EXISTENCE RESULTS FOR NONLINEAR IMPULSIVE $q_k$-INTEGRAL BOUNDARY VALUE PROBLEMS

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Abstract. We investigate a nonlinear impulsive $q_k$-integral boundary value problem by means of Leray–Schauder degree theory and contraction mapping principle. The conditions ensuring the existence and uniqueness of solutions for the problem are presented. An illustrative example is discussed.

1. Introduction

We investigate the existence and uniqueness of solutions for a nonlinear impulsive $q_k$-integral boundary value problem

\begin{equation}
\begin{aligned}
D_{q_k} u(t) &= f(t, u(t)), \quad 0 < q_k < 1, \quad t \in J', \\
\Delta u(t_k) &= I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\
u(T) &= \sum_{i=0}^{m} \int_{t_i+1}^{t_i} g(s, u(s)) dq_i s,
\end{aligned}
\end{equation}

where $D_{q_k}$ are $q_k$-derivatives $(k = 0, 1, 2, \ldots, m)$, $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $J = [0, T]$, $T > 0$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, and $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k$ $(k = 1, 2, \ldots, m)$ respectively.

The study of $q$-difference equations, initiated with the pioneer work of Jackson [1], has been developed over the years. The concept of $q$-calculus corresponds to the classical calculus without the idea of limit. This subject is also known as quantum calculus and finds its applications in a variety of disciplines such as special functions, super-symmetry, control theory, operator theory, combinatorics, initial and boundary value problems of $q$-difference equations, etc. For the systematic development of $q$-calculus, we refer the reader to the books \cite{2, 3, 4} and papers \cite{5, 6, 7, 8, 9, 10}.

2010 Mathematics Subject Classification: 39A13; 05A13.

Key words and phrases: $q_k$-difference equation, impulse, $q_k$-integral boundary conditions, Leray–Schauder degree theory, contraction mapping principle.

Partially supported by National Natural Science Foundation of China (No. 11501342) and the Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi (Nos. 2014135 and 2014136).

Communicated by Stevan Pilipović.
The importance of q-difference equations lies in the fact that these equations are always completely controllable and appear in the q-optimal control problems [11]. The variational q-calculus is regarded as a generalization of the continuous variational calculus due to the presence of an extra-parameter q that may be physical or economical in its nature. The variational calculus on the q-uniform lattice includes the study of the q-Euler equations and its applications to the isoperimetric and Lagrange problems and commutation equations. In other words, it suffices to solve the q-Euler-Lagrange equation for finding the extremum of the functional involved instead of solving the Euler–Lagrange equation [12]. Further details can be found in [13–16].

The initial and boundary value problems of impulsive fractional differential equations have been extensively investigated by many researchers, for instance, see [17–25] and references therein. In a recent paper [26], the authors discussed the existence and uniqueness of solutions for impulsive qk-difference equations.

Motivated by [26], the present work is devoted to the study of impulsive qk-difference equations with integral boundary condition. The paper is organized as follows. In Section 2, we present some basic concepts of the topic and an auxiliary lemma. Section 3 contains the main results, while an illustrative example is discussed in Section 4.

2. Preliminaries

Let us set \( J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, T] \) and introduce the space: \( PC(J, \mathbb{R}) = \{ u : J \rightarrow \mathbb{R} | u \in C(J_k), k = 0, 1, \ldots, m, \text{ and } u(t_k^+) \text{ exist for } k = 1, 2, \ldots, m \} \) with the norm \( \| u \| = \sup_{t \in J} |u(t)| \). Obviously \( PC(J, \mathbb{R}) \) is a Banach space.

