A CHARACTERIZATION OF PGL(2, p^n) BY SOME IRREDUCIBLE COMPLEX CHARACTER DEGREES

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Abstract. For a finite group $G$, let $\text{cd}(G)$ be the set of irreducible complex character degrees of $G$ forgetting multiplicities and $X_1(G)$ be the set of all irreducible complex character degrees of $G$ counting multiplicities. Suppose that $p$ is a prime number. We prove that if $G$ is a finite group such that $|G| = |\text{PGL}(2, p)|$, $p \in \text{cd}(G)$ and $\max(\text{cd}(G)) = p + 1$, then $G \cong \text{PGL}(2, p)$, $\text{SL}(2, p)$ or $\text{PSL}(2, p) \times A$, where $A$ is a cyclic group of order $(2, p - 1)$. Also, we show that if $G$ is a finite group with $X_1(G) = X_1(\text{PGL}(2, p^n))$, then $G \cong \text{PGL}(2, p^n)$. In particular, this implies that PGL(2, p^n) is uniquely determined by the structure of its complex group algebra.

1. Introduction and preliminaries

Throughout this paper, let $G$ be a finite group, $p$ a prime number, $n$ a natural number and let all characters of the groups be complex characters (that is, characters afforded by irreducible complex representations). The set of irreducible characters of $G$ is denoted by $\text{Irr}(G)$ and we write $\text{cd}(G)$ for the set of irreducible character degrees of $G$ forgetting multiplicities. Denote by $X_1(G)$ the first column of the ordinary character table of $G$. Thus $X_1(G)$ can be considered as the set of all irreducible character degrees of $G$ counting multiplicities.

It is known that non-abelian simple groups are uniquely determined by their character tables. It was shown in [9] that the symmetric groups are also uniquely determined by their character tables. Hupert [5] conjectured that if $G$ is a finite group and $S$ is a finite non-abelian simple group such that $\text{cd}(G) = \text{cd}(S)$, then $G \cong S \times A$, where $A$ is an abelian group. He verified the conjecture for the Suzuki groups, the family of simple groups $\text{PSL}(2, p)$, for even $q$, and many of the sporadic simple groups. The authors proved in [12, 8, 3] that each Mathieu-groups, $\text{PSL}(2, p)$, can be uniquely determined by their orders and their largest and second largest irreducible character degrees, respectively.

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Here we prove the following.

**Theorem 1.1.** If $|G| = |\text{PGL}(2,p)|$ and

1. $p \in \text{cd}(G)$,  
2. $\max(\text{cd}(G)) = p + 1$, 

then $G \cong \text{PGL}(2,p), \text{SL}(2,p)$ or $\text{PSL}(2,p) \times A$, where $A$ is a cyclic group of order $(2, p - 1)$.

Tong-Viet [10] shows that the simple classical groups of Lie type are uniquely determined by the first column of their character tables. Here we prove

**Theorem 1.2.** For the natural number $n$, $\text{PGL}(2,p^n)$ is uniquely determined by the first column of its character table.

Let $C$ be the complex number field. Denote by $CG$ the group algebra of $G$. The Brauer’s Problem asks which groups can be determined by the structure of their complex group algebras. As a consequence of our results, we show that $\text{PGL}(2,p^n)$ is uniquely determined by the structure of its complex group algebra.

Throughout the paper, we use the following notations: For a natural number $n$, $\pi(n)$ is the set of prime divisors of $n$ and $\pi(G)$ is $\pi(|G|)$. For a prime $r$, the set of $r$-Sylow subgroups of $G$ is denoted by $\text{Syl}_r(G)$ and $n_r(G) = |\text{Syl}_r(G)|$. Let $s$ be a prime and let $m$ be a natural number. We use $s^e | m$ when $s^e | m$ but $s^{e+1} \nmid m$. The $s$-part of $m$ is denoted by $|m|_s$, i.e., $|m|_s = s^e$ if $s^e | m$. If $\gcd(s,m) = 1$ and $s$ is odd, then we denote by $e(s,m)$ multiplicative order of $m$ modulo $s$, i.e., the smallest natural number $n$ satisfying the condition $m^n \equiv 1 \pmod{s}$. Also, we write $H \triangleleft G$ if $H$ is a characteristic subgroup of $G$. Set $H_G = \cap_{g \in G} H^g$. If $\chi = \sum_{i=1}^N n_i \chi_i$, where for every $1 \leq i \leq N$, $\chi_i \in \text{Irr}(G)$, then those $\chi_i$ with $n_i > 0$ are called irreducible constituents of $\chi$.

