcal{U} \).

Now as \( \mathcal{U}'' \prec \mathcal{U} \) and \( \mathcal{U} \prec \mathcal{U}_k \) (where \( t \) is a positive integer greater than \( k \)), by Lemma 3.1, \( \mathcal{U}'' \prec \mathcal{U} \prec \mathcal{U}_k \).

Again for \( \mathcal{V} \cap \mathcal{U} \in \mathcal{U}'' \cap \mathcal{U}_k \), we have \( \text{St}(\mathcal{V} \cap \mathcal{U} ; \mathcal{U}'' \cap \mathcal{U}_k) \subset \text{St}(\mathcal{U} ; \mathcal{U}'') \subset \mathcal{U}' \) (for some \( \mathcal{U}' \in \mathcal{U}'' \) as \( \mathcal{U}'' \prec \mathcal{U}' \)). So \( \mathcal{U}'' \prec \mathcal{U} \prec \mathcal{U}' \).

Case (iii): Let \( \mathcal{U}' \prec \mathcal{U}_k \), \( \mathcal{U}'' \prec \mathcal{U}_k \in \{ \mathcal{V} \cap \mathcal{U}_l : t = 1, 2, \ldots ; \mathcal{V} \in \mu \} \). Now, for \( \mathcal{U}' \cap \mathcal{U}_l \in \mu \), there exists a \( \mathcal{U}'' \in \mathcal{U} \), such that \( \mathcal{U}'' \prec \mathcal{U}', \mathcal{U}'' \prec \mathcal{U} \).

Also Lemma 2.2 ensures that, for a positive integer \( t \) greater than both \( k \) and \( l \), \( \mathcal{U}_l \prec \mathcal{U}_k \) and \( \mathcal{U}_k \prec \mathcal{U} \).

From (a), (b) and Lemma 3.1 we get \( \mathcal{U}'' \prec \mathcal{U} \prec \mathcal{U}_k \) as well as \( \mathcal{U}'' \prec \mathcal{U} \prec \mathcal{U}_k \). Here we note that \( \mathcal{U}'' \prec \mathcal{U} \prec \mathcal{U}_k \). The proof of case (iv) can be done similarly.

So we have for any \( \mathcal{W}_1, \mathcal{W}_2 \in \mu_1 \), there exists a \( \mathcal{W}_3 \in \mu_1 \) such that \( \mathcal{W}_3 \prec \mathcal{W}_1, \mathcal{W}_3 \prec \mathcal{W}_2 \). Hence \( \mu_1 \) is a base for some covering uniformity on \( X \). Now \( \mu_1 \) generates the topology \( \tau_{\mu_1} \) on \( X \). Since \( \mu \subset \mu_1 \), the topology generated by \( \mu \) i.e., the topology \( \tau \) is weaker than \( \tau_{\mu_1} \). Now \( \{ \text{St}(x; \mathcal{U}) : \mathcal{U} \in \mu_1 \} \) forms a local base at \( x \in X \) in \( (X, \tau_{\mu_1}) \) and also \( \text{St}(x; \mathcal{U}) \) is infinite for each \( x \in X \) and for each \( \mathcal{U} \in \mu_1 \). So \( (X, \tau_{\mu_1}) \) contains no isolated points. Also \( \tau_{\mu_1} \) is \( T_2 \) and uniformizable. Hence the Lemma follows.

**Theorem 3.1.** For a Tychonoff space \((X, \tau)\), the following statements are equivalent:

(i) \((X, \tau)\) is maximal Tychonoff.

(ii) Every normal isolator cover \( \mathcal{U} \) of \((X, \tau)\) has an open refinement \( \mathcal{V} \), which is also a cover of \((X, \tau)\).

(iii) Every normal isolator cover \( \mathcal{U} \) of \((X, \tau)\) has an open star refinement \( \mathcal{V} \), which is also a cover of \((X, \tau)\).

**Proof.** We shall proceed to prove in the following manner: (i)\(\iff\) (ii), (ii)\(\iff\) (iii).

(i)\(\Rightarrow\) (ii): Let \((X, \tau)\) be a maximal Tychonoff space and also let \( \mathcal{U} \) be a normal isolator cover of \((X, \tau)\) and \( \mathcal{U} = \mathcal{U}_1, \mathcal{U}_2, \ldots \) be the corresponding normal sequence of isolator covers of \((X, \tau)\).

Now we shall consider the collection \( \mu_1 \) of covers consisting of all normally open covers of \((X, \tau)\), the covers \( \{ \mathcal{U}_1, \mathcal{U}_2, \ldots \} \) and the covers \( \{ \mathcal{V} \cap \mathcal{U}_k : k = 1, 2, \ldots ; \mathcal{V} \in \mu \} \), where \( \mu \) is the collection of all normally open covers of \((X, \tau)\).
So by Lemma 3.2, $\mu'_1$ is a base for some uniformity on $X$, which generates a stronger Tychonoff (or uniformizable) topology $\tau_{\mu'_1}$ on $X$ (i.e. $\tau_{\mu'_1} \supset \tau$) such that $(X, \tau_{\mu'_1})$ is $T_2$ and contains no isolated points. Then $\tau = \tau_{\mu'_1}$, by the maximality of $\tau$ as a Tychonoff (or uniformizable) topology. So we can write $\tau_{\mu'_1} \subset \tau$.