Next we recall some basic concepts of qk-calculus [26]. For \( 0 < q_k < 1 \) and \( t \in J_k \), we define the qk-derivatives of a real valued continuous function \( f \) as

\[
D_{q_k} f(t) = \frac{f(t) - f(q_k t + (1 - q_k) t_k)}{(1 - q_k)(t - t_k)}, \quad D_{q_k} f(t_k) = \lim_{t \to t_k} D_{q_k} f(t).
\]

Higher order qk-derivatives are given by

\[
D_{q_k}^0 f(t) = f(t), \quad D_{q_k}^n f(t) = D_{q_k} D_{q_k}^{n-1} f(t), \quad n \in \mathbb{N}, \quad t \in J_k.
\]

The qk-integral of a function \( f \) is defined by

\[
(2.2) \quad t_k I_{q_k} f(t) := \int_{t_k}^t f(s) d_{q_k} s = (1-q_k)(t-t_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n) t_k), \quad t \in J_k,
\]

provided the series converges. If \( a \in (t_k, t) \) and \( f \) is defined on the interval \((t_k, t)\), then

\[
\int_a^t f(s) d_{q_k} s = \int_{t_k}^t f(s) d_{q_k} s - \int_{t_k}^a f(s) d_{q_k} s.
\]

Observe that

\[
D_{q_k} (t_k I_{q_k} f(t)) = D_{q_k} \int_{t_k}^t f(s) d_{q_k} s = f(t),
\]
Repeating the above process, it is found that
\[
\begin{align*}
t_k \mathcal{I}_{qk}(D_{qk} f(t)) &= \int_{t_k}^{t} D_{qk} f(s) \, dq_k s = f(t), \\
a \mathcal{I}_{qk}(D_{qk} f(t)) &= \int_{a}^{t} D_{qk} f(s) \, dq_k s = f(t) - f(a), \quad a \in (t_k, t).
\end{align*}
\]

In the case \( t_k = 0 \) and \( q_k = q \) in (2.1) and (2.2), then \( D_{qk} f = D_q f \), \( t_k \mathcal{I}_{qk} f = a \mathcal{I}_{qk} f \), where \( D_q \) and \( a \mathcal{I}_{q} \) are the well-known \( q \)-derivative and \( q \)-integral of the function \( f(t) \) and are defined by
\[
D_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}, \quad a \mathcal{I}_{q} f(t) = \int_{0}^{t} f(s) \, dq s = \sum_{n=0}^{\infty} t(1 - q)^n f(tq^n).
\]

**Lemma 2.1.** For a given \( \sigma(t) \in C(J, \mathbb{R}) \), a function \( u \in PC(J, \mathbb{R}) \) is a solution of the following impulsive \( q_k \)-integral boundary value problem
\[
\begin{align*}
D_{q} u(t) &= \sigma(t), \quad 0 < q_k < 1, \quad t \in J', \\
\Delta u(t_k) &= I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\
u(T) &= \sum_{i=0}^{m} I_{k+1}(s, u(s)) \, dq_k s,
\end{align*}
\]
if and only if \( u \) satisfies the \( q_k \)-integral equation
\[
\begin{align*}
u(t) &= \begin{cases}
\int_{0}^{t} \sigma(s) \, dq_k s + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} [g(s, u(s)) - \sigma(s)] \, dq_k s - \sum_{i=1}^{m} I_i(u(t_i)), & t \in J_0; \\
\int_{t_k}^{t} \sigma(s) \, dq_k s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \sigma(s) \, dq_k s + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} [g(s, u(s)) - \sigma(s)] \, dq_k s - \sum_{i=1}^{m} I_i(u(t_i)), & t \in J_k.
\end{cases}
\end{align*}
\]