In the following lemmas, for $\chi \in \text{Irr}(G)$ and the normal subgroup $N$ of $G$, $\chi_N$ is the restriction of $\chi$ to $N$ and for $\theta \in \text{Irr}(N)$, $\theta^G$ is the induced character on $G$. For Theorem 1.1, we need some facts about the relation between Irr$(G)$ and Irr$(G/N)$, when for some $\chi \in \text{Irr}(G)$, $\chi_N = \theta \in \text{Irr}(N)$.

**Lemma 1.1.** (Gallagher’s Theorem) [6] Corollary 6.17] Let $N \trianglelefteq G$ and let $\chi \in \text{Irr}(G)$ be such that $\chi_N = \theta \in \text{Irr}(N)$. Then the characters $\beta \chi$ for $\beta \in \text{Irr}(G/N)$ are irreducible distinct for distinct $\beta$ and are all of the irreducible constituents of $\theta^G$.

In order to find the normal abelian subgroups of the given groups in Theorems 1.1 and 1.2, we need the following well-known lemma.

**Lemma 1.2** (Ito’s Theorem). [6] Theorem 6.15] Let $A \trianglelefteq G$ be abelian. Then $\chi(1) | [G:A]$, for all $\chi \in \text{Irr}(G)$.

The interest of Lemma 1.3 is that it allows one to obtain some information about cd of the normal subgroup $N$ of $G$ by considering some elements of cd$(G)$ and $[G:N]$, which will be needed in the proofs of Theorems 1.1 and 1.2.

**Lemma 1.3.** [6] Theorem 6.2 and Corollary 11.29] Let $N \trianglelefteq G$ and $\chi \in \text{Irr}(G)$. Let $\theta$ be an irreducible constituent of $\chi_N$ and suppose that $\theta_1 = \theta, \ldots, \theta_t$ are distinct conjugates of $\theta$ in $G$. Then $\chi_N = e \sum_{i=1}^t \theta_i$, where $e = [\chi_N, \theta]$. Also, $\chi(1)/\theta(1) | [G:N]$. 


Applying Lemma 1.4 to the proof of Theorem 1.1 (Case b) leads us to obtain some prime divisors of the elements of cd of some normal subgroups of $G$, by considering their normal abelian Sylow subgroups.

**Lemma 1.4 (Ito–Michler’s Theorem).** [4] Theorem 19.10 and Remark 19.11
Let $p(G)$ be the set of all prime divisors of the elements of cd$(G)$. Then $p \notin p(G)$ if and only if $G$ has a normal abelian $p$-Sylow subgroup.

In this paper, we need cd$(\text{SL}(2,q))$, cd$(\text{PSL}(2,q))$, cd$(\text{PGL}(2,q))$ and cd$(G)$, where $G$ is an extension of $\text{PSL}(2,q)$, frequently. So we bring them in Lemma 1.5 for making it easy to use.

**Lemma 1.5.** [11] Theorem A and Corollary C
If $q$ is a power of an odd prime number, then

(i) $\text{cd}(\text{SL}(2,q)) = \{1, q − 1, (q−1)/2, q, q+1, (q+1)/2\}$;
(ii) $\text{cd}(\text{PSL}(2,q)) = \{1, q − 1, q, q+1, (q+\varepsilon)/2\}$, where $\varepsilon = (-1)^{(q−1)/2}$;
(iii) $\text{cd}(\text{PGL}(2,q)) = \{1, q − 1, q, q+1\}$;
(iv) if $q>3$ and $\text{PSL}(2,q) \leq G \leq \text{Aut}(\text{PSL}(2,q))$ such that $|G:\text{PSL}(2,q)| = 2$ and $G \neq \text{PSL}(2,q)$, then $2(q−1) \in \text{cd}(G)$.

