ON THE LOCATION OF THE ZEROS
OF CERTAIN POLYNOMIALS

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ABSTRACT. We extend Aziz and Mohammad’s result that the zeros, of a polynomial
\( P(z) = \sum_{j=0}^{n} a_j z^j \), with moduli greater than \( t(n-1)/n \) are simple, to polynomials with complex coefficients. Then we improve their result that the polynomial \( P(z) \), of degree \( n \), with complex coefficients, does not vanish in the disc
\[
|z - ae^{i\alpha}| < a/(2n); a > 0, \max_{|z|=a} |P(z)| = |P(ae^{i\alpha})|,
\]
for \( r < a < 2 \), \( r \) being the greatest positive root of the equation
\[
x^n - 2x^{n-1} + 1 = 0,
\]
and finally obtained an upper bound, for moduli of all zeros of a polynomial, (better, in many cases, than those obtainable from many other known results).

1. Introduction and statement of results

While thinking about zeros of a polynomial, Aziz and Mohammad [1] proved the following result, thereby suggesting that certain zeros of a polynomial with non-negative increasing coefficients may be simple.

**Theorem A.** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) such that
\[
a_0 \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0.
\]
Then all the zeros of \( P(z) \), of modulus \( \geq n/(n+1) \), are simple.

In [2] Aziz and Mohammad obtained, a generalization as well as a refinement, of Theorem A.

**Theorem B.** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n > 1 \), such that for some \( t > 0 \) one has \( ta_j \geq a_{j-1} > 0, j = 2, 3, \ldots, n, a_0 \) may be a real or a complex number. Then all the zeros of \( P(z) \), of modulus greater than \( t(n-1)/n \), are simple.

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287
In this paper, we extend Theorem B to polynomials with complex coefficients. More precisely we prove

**Theorem 1.1.** If \( p(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial of degree \( n(> 1) \) such that for certain \( t > 0 \)
\[
(1.1) \quad t|a_j| \geq |a_{j-1}|, \quad j = 2, 3, \ldots, n
\]
and for certain real \( \alpha \) and \( \beta \): \( |\text{Arg} a_j - \beta| \leq \alpha \leq \pi/2, \quad j = 1, 2, 3, \ldots, n \), then \( p(z) \) can not have a zero of order \((\geq 2)\), with modulus greater than
\[
\frac{t(n-1)}{n} \left\{ \cos \alpha + \sin \alpha + \frac{2\sin \alpha}{n|a_n|} \sum_{j=0}^{n-2} \left( \frac{n}{t(n-1)} \right)^{n-1-j} (j+1)|a_{j+1}| \right\}.
\]
In other words all the zeros of \( p(z) \), with moduli greater than
\[
\frac{t(n-1)}{n} \left\{ \cos \alpha + \sin \alpha + \frac{2\sin \alpha}{n|a_n|} \sum_{j=0}^{n-2} \left( \frac{n}{t(n-1)} \right)^{n-1-j} (j+1)|a_{j+1}| \right\},
\]
are simple.

**Theorem 1.2.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n(> 1) \), with \( \text{Re} a_j = \alpha_j, \text{Im} a_j = \beta_j, \quad j = 0, 1, \ldots, n \). If for certain \( t > 0 \)
\[
(1.2) \quad t\alpha_j \geq \alpha_{j-1} \geq 0, \quad j = 2, 3, \ldots, n \quad \text{and} \quad \alpha_n > 0,
\]
then \( p(z) \) can not have a zero of order \((\geq 2)\), with modulus greater than
\[
\frac{t(n-1)}{n} \left\{ 1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left( \frac{n}{t(n-1)} \right)^{n-1-j} (j+1)|\beta_{j+1}| \right\}.
\]
In other words all the zeros of \( p(z) \), with moduli greater than \((1.3)\) are simple.

Next we obtain the following result for polynomials with complex coefficients, similar to Theorem B, but not an extension of Theorem B to polynomials with complex coefficients.

**Theorem 1.3.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n(> 1) \) such that for certain \( t > 0 \): \( t|a_j| \geq |a_{j-1}|, \quad j = 2, 3, \ldots, n \). If \( k \) is the greatest positive root of the trinomial equation \( x^n - 2x^{n-1} + 1 = 0 \), then \( p(z) \) can not have a zero of order \((\geq 2)\), with modulus greater than \( kt(n-1)/n \). In other words all the zeros of \( p(z) \), with moduli greater than \( kt(n-1)/n \), are simple.

Further in [2], Aziz and Mohammad also obtained a zero free region for polynomials with complex coefficients.

**Theorem C.** Let \( P(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial with complex coefficients. Then for no real \( a > 0 \), \( P(z) \) vanishes in the disk \( |z - ae^{i\alpha}| < a/(2n) \), where
\[
\text{Max} |P(z)| = |P(ae^{i\alpha})|.
\]
We obtain a refinement of Theorem C, for \( r < a < 2 \), \( r \) being the greatest positive root of the equation \( x^n - 2x^{n-1} + 1 = 0 \).
Theorem 1.4. Let \( p(z) \) be a polynomial of degree \( n \geq 2 \) and \( K \), the greatest positive root of the equation \( x^{n+1} - 2x^n + 1 = 0 \). If
\[
\max_{|z|=a} |p(z)| = |p(\aei\alpha)|, \quad a > 0,
\]
then for no \( a > 0 \), \( p(z) \) vanishes in the disc \( |z - \aei\alpha| < R_a \).