**Proof.** Let \( u \) be a solution of \( q_k \)-integral boundary value problem (2.3). For \( t \in J_0 \), applying the operator \( a \mathcal{I}_{qk} \) on both sides of \( D_{qk} u(t) = \sigma(t) \), we get
\[
u(t) = u(0) + a \mathcal{I}_{qk} \sigma(t) = u(0) + \int_{0}^{t} \sigma(s) \, dq_k s.
\]
Thus, \( u(t_1^+) = u(0) + \int_{0}^{t_1} \sigma(s) \, dq_k s \). For \( t \in J_1 \), applying the operator \( t_1^- \mathcal{I}_{q1} \) on both sides of \( D_{q1} u(t) = \sigma(t) \) yields
\[
u(t) = u(t_1^-) + \int_{t_1^-}^{t} \sigma(s) \, dq_1 s.
\]
Taking into account the condition: \( \Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1)) \), we obtain
\[
u(t) = u(0) + \int_{t_1}^{t} \sigma(s) \, dq_1 s + \int_{0}^{t_1} \sigma(s) \, dq_k s + I_1(u(t_1)), \quad \forall t \in J_1.
\]
Repeating the above process, it is found that
\[
u(t) = u(0) + \int_{t_k}^{t} \sigma(s) \, dq_k s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \sigma(s) \, dq_k s + \sum_{i=1}^{k} I_i(u(t_i)), \quad t \in J_k.
\]
Substituting \( t = T \) in (2.5), we have

\[
(2.6) \quad u(T) = u(0) + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \sigma(s) d_{q_i}s + \sum_{i=1}^{m} I_i(u(t_i)).
\]

Using the boundary condition given by (2.3) in (2.6), we obtain

\[
u(t) = \int_{t_k}^{t} \sigma(s) d_{q_k}s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \sigma(s) d_{q_i}s + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} [g(s, u(s)) - \sigma(s)]d_{q_i}s
- \sum_{i=k+1}^{m} I_i(u(t_i)), \quad t \in J_k.
\]

Conversely, assume that \( u \) satisfies \( q_k \)-integral equation (2.4). Then, by applying the operator \( D_{q_k} \) on both sides of (2.4) and using \( t = T \), we obtain (2.6). \( \Box \)

3. Main results

By Lemma 2.1, the nonlinear impulsive \( q_k \)-integral boundary value problem (1.1) can be transformed into an equivalent fixed point problem: \( u = Gu \), where the operator \( \mathcal{G} : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R}) \) is defined by

\[
(\mathcal{G}u)(t) = \int_{t_k}^{t} f(s, u(s)) d_{q_k}s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f(s, u(s)) d_{q_i}s
+ \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} [g(s, u(s)) - f(s, u(s))]d_{q_i}s - \sum_{i=k+1}^{m} I_i(u(t_i)).
\]

One can notice that the existence of a fixed point of the operator \( \mathcal{G} \) implies the existence of a solution of problem (1.1).

To show the existence of solutions for problem (1.1), we rely on Leray–Schauder degree theory and Banach fixed point theorem.

**THEOREM 3.1.** Assume that \((H_1)\) there exist nonnegative constants \( a, b, c, d \) and \( e \) such that \( \frac{2a+eT+c+me}{(1-2e)(d+b+e)} > 0 \) and

\[
|f(t, u)| \leq a + b|u|, \quad |g(t, u)| \leq c + d|u|, \quad |I_k(u)| \leq e, \quad k = 1, 2, \ldots, m,
\]

for all \( t \in J, u \in \mathbb{R} \). Then impulsive \( q_k \)-integral boundary value problem (1.1) has at least one solution.

**PROOF.** In the first step, it will be shown that the operator \( \mathcal{G} : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R}) \) is completely continuous. Let \( \mathcal{H} \subset PC(J, \mathbb{R}) \) be bounded. Then, for \( \forall t \in J, u \in \mathcal{H} \), we have \( |f(t, u)| \leq \mathcal{L}_1, \ |g(t, u)| \leq \mathcal{L}_2, \ |I_k(u)| \leq \mathcal{L}_3 \), where \( \mathcal{L}_i \) (\( i = 1, 2, 3 \)) are constants and \( k = 1, 2, \ldots, m \). Hence, for \( (t, u) \in J \times \mathcal{H} \), the following inequality holds