Since for every odd prime divisor $r$ of $|\text{PGL}(2,p^n)|$, $\text{PGL}(2,p^n)$ has exactly one irreducible character degree divisible by $r$, we may apply the following lemma to the proof of Step 3 of Theorem 1.2.

**Lemma 1.6.** [7] Theorem C and Corollary 7.5
Let $G$ be a finite group with exactly one irreducible character degree divisible by $p$. Assume that $G$ is not $p$-solvable, and let $U = \text{O}_p(G)$ and $K/U = \text{O}_p(G/U)$. Then $K$ is the unique largest normal $p$-solvable subgroup of $G$. Also, $G/K$ has a simple socle $S/K$, and $[G:S]$ is not divisible by $p$. In particular, $S/K \cong M_{11}$, $J_1$ or $\text{PSL}(2,q)$, where $q$ is a power of the prime $r$.

Lemmas 1.7 and 1.8 will be needed in Step 3 of the proof of Theorem 1.2 and the proof of Theorem 1.11 respectively.

**Lemma 1.7.** [6] Theorem 12.15
If $|\text{cd}(G)| \leq 3$, then $G$ is solvable.

**Lemma 1.8.** [12] Let $G$ be a nonsolvable group. Then $G$ has a normal series $1 \triangleleft H \triangleleft K \triangleleft G$ such that $K/H$ is a direct product of isomorphic non-abelian simple groups and $|G/K||\text{Out}(K/H)|$.

The following lemma follows immediately by checking the order of finite simple groups of Lie type over a finite field of order $q$ for showing that the non-abelian chief factor of $G$ is isomorphic to $\text{PSL}(2,p)$.

**Lemma 1.9.** Let $H$ be a finite simple group of Lie type over a finite field of order $q$, where $q = r^i$ for a prime $r$. If $p \in \pi(H)$ and $e(p,q) = i$, then $(q^i−1) < |H|_r$ except in the following cases:

(i) $i = 2$ and $H = \text{PSL}(2,q)$;
(ii) $i = 6$ and $H = \text{PSU}(3,q)$;
(iii) $i = 4$ and $H = ^2B_2(q)$, where $q = 2^{2m+1}$, $m \geq 1$;
(vi) $i = 6$ and $H = ^2G_2(q)$, where $q = 3^{2m+1}$, $m \geq 1$. 

2. Proof of Theorem 1.1

Throughout this section, let $G$ be a group satisfying the conditions of the main theorem. Since $p, p + 1 \in \text{cd}(G)$, fix $\chi, \phi \in \text{Irr}(G)$ such that $\chi(1) = p$ and $\phi(1) = p + 1$.

I. Let $p = 3$ and $P \in \text{Syl}_p(G)$. If $n_3(G) = 1$, then, since $P$ is a cyclic group of order 3, Ito’s theorem forces $3 = \chi(1) \mid [G : P] = 8$, which is impossible. Thus $n_3(G) = 4$ and hence, $P_G = 1$, so $G = G/P_G \hookrightarrow S_4$. But $[G] = [S_4]$, so $G \cong S_4 \cong \text{PGL}(2, 3)$, as claimed. The same reasoning completes the proof in the case when $p = 2$.

II. Let $p > 3$. We claim that $G$ is not solvable. On the contrary, suppose that $G$ is solvable. We are going to get a contradiction in the following cases:

Case a. Let $(p - 1)/2$ be even. Let $H$ be a Hall subgroup of $G$ of order $2p(p - 1)$. Thus $|G : H| = (p + 1)/2$. Hence $G/H_G \hookrightarrow S_{(p + 1)/2}$. Since $p > (p + 1)/2$, $p \in \pi(H_G)$. Let $P \in \text{Syl}_p(H_G)$. Since $|P| = p$, $P$ is abelian. Also, $n_p(H_G) = kp + 1 \mid (p - 1)(p + 1)/r^a$. First suppose that $k \geq 1$. Then there exists a natural number $t$ such that $r^a(kp + 1) = (p - 1)(p + 1)$. Therefore, $p \mid tr^a + 1$ and hence there exists a natural number $s$ such that $ps = r^a + 1$. Hence, $(ps - 1)(kp + 1) = p^2 - 1$, which implies $k = s = 1$. Therefore, $r^a \mid p - 1$. On the other hand, $r \mid p + 1$, so $r \mid \gcd(p - 1, p + 1) = 2$, which is a contradiction. It follows that $n_p(H_G) = 1$. Thus $P \leq G$ and hence, Ito’s theorem implies that $p = \chi(1) \mid [G : P]$, which is impossible.

Let $p + 1 = 2^a$, for some natural number $a$. Let $H$ be a Hall subgroup of $G$ of order $2p(p + 1) = 2^a + 1$. Thus $|G : H| = (p - 1)/2$ and $G/H_G \hookrightarrow S_{(p - 1)/2}$. Since $|P| = p$, $P$ is abelian. If $n_p(H_G) = 1$, then $P \leq G$, so applying Ito’s theorem to $P$ and $\chi$ leads us to get a contradiction. Therefore, $n_p(H_G) \neq 1$, so we can see at once that $n_p(H_G) = p + 1 = 2^a$ and hence, $|H : H_G| = 2$. Let $\theta \in \text{Irr}(H_G)$ such that $\theta(H_G, \theta) \neq 0$. Then Lemma 1.3 shows that $p + 1 = \phi(1) \mid \theta(1)[G:H_G] = \theta(1)[H : H_G], \text{so either }|H| = |H_G|; p + 1 \mid \theta(1) \text{ or } |H_G| = |H|/2 \text{ and } (p + 1)/2 \mid \theta(1)$. Also, $p \in \pi(H_G)$ and $n_p(H_G) \neq 1$, so Ito–Michler’s Theorem guarantees that there exists $\eta \in \text{cd}(H_G)$ such that $p \mid \eta(1)$. It is known that $\Sigma_{\alpha \in \text{Irr}(H_G)} \alpha(1) = |H_G|$. Thus either $|H_G| = |H|$ and $p^2 + (p + 1)^2 \leq |H_G|$, or $|H_G| = |H|/2$ and $p^2 + ((p + 1)/2) \leq |H_G|$. This forces either $p^2 + (p + 1)^2 \leq 2p(p + 1)$ or $p^2 + (p + 1)^2/2 \leq p(p + 1)$, which is impossible.

Therefore, $G$ is not solvable. Now, Lemma 1.3 shows that $G$ has a normal series as $1 \leq H \leq K \leq G$ such that $K/H$ is a direct product of $m$ copies of a non-abelian simple group $S$. Since $p \mid |G|$, we deduce that exactly one of the following holds:

\[ p \mid |G/K|, \quad p \mid |H|, \quad p \mid |K/H|. \]
Thus the proof falls into the following cases:

1. Let $p \mid |G/K|$. We know that $K/H$ is isomorphic to $m$ copies of a non-abelian simple group $S$. Thus $\text{Out}(K/H) \cong \text{Out}(S) \wr S_m$. Also Lemma 1.3 shows that $|G/K| \mid |\text{Out}(K/H)|$. Therefore $p \mid |\text{Out}(S)|$ or $p \mid |S_m|$. If $p \mid |S_m|$, then $m \geq p$. But the order of the smallest simple group is 60 and hence, $60^p \leq |K/H|$. It follows that $60^p \leq p(p^2 - 1)$, a contradiction. Hence, $p \mid |\text{Out}(S)|$ and $p \not\mid |S|$. Now, considering the order of the outer automorphism groups of alternating groups and simple sporadic groups leads us to see that $S$ is a simple group of Lie type over a finite field of order $q$, where $q = p^0_0$ for some prime number $p_0$ and some natural number $f$ such that $p \mid f$ (see [2]). Since $p \not\mid |S|$ and $|S| \mid |G|$, $|S| \mid (p^2 - 1)$ and since $p \geq 5$, $q \mid |S|$ and $p_0 \geq 2$, we deduce that $2p \leq p_0^0 \leq p_0^0 = q \leq p^2 - 1$, which is a contradiction.