Now $\{\text{St}(x; W) : W \in \mu'_1\}$ is a local base at $x \in X$ in $(X, \tau_{\mu'_1})$ and $\{\text{St}(x; W') : W' \in \mu\}$ is a local base at $x \in X$ in $(X, \tau)$. By the Hausdorff criterion, for $U_0 \in \mu'_1$, there exists a $V \in \mu$ such that $\text{St}(x; V) \subset \text{St}(x; U_2)$. Take a $V_2 \subset V$ containing $x$. Then $V_2 \subset \text{St}(x; V) \subset \text{St}(x; U_2) \subset \text{St}(U_2; U_2)$ [for some $U_2 \in U_2$ containing $x$]. As $U_2 \not\subset U_1 = \mathcal{U}$, there exists some $U_2 \subset U_1 = \mathcal{U}$ such that $\text{St}(U_2; U_2) \subset U_x$. So $V_2 \subset U_x$. Therefore the cover $\mathcal{W} = \{V_2 : x \in X\}$ is the required open cover of $(X, \tau)$, which is a refinement of $\mathcal{U}$.

$(ii) \Rightarrow (i)$: Let the condition holds and if possible let $\tau_1$ be a Tychonoff (or uniformizable) topology that contains no isolated points satisfying $\tau \subset \tau_1$. It is sufficient to prove that $\tau_1 \subset \tau$.

Let $U \in \tau_1$. Consider the collection $\mu_1$ of all normally open covers of $(X, \tau_1)$. Then for $x \in U \in \tau_1$, there exists a $U' \in \mu_1$ such that $x \in \text{St}(x; U') \subset U$. But as $U'$ is a normal isolator cover of $(X, \tau_1)$, it is therefore so in $(X, \tau)$, as well. Hence the assumption shows the existence of an open cover $\mathcal{W}$ of $(X, \tau)$ satisfying $\mathcal{W} \subset U'$. Obviously $x \in \text{St}(x; \mathcal{W}) \subset \text{St}(x; U') \subset U$ and hence $U \in \tau$ as $U$ is a neighborhood of $x$ in $(X, \tau)$. So $\tau_1 \subset \tau$ and hence $(i)$ is followed.

$(ii) \Rightarrow (iii)$: Let $(ii)$ holds i.e. every normal isolator cover of $(X, \tau)$ has a $\tau$-open refinement. Let $\mathcal{U}$ be a normal isolator cover of $(X, \tau)$ with $\ldots, U_3, U_2, U_1 = \mathcal{U}$ be the corresponding normal sequence of isolator covers. Since $U_2$ is also a normal isolator cover, by $(ii)$, it has a $\tau$-open refinement $\mathcal{V}$. Now for $V \in \mathcal{V}$, there exists a $U_3 \in U_3$, such that $V \subset U_2$. So $\text{St}(V; \mathcal{V}) \subset \text{St}(U_2; U_2)$, as every member of $\mathcal{V}$ which intersects $V$ must be contained in a member of $U_2$ intersecting $U_2$. Since $U_2 \not\subset \mathcal{U}$, $\text{St}(V; \mathcal{V}) \subset \text{St}(U_2; U_2) \subset U$ for some $U \in \mathcal{U}$. That is for $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ such that $\text{St}(V; \mathcal{V}) \subset U$ and hence $\mathcal{V}$ is a star refinement of $\mathcal{U}$.

$(iii) \Rightarrow (ii)$: As every star refinement of a cover $\mathcal{W}$ is obviously a refinement of $\mathcal{W}$, $(iii) \Rightarrow (ii)$ follows obviously. \[ \square \]

**Corollary 3.1.** In a maximal Tychonoff space $(X, \tau)$, a subset $G$ of $X$ is open if and only if for every $x \in G$ there exists a normal isolator cover $\mathcal{U}$ of $(X, \tau)$ such that $x \in \text{St}(x; \mathcal{U}) \subset G$.

**Proof.** Let $G$ be an open subset of $(X, \tau)$ and $x \in G$. Now we know that the collection of all normally open covers $\mu$ (say) of $(X, \tau)$ forms a base for a uniformity on $X$, which generates $\tau$ on $X$ where $\beta_x = \{\text{St}(x; V) : V \in \mu\}$ forms a local base at $x \in X$ in $(X, \tau)$. Hence the necessary part is followed as every normally open cover is also a normal isolator cover.