\[
||\mathcal{G}u||(t) \leq \int_{t_k}^{t} |f(s, u(s))|d_{q_k}s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} |f(s, u(s))|d_{q_i}s
+ \sum_{i=k+1}^{m} I_i(u(t_i)).
\]
consequence, it follows that the operator $G$ is compact. Therefore, by the Arzelá–Ascoli theorem, the operator $t \mapsto f, g$ is completely continuous. Furthermore, for any $t', t'' \in J_k \ (k = 0, 1, 2, \ldots, m)$ such that $t' < t'' < T$, we have

$$\begin{align*}
|G(t') - G(t'')| &\leq \left| \int_{t_k}^{t''} f(s, u(s)) \, dq_s - \int_{t_k}^{t'} f(s, u(s)) \, dq_s \right| \\
&\leq \int_{t'}^{t''} |f(s, u(s))| \, dq_s \leq L_1 (t'' - t').
\end{align*}$$

As $t' \to t''$, the right-hand side of (3.1) tends to zero. Thus, $G(H)$ is a relatively compact set. Therefore, by the Arzelá–Ascoli theorem, the operator $G$ is compact. Also, continuity of functions $f, g$ and $I_k$ imply that $G$ is a continuous operator. In consequence, it follows that the operator $G$ is completely continuous.

Now let us define $H(\lambda, u) = \lambda G u$, $u \in PC(J, \mathbb{R})$, $\lambda \in [0, 1]$ and note that $h_\lambda(u) = u - H(\lambda, u) = u - \lambda G u$ is completely continuous.

Next, we fix $R = \frac{(2a+c)T + m \varepsilon}{(2b+d)T} + 1$ and define a set $B_R = \{ u \in PC(J, \mathbb{R}) \mid \|u\| < R \}$. To arrive at the desired conclusion, it is sufficient to show that $G : B_R \to PC(J, \mathbb{R})$ satisfies

$$\lambda \neq \lambda G u, \quad \forall u \in \partial B_R \quad \forall \lambda \in [0, 1].$$

Suppose that (3.2) is not true. Then, there exists some $\lambda \in [0, 1]$ such that $u = \lambda G u$ for any $u \in \partial B_R$ and $t \in J$. Thus, we have

$$\begin{align*}
|u(t)| &= |\lambda(Gu)(t)| \leq \int_{t_k}^{t} |f(s, u(s))| \, dq_s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} |f(s, u(s))| \, dq_s \\
&\quad + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} [g(s, u(s)) + |f(s, u(s))|] \, dq_s + \sum_{i=k+1}^{m} |I_i(u(t_i))| \\
&\leq \int_{t_k}^{t} (a + b|u(s)|) \, dq_s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} (a + b|u(s)|) \, dq_s \\
&\quad + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} (c + d|u(s)| + a + b|u(s)|) \, dq_s + \sum_{i=k+1}^{m} c \\
&\leq (a + b\|u\|) \left[ (t - t_k) + \sum_{i=0}^{k-1} (t_{i+1} - t_i) \right].
\end{align*}$$
\[ \|u\| = \|2b + d\|u\| + (2a + c)T + me, \]

which leads to a contradiction: \( \|u\| \leq \frac{(2a + c)T + me}{1 - (2b + d)} < R. \) Hence our supposition is false and (3.2) is true. Applying the homotopy invariance of topological degree, it follows that

\[ \text{deg}(h, B, 0) = \text{deg}(I - \lambda G, B, 0) = \text{deg}(h_1, B, 0) \]

\[ = \text{deg}(h_0, B, 0) = \text{deg}(I, B, 0) = 1 \neq 0, \quad 0 \in B, \]

where \( I \) is the unit operator. Since \( \text{deg}(I - G, B, 0) = 1 \), the operator \( G \) has at least one fixed point in \( B \) by the solvability of topological degree. Thus, the impulsive \( q \)-integral boundary value problem (1.1) has at least one solution in \( B_r \).

To prove the uniqueness of solutions, we list the following assumptions:

(H2) there exist nonnegative continuous functions \( M(t) \) and \( N(t) \) such that

\[ |f(t, u) - f(t, v)| \leq M(t)|u - v|, \quad |g(t, u) - g(t, v)| \leq N(t)|u - v|, \]

for all \( t \in J, \ u, v \in \mathbb{R} \).