2. Let $p \mid |H|$. Let $\theta \in \text{Irr}(H)$ be a constituent of $\chi_H$. Then Lemma 1.3 implies that $\chi(1)/\theta(1) \mid |G:H|$, and since $p \not\mid |G:H|$, $\theta(1) = p$. So $\chi_H = \theta$ and now, Gallagher’s theorem shows that for every $\beta \in \text{Irr}(G/H)$, $\beta \chi \in \text{Irr}(G)$. So for every $\beta \in \text{Irr}(G/H)$, $p\beta(1) \in \text{cd}(G)$. But by our assumption, max($\text{cd}(G)) = p + 1$, so for every $\beta \in \text{Irr}(G/H)$, $\beta(1) = 1$ and hence, $G/H$ is abelian, which is a contradiction.

3. Let $p \mid |K/H|$. Since $K/H$ is isomorphic to the direct product of $m$ copies of $S$, we must have $p^m \mid |K/H|$. But we know that $p \mid \text{out}(G)$. This implies that $K/H$ is a simple group such that $p$ is the maximal prime divisor of its order. Also $|K/H| \mid p(p^2 - 1)$. Now, these conditions on $K/H$ rule out the case that $K/H$ is a sporadic simple group.

If $K/H$ is an alternating group, then $S \cong A_n$, for some $n \geq 5$, so $p \leq n$ and $n! = |A_n| = |S| \leq p(p^2 - 1) \leq n(n^2 - 1)$. This implies that $p = n = 5$ and hence, $K/H \cong A_5 \cong \text{PSL}(2, 5)$.

Let $K/H$ be a finite simple group of Lie type over a finite field of order $q$, where $q = r^m$ for a prime $r$. If $p \neq r$, then suppose $e(p, q) = i$. Since $|K/H| \mid |G|$, we deduce that one of the following holds:

1. Let $|K/H| = |p^2 - 1|$. Then $r = 2$ and since gcd($p - 1, p^2 + 1$) = 2, we can see that $|p^2 - 1| = 2|p - 1|$, or $2|p + 1|$. If $4 \mid q$, then either $i = 1$ or $p \not\mid q - 1$ and hence, $p \mid (q^i - 1)/(q - 1)$. If $i = 1$, then since $|K/H| = q^2$, we can see that $q - 1 \mid |K/H| - 1$, so $p \mid |K/H| - 1 = |p^2 - 1| - 1$, which is impossible. So $i \neq 1$ and hence, $p \mid (q^i - 1)/(q - 1)$. Thus $3p \leq |(q - 1)p| \leq q^i - 1$ and $|K/H| \leq 2(p + 1)$. Therefore, $|K/H| > |K/H|$, and so, $K/H$ is isomorphic to one of the groups obtained in Lemma 1.3. If $i = 6$ and $K/H \cong \text{PSU}(3, q)$, then $p \mid (q^i + 1)/(q + 1)$. Thus $5p \leq (q^3 + 1) < 2q^3 < 2|K/H|$, or $4(p + 1)$, which is impossible. If $i = 4$ and $K/H \cong 2B_2(q)$, where $q = 2^{2m+1}$, then since $2m + 1$ is odd, $q^2 + 1$ is not prime, so we can see that $p \not\mid q^2 + 1$ and hence, $3p \leq q^2 + 1 \leq 2(p^2 + 1) + 1$, which is impossible. Thus $K/H \cong \text{PSL}(2, q)$. Now let $q = 2$. If $p \neq q^i + 1$, then we can see that $3p \leq q^i - 1$. Now applying the previous argument leads us to get a contradiction. If $p = 2^i - 1$, then $i$ is prime and $2^i = p + 1$. Since $|K/H| = |p^2 - 1|$, we deduce that $|K/H| = 2^{i+1}$. But $p \geq 5$, so $i \geq 3$. Thus checking the order of finite simple groups of Lie type leads us to get a contradiction.
2. Let \(|K/H|_r \mid (p+1)/2\) or \(|K/H|_r \mid (p-1)/2\). Then \(p \leq q^i - 1\) and \(|K/H|_r \leq (p+1)/2\). Thus \((q^i - 1) > |K/H|_r\) and so, \(K/H\) is isomorphic to one of the groups obtained in Lemma 1.3. If \(i = 6\) and \(K/H \cong \text{PSU}(3,q)\), then \(p \leq (q^i + 1) \leq |K/H|_r + 1 \leq (p+1)/2 + 1\), which is impossible. If \(i = 4\) and \(K/H \cong 2B_2(q)\), where \(q = 2^{2m+1}\), then since \(p \mid q^2 + 1\), \(p \leq q^2 + 1 = |K/H|_r + 1 \leq (p+1)/2 + 1\) and hence, \(p \leq 3\), which is impossible. The same reasoning rules out the case that \(i = 6\) and \(K/H \cong 2G_2(q)\), where \(q = 3^{2m+1}\). Thus \(K/H \cong \text{PSL}(2,q)\).