Conversely, let $G$ be a subset of $X$ such that for every $x \in G$ there exists a normal isolator cover $\mathcal{U}$ of $(X, \tau)$ with $x \in \text{St}(x; \mathcal{U}) \subset G$. Since $(X, \tau)$ is maximal Tychonoff, then by Theorem 3.1 the normal isolator cover $\mathcal{U}$ has a $\tau$-open refinement $\mathcal{V}$ (say). Now if $U$ is a member of $\mathcal{V}$ containing $x$, then obviously...
Theorem 3.2. Let \((X, \tau)\) be a maximal Tychonoff space. Then for every nonempty proper subset \(G\) of \(X\) containing no isolated points in \(G\) (with the induced subspace topology), \(\text{cl}(G)\) is open.

Proof. Let \(G\) be a nonempty proper subset of \(X\) containing no isolated points in \((X, \tau)\). If \(\text{cl}(G) = X\), then \(\text{cl}(G)\) is obviously open; so let \(\text{cl}(G) \subsetneq X\). Then \(X - \text{cl}(G)\) is an open set and obviously contains no isolated points in \((X, \tau)\). Also \(\text{cl}(G)\) obviously contains no isolated points in \((X, \tau)\).

Now we consider the cover \(\mathcal{U} = \{\text{cl}(G), X - \text{cl}(G)\}\) of \(X\). Here we see that for any normally open cover \(V\) of \((X, \tau)\) if \(x\) belongs to \(X - \text{cl}(G)\), then \(\text{St}(x; \mathcal{U} \cap V)\) being a union of open sets is itself an open set and hence is infinite. Let \(x \in \text{cl}(G)\). Now if \(x \in G\), then \(x\) is not an isolated point of \(G\). So when an open set containing \(x\) intersects \(G\), it intersects at infinite number of points. Again if \(x\) is a limit point of \(G\), then as \((X, \tau)\) is \(T_1\), every open set containing \(x\) intersects \(G\) at infinite number of points. So in both cases \(\text{St}(x; \mathcal{U} \cap V)\) is infinite. Hence \(\mathcal{U}\) is an isolator cover of \((X, \tau)\). Also \(\ldots, \mathcal{U}, \mathcal{U}, \mathcal{U}\) is a normal sequence and so \(\mathcal{U}\) is a normal isolator cover of \((X, \tau)\). Thus by Theorem 3.1, \(\mathcal{U}\) has a \(\tau\)-open refinement. So \(\text{cl}(G)\) is obviously \(\tau\)-open. Hence the result follows.

L. Feng and S. García-Ferreira, in their paper [12], have established that every maximal Tychonoff space is extremally disconnected [12, Lemma 1.6]. Here an alternative proof of this result is established.

Corollary 3.2. A maximal Tychonoff space is extremally disconnected.

Proof. As in a maximal Tychonoff space, every nonempty proper open set contains no isolated points in that space, so by Theorem 3.2, the closure of every open set is open and hence a maximal Tychonoff space is extremally disconnected.

Corollary 3.3. A maximal Tychonoff space is zero dimensional space.

Proof. The proof follows immediately, since an extremally disconnected regular space is zero-dimensional.

Theorem 3.3. In a maximal Tychonoff space \((X, \tau)\), for any proper nonempty subset \(G\) containing no isolated points in \((X, \tau)\), \(\text{cl}(G)\) is maximal Tychonoff.

Proof. Let \((X, \tau)\) be maximal Tychonoff and \(G\) be a proper nonempty subset of \(X\) such that \(G\) contains no isolated points and also let \(\text{cl}(G) \neq X\). Then obviously \((\text{cl}(G), \tau_{\text{cl}(G)})\) is \(T_2\), uniformizable and also contains no isolated points.
Now if $U$ is any normal isolator cover of $(\operatorname{cl}(G), \tau_{\operatorname{cl}(G)})$, then $U \cup \{X - \operatorname{cl}(G)\}$ is also a normal isolator cover of $(X, \tau)$.

As $(X, \tau)$ is maximal Tychonoff, by Theorem 3.1 $U \cup \{X - \operatorname{cl}(G)\}$ has a $\tau$-open refinement $V'$, which is also a cover of $X$. Now $V = V' \cap \{V \in V' : V \cap \operatorname{cl}(G)\}$ is an open cover of $\operatorname{cl}(G)$ and it refines $U$. So $V$ is a $\tau_{\operatorname{cl}(G)}$-open refinement of $U$ i.e., every normal isolator cover of $(\operatorname{cl}(G), \tau_{\operatorname{cl}(G)})$ has a $\tau_{\operatorname{cl}(G)}$-open refinement. Hence by Theorem 3.1 $(\operatorname{cl}(G), \tau_{\operatorname{cl}(G)})$ is a maximal Tychonoff space.

Corollary 3.4. In a maximal Tychonoff space $(X, \tau)$, for every proper non-empty open set $A$, $\operatorname{cl}(A)$ is maximal Tychonoff.

Proof. As every proper nonempty open set contains no isolated points, the proof follows from Theorem 3.3.

Corollary 3.5. A topological space $(X, \tau)$ is maximal Tychonoff if and only if every nonempty open subset is maximal Tychonoff.

Proof. The proof follows from Corollary 3.4.

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