(H3) there exists a positive constant \( K \) such that

\[ |I_k(u) - I_k(v)| \leq K|u - v|, \quad u, v \in \mathbb{R}, \quad k = 1, 2, \ldots, m. \]

In the sequel, we set

\[ M^* = \max_{t \in J} |f(t, 0)|, \quad N^* = \max_{t \in J} |g(t, 0)|, \quad \gamma = \sum_{i=0}^{m} t_i q_i (2M + N)(t_{i+1}) + mK, \]

\[ \beta = (2M^* + N^*)T, \quad B_r = \{ u \in PC(J, \mathbb{R}) \mid \|u\| \leq r \}, \quad r \geq \frac{\beta}{1 - \gamma}. \]

**Theorem 3.2.** Let \( \gamma < 1 \) and the conditions (H2) - (H3) hold. Then the impulsive \( q \)-integral boundary value problem (1.1) has a unique solution in \( B_r \).

**Proof.** Firstly, we show that the operator \( G \) maps \( B_r \) into itself. For \( \forall t \in J_k, u \in B_r \), by (H2) and (H3), we find that

\[ \|G(u)(t)\| \leq \int_{t_k}^{t} |f(s, u(s))|d_q s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} |f(s, u(s))|d_q s \]

\[ + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} |g(s, u(s))|d_q s + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} |I_i(u(t_i))|d_q s \]

\[ \leq \int_{t_k}^{t} \left[ |f(s, u(s))| - f(s, 0) + |f(s, 0)| \right]d_q s \]

\[ + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \left[ |f(s, u(s)) - f(s, 0)| + |f(s, 0)| \right]d_q s \]
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This implies that $\|Gv - Gw\| \leq \gamma \|v - w\|$. Clearly $G$ is a contraction in view of the assumption $\gamma < 1$. Hence, the conclusion of Theorem 3.2 follows by contraction mapping principle due to Banach. □

4. Example

Consider the following nonlinear impulsive $q_k$-integral boundary value problem

$$D_{\frac{1}{1+k}} \left( \frac{k}{1+k} \right) u(t) = 5 + \frac{u(t)}{3 + u^2(t)}, \quad t \in [0, 1], \quad t \neq \frac{k}{1+k},$$

$$\Delta u \left( \frac{k}{1+k} \right) = 10 \sin u \left( \frac{k}{1+k} \right), \quad k = 1, 2,$$

$$u(1) = \int_{0}^{1/2} \left( 3s + \frac{1}{5} u(s)e^{-u^2(s)} \right) d_{2/3}s + \int_{1/2}^{3/3} \left( 3s + \frac{1}{5} u(s)e^{-u^2(s)} \right) d_{1/2}s + \int_{2/3}^{1} \left( 3s + \frac{1}{5} u(s)e^{-u^2(s)} \right) d_{2/3}s.$$

Here, $q_k = \frac{2}{1+k} (k = 0, 1, 2)$, $t_k = \frac{k}{1+k} (k = 1, 2)$, $f(t, u) = 5 + \frac{u}{3 + u^2}$, $I_k(u) = 10 \sin u$, $g(t, u) = 3t + \frac{1}{5} u e^{-u^2}$. Clearly $|f(t, u)| \leq 5 + \frac{1}{5} |u|$, $|g(t, u)| \leq 3 + \frac{1}{5} |u|$, $|I_k(u)| \leq 10$. Selecting $a = 5$, $b = \frac{1}{5}$, $c = 3$, $d = \frac{1}{5}$ and $e = 10$, all the conditions of Theorem 3.2 hold. Hence, by the conclusion of Theorem 3.2 there exists at least one solution for the problem (4.1).

Acknowledgment. We thank the anonymous reviewer for his/her constructive comments that led to improvement of the original manuscript.

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