3. Let \(|K/H|_r = |p-1|_2\) or \(|K/H|_r = |p+1|_2\). If \(|K/H|_r \neq |p-1|_2\) or \(|K/H|_r \neq |p+1|_2\), then we can see that \(|K/H|_r \leq (p+1)/3\) and hence, applying the same argument as that used in 2 leads us to \(K/H \cong \text{PSL}(2,q)\). Thus for some natural number \(t\), \(|K/H|_r = 2^t\) and either \(|K/H|_r = p-1\) or \(|K/H|_r = p+1\).

Since \(p \mid |K/H|\), checking the order of finite simple groups of Lie type shows that either \(p=7\) and \(K/H \cong \text{PSL}(3,2) \cong \text{PSL}(2,7)\) or \(K/H \cong \text{PSL}(2,q)\).

But if \(K/H \cong \text{PSL}(2,q)\), where \(p \mid q\), then since \(p \in \pi(K/H)\), either \(p \mid q-1\) or \(p \mid q+1\), so we have the following possibilities:

- If \(p = q-1\), then \(q = 2^\alpha\), for some natural number \(\alpha\) and
  \(|G| = p(p-1)(p+1) = q(q-1)(q-2) < q(q^2 - 1) = |K/H|\),
  which is impossible.

- If \(p \mid q-1\) and \(p \leq (q-1)/2\), then
  \(|G| = p(p-1) \leq ((q-1)/2)((q+1)/2)((q-3)/2) < q(q^2 - 1)/2 = |K/H|,
  which is a contradiction.

- If \(p \mid q+1\), then applying the same argument as above leads to \(q = 4\) and \(p = 5\). Thus \(K/H \cong \text{PSL}(2,4) \cong \text{PSL}(2,5)\).

These show that \(p \mid q\). Since \(p \mid |G|\), we deduce that \(p = q\). Thus considering the order of finite simple groups of Lie type over a finite field of order \(p\) forces \(K/H = \text{PSL}(2,p)\), and so \(|H| = 2\) or \(|G/K| = 2\). Let for some natural number \(d\) with \(d \mid 2n\), \(dp(q-1)\) or \(dp(q+1)\) belongs to \(\text{cd}(G/K) = \text{cd}(G)\), \(|G/K| = 1\) and hence, \(G = K\) and \(G \cong \text{PSL}(2,p)\). Thus either \(G \cong \text{PSL}(2,p)\) or \(G \cong \text{SL}(2,p)\). If \(|G/K| = 2\), then \(|H| = 1\) and hence, \(K = \text{PSL}(2,p)\) and \(G = \text{PSL}(2,p) : 2 = \text{PGL}(2,p)\). Thus the theorem is proved.

**Corollary 2.1.** Let \(|G| = |\text{PGL}(2,p)|\). If \(\text{cd}(G) = \text{cd}(\text{PGL}(2,p))\), then \(G\) is isomorphic to \(\text{PGL}(2,p)\).