Finally we obtain an upper bound for moduli of all zeros of a polynomial (better in many cases than those obtainable from many other known results).

Theorem 1.5. Let \( p(z) = a_0 z^n + \sum_{k=m}^{n} a_k z^{n-k} \) be a polynomial, with
\[
r = \max_{m \leq k \leq n-1} \left| \frac{a_{k+1}}{a_k} \right|
\]
and let \( \xi \) be unique positive root of the equation
\[
x^m - r x^{m-1} - \frac{|a_m|}{|a_0|} = 0.
\]
Then all the zeros of \( p(z) \) lie in \( |z| < \xi \).

By taking \( m = 1 \) in Theorem 1.5 we get

Corollary 1.1. Let \( p(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \) be a polynomial, with
\[
R = \max \left( \left| \frac{a_2}{a_1} \right|, \left| \frac{a_3}{a_2} \right|, \ldots, \left| \frac{a_n}{a_{n-1}} \right| \right)
\]
Then all the zeros of \( p(z) \) lie in \( |z| < R + |a_1/a_0| \).

Remark 1.1. In many cases Corollary 1.1 gives better upper bounds than those obtainable by other known results. For the polynomial
\[
p_1(z) = a_0 z^5 + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 z + a_5,
\]
with \( |a_0| = 1, |a_1| = 2, |a_2| = 3, |a_3| = 4, |a_4| = 5, |a_5| = 6 \), all the zeros lie in
(i) \( |z| < 7 \), by Cauchy [13], [10] Theorem (27.2), [11], [12],
(ii) \( |z| \leq 4 \), by Kojima [15], [9],
(iii) \( |z| \leq R, R > 3.6 \), by Govil and Rahman [6] Theorem 1],
(iv) \( |z| \leq R, 6.9 < R < 7 \), by Dehmer [4] Theorem 3,2],
(v) \( |z| \leq R, R > 3.8 \), by Jain [7] Corollary 1],
(vi) \( |z| \leq R, 3.9 < R < 4 \), by Dehmer and Mowshowitz [5] Theorem 2],
(vii) \( |z| \leq 4 \), by Dehmer and Mowshowitz [5] Theorem 4],
(viii) \( |z| < 3.5 \), by Corollary 1.1.
2. Lemmas

For the proofs of the theorems we require the following lemmas.

**Lemma 2.1.** If \( p(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial with complex coefficients such that for certain real \( \beta \) and \( \alpha \) one has \( |\text{Arg} a_j - \beta| \leq \alpha \leq \pi/2 \), \( j = 0, 1, \ldots, n \) and \( |a_n| \geq |a_{n-1}| \geq \cdots \geq |a_0| \), then \( p(z) \) has all its zeros on or inside the circle
\[
|z| = \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{|a_n|} \sum_{j=0}^{n-1} |a_j|.
\]

This lemma is due to Govil and Rahman [6].

**Lemma 2.2.** If \( p(z) = \sum_{j=0}^{n} a_j z^j \) is a polynomial such that for certain \( a > 0 \)
\[
|a_n| \geq a |a_{n-1}| \geq a^2 |a_{n-2}| \geq \cdots \geq a^{n-1} |a_1| \geq a^n |a_0| \]
and for certain real \( \beta \) and \( \alpha \)
\[
|\text{Arg} a_j - \beta| \leq \alpha \leq \pi/2, \quad j = 0, 1, \ldots, n
\]
then \( p(z) \) has all its zeros in
\[
|z| \leq 1 + \frac{2 \sin \alpha}{|a_n|} \sum_{j=0}^{n-1} a^{n-j} |a_j|.
\]

**Proof of Lemma 2.2.** Using (2.1) and (2.2) we can say that polynomial \( p(z/a) \) satisfies the hypotheses of Lemma 2.1 and therefore has all its zeros in
\[
|z| \leq \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{|a_n|} \sum_{j=0}^{n-1} a^{n-j} |a_j|.
\]

Accordingly the polynomial \( p(z) \equiv p(\frac{a^j}{a}) \) will have all its zeros in
\[
|z| \leq \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{|a_n|} \sum_{j=0}^{n-1} a^{n-j} |a_j|. \tag{2.3}
\]

**Lemma 2.3.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial. If \( \text{Re} a_j = \alpha_j, \; \text{Im} a_j = \beta_j, \; j = 0, 1, \ldots, n \) and \( \alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 \geq 0, \; \alpha_n > 0 \), then \( p(z) \) has all its zeros in
\[
|z| \leq 1 + 2 \sum_{j=0}^{n} |\beta_j|. \tag{2.4}
\]

This lemma is due to Govil and Rahman [6].