**Proof.** It follows immediately from the proof of Theorem 1.1

**3. Proof of Theorem 1.2**

First let \(n = 1\). Since \(X_1(\text{PGL}(2,p)) = X_1(\text{G}(G))\), \(\text{cd}(G) = \text{cd}(\text{PGL}(2,p))\) and \(|G| = |\text{PGL}(2,p)|\), because \(|G| = \sum_{\chi \in \text{Irr}(G)} \chi(1)^2\). Thus Corollary 2.1 completes the proof. Now let \(n > 1\). Since \(\text{cd}(G) = \text{cd}(\text{PGL}(2,p^n))\), we deduce by Lemma 1.3(iii) that \(\text{cd}(G) = \{1, p^n - 1, p^{n+1}, p^n\}\). Thus for every \(r \in \pi(G) - \pi((2,p-1))\), \(G\) has exactly one irreducible character degree divisible by \(r\). We are going to complete the proof in the following steps.
Step 1. If $M$ is a nontrivial normal solvable subgroup of $G$, then $p$ is odd, $M = Z(G)$ and $|M| = 2$.

Proof. Let $N$ be a normal minimal subgroup of $G$ such that $N \leq M$. Then there exists $r \in \pi(G)$ such that $N$ is an $r$-elementary abelian subgroup. Thus Ito’s theorem and our assumption force $p^n, p^n - 1, p^n + 1 \mid [G : N]$ and hence $p^n(p^{2n} - 1)/(2, p - 1) \mid [G : N] = |\text{PGL}(2, p^n)|/[N]$. Therefore, $|N| = (2, p - 1)$.

Since $M \neq 1$, we deduce that $|N| \neq 1$ and hence, $2 \mid p - 1$ and $|N| = 2$. This forces $N \leq Z(G)$. Also, applying Ito’s theorem and our assumption force $p^n, p^n - 1, p^n + 1 \mid [G:Z(G)]$ and hence, $|Z(G)| = |N| = 2$. We claim that $M = N$. If not, then we can assume that $L/N$ is a normal minimal subgroup of $G/N$ such that $L/N \leq M/N$. Thus there exists $s \in \pi(G)$ such that $L/N$ is an $s$-elementary abelian subgroup. If $s \neq 2$, then since $N = Z(G)$, we deduce that $G$ contains a normal subgroup $K$ such that $K \cong L/N$, which is a contradiction with the above statements. Thus $s = 2$. If $|L| = 4$, then Ito’s theorem and our assumption guarantee $p^n(p^{2n} - 1)/2 \mid |G:L|$, which is a contradiction. Thus $|L| > 4$. Now for $\varepsilon = \pm$, let $\chi_\varepsilon \in \text{Irr}(G)$ such that $\chi_\varepsilon(1) = p^n + \varepsilon 1$ and $\theta_\varepsilon \in \text{Irr}(L)$ such that $[\chi_\varepsilon, \theta_\varepsilon] \neq 0$. Thus Lemma 1.4 shows that $\chi_\varepsilon(1)/\theta_\varepsilon(1) \mid |G:L|$ and hence there exists $\theta \in \text{Irr}(L)$ such that $|L|/2 \mid \theta(1)$. On the other hand, $L/N$ is 2-elementary abelian and $|L/N| \geq 4$. Thus there exists $xN \in L/N$ such that $O(xN) = 2$ and $(xN) \neq L/N$. Therefore, $(xN)$ is a normal abelian subgroup of $L$ of order 4 and hence Ito’s theorem shows that $\theta(1) \mid |L|/4$, which is a contradiction. Therefore, $M = N$, as claimed.

Step 2. There exists $r \in \pi(G) - \{2\}$ such that $G$ is not an $r$-solvable group.

Proof. Since by Step 1, $G$ is not solvable, the result follows immediately from the definition of $r$-solvable groups.

Step 3. $G \cong \text{PGL}(2, p^n)$.

Proof. By Step 2, there exists $r \in \pi(G) - \{2\}$ such that $G$ is not $r$-solvable. Also, $\text{cd}(G) = \{1, p^n, p^n + 1, p^n - 1\}$. Thus Lemma 1.6 shows that if $U = O_r(G)$ and $K/U = O_r(G/U)$, then $G/K$ has a simple socle $S/K$ (which is isomorphic to $M_{11}$, $J_1$ or $\text{PSL}(2, q)$, and $[G:S]$ is not divisible by $p$. Since $r \neq 2$, step 1 shows that $U = 1$. Also, $\text{cd}(G/K) = \{\chi(1) : \chi \in \text{Irr}(G), K \leq \ker(\chi)\}$ and Lemma 1.7 shows that $|\text{cd}(G/K)| \geq 4$. Therefore, $\text{cd}(G/K) = \text{cd}(G)$. Thus $p^n, p^n + 1, p^n - 1 \mid |G/K|$. This shows that $|G|/2 \mid |G/K|$. Thus considering the order of $\text{Aut}(M_{11})$ and $\text{Aut}(J_1)$ guarantees that $S/K$ is not isomorphic to $M_{11}$ and $J_1$ and hence, $S/K \cong \text{PSL}(2, q)$. If $p \mid |G:S|$, then Theorem A in 11 shows that for some natural number with $d \mid 2n$, $dp(q - 1) = dp(q + 1)$ belongs to $\text{cd}(G/K) = \text{cd}(G)$, which is a contradiction. Thus $p \nmid |G:S|$. So $p^n \in \text{cd}(S/K)$ and $p^n \mid |S/K|$. If $p \nmid q$, then considering the elements of $\text{cd}(S/K)$ mentioned in Lemma 1.6(i) shows that $p^n \mid q + 1$ or $p^n \mid q - 1$. If $p^n = q + 1$ or $q - 1$, then $|S/K| = p^n(p^n - 1)(p^n - 2)$ or $p^n(p^n - 1)(p^n + 1)$ which divides $|G|$ and hence $p^n - 2 \mid p^n + 1$ or $p^n + 2 \mid p^n - 1$, which is impossible. Thus $p^n \mid q$ and since $p^n \mid |G|$, we deduce that $p^n = q$ and hence, $S/K \cong \text{PSL}(2, p^n)$. If $p = 2$, then $|S/K| = |G|$ and hence, $S = G$, as claimed. Now let $p$ be odd. If $K \neq 1$, then $G/S = 1$ and by step 1, $K = Z(G)$, which is a cyclic group of order 2 and hence $G \cong \text{SL}(2, p^n)$ or $\text{PSL}(2, p^n) \times Z(G)$. But $\text{cd}(\text{SL}(2, p^n)), \text{cd}(\text{PSL}(2, p^n)) \neq \text{cd}(\text{PGL}(2, p^n))$, by Lemma
which is a contradiction. Thus $K = 1$ and $|G/S| = 2$. Since $K = 1$ and $S$ is a socle of $G$, we can see that $C_G(S) = 1$ and hence, $G/S \cong \text{Out}(S)$.

But we know that if $p$ is an odd prime, then $\text{Out}(S) = \text{Out}(\text{PSL}(2, p^n)) = (\langle \delta \rangle \times \langle \gamma \rangle)$, where $\delta$ is a diagonal automorphism of order 2 and $\gamma$ is a field automorphism of order $n$. Also, $\text{PSL}(2, p) \cdot \langle \delta \rangle = \text{PGL}(2, p)$. If $G \not\cong \text{PGL}(2, p^n)$, then since $[G : S] = 2$, we deduce that $G$ contains a field automorphism $\phi$ of order 2 and hence, $G = \text{PGL}(2, p^n) \cdot \langle \phi \rangle$. Thus Lemma 1.5(iv) shows that $2(p^n - 1) \in \text{cd}(G) = \text{cd}(\text{PGL}(2, p^n))$, which is a contradiction. This shows that $G \cong \text{PGL}(2, p^n)$, as claimed.

Remark 3.1. By Molien’s theorem [1, Theorem 2.13] $X_1(\text{PGL}(2, p^n)) = X_1(G)$ if and only if $C_{\text{PGL}(2, p^n)} = C_G$. Thus Theorem 1.2 shows that if $C_G = C_{\text{PGL}(2, p^n)}$, then $G \cong \text{PGL}(2, p^n)$.

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