**Lemma 2.4.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) (\( > 1 \)) with
\[
\text{Re} a_j = \alpha_j, \quad \text{Im} a_j = \beta_j, \quad j = 0, 1, \ldots, n.
\]
If for certain \( a > 0 \)
\[
\alpha_n \geq a \alpha_{n-1} \geq a^2 \alpha_{n-2} \geq \cdots \geq a^{n-1} \alpha_1 \geq a^n \alpha_0 \geq 0, \quad \alpha_n > 0
\]

\( \alpha_n \geq a \alpha_{n-1} \geq a^2 \alpha_{n-2} \geq \cdots \geq a^{n-1} \alpha_1 \geq a^n \alpha_0 \geq 0, \quad \alpha_n > 0 \)
then \( p(z) \) has all its zeros in
\[
|z| \leq \frac{1}{a} \left( 1 + \frac{2}{\alpha_n} \sum_{j=0}^{n} a^{n-j}|\beta_j| \right).
\]

**Proof of Lemma 2.4.** Using (2.4) and (2.3) we can say that the polynomial \( p(z/a) \) satisfies the hypotheses of Lemma 2.3 and therefore has all its zeros in
\[
|z| \leq 1 + \frac{2}{\alpha_n} \sum_{j=0}^{n} a^{n-j}|\beta_j|.
\]
Accordingly the polynomial \( p(z) \equiv p(az/a) \) will have all its zeros in
\[
|z| \leq \frac{1}{a} \left( 1 + \frac{2}{\alpha_n} \sum_{j=0}^{n} a^{n-j}|\beta_j| \right).
\]

**Lemma 2.5.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial with complex coefficients such that for certain \( a > 0 \)
\[
|a_n| \geq a|a_{n-1}| \geq a^2|a_{n-2}| \geq \cdots \geq a^{n-1}|a_1| \geq a^n|a_0|.
\]
Then \( p(z) \) has all its zeros in \( |z| \leq K_1/a \), where \( K_1 \) is the greatest positive root of the trinomial equation \( K^{n+1} - 2K^n + 1 = 0 \).

This lemma is due to Govil and Rahman [6].

**Lemma 2.6.** Let \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \), be a polynomial with complex coefficients. Then for every positive real number \( r \), all the zeros of \( P(z) \) lie in the disc
\[
|z| \leq \max \left\{ r, \sum_{j=0}^{n-1} \frac{|a_j/a_n|}{r^{n-j-1}} \right\}.
\]
This lemma is due to Aziz and Mohammad [2].

**Lemma 2.7.** Let \( p(z) \) be a polynomial of degree \( n \). Then
\[
\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.
\]
This lemma is due to Bernstein [13].

On applying Lemma 2.7 to the polynomial \( p( rz ) \), we get

**Lemma 2.8.** Let \( p(z) \) be a polynomial of degree \( n \) \(( \geq 1 )\). Then for every positive \( r \) we have \( \max_{|z|=r} |p'(z)| \leq (n/r) \max_{|z|=r} |p(z)| \).

By repeated application of Lemma 2.8 we get

**Lemma 2.9.** Let \( p(z) \) be a polynomial of degree \( n \) \(( \geq 1 )\). Then for every positive \( r \) we have
\[
\max_{|z|=r} |p^k(z)| \leq \frac{n(n-1) \ldots (n-k+1)}{r^k} \max_{|z|=r} |p(z)|, \quad k = 1, 2, \ldots, n.
\]
3. Proofs of the theorems

Proof of Theorem 1.1 We have

\[ p'(z) = \sum_{j=0}^{n-1} (j+1)a_{j+1}z^j = \sum_{j=0}^{n-1} b_jz^j, \text{ say.} \]

Now

\[ t|a_{j+1}| \geq |a_j|, \quad j = 1, 2, \ldots, n-1, \quad (\text{by } (1.1)). \]

Further \( \frac{n-1}{n} \geq \frac{1}{j+1}, \quad j = 1, 2, \ldots, n-1, \) which, by (3.2), makes it possible to write \( t^{n-1}(j+1)|a_{j+1}| \geq j|a_j|, \quad j = 1, 2, \ldots, n-1, \) i.e., \( |b_j| \geq \frac{n}{t(n-1)}|b_{j-1}|, \quad j = 1, 2, \ldots, n-1, \quad (\text{by } (3.1)). \)

We can now apply Lemma 2.2 to the polynomial \( p'(z) \), of degree \( n-1 \) and say that all the zeros of \( p'(z) \) lie in

\[ |z| \leq \frac{t(n-1)}{n}\left\{ \cos \alpha + \sin \alpha \sum_{j=0}^{n-2} \left( \frac{n}{t(n-1)} \right)^{n-j} (j+1)|a_{j+1}| \right\}. \]

Therefore all the zeros of \( p(z) \), with moduli greater than

\[ \frac{t(n-1)}{n}\left\{ \cos \alpha + \sin \alpha \sum_{j=0}^{n-2} \left( \frac{n}{t(n-1)} \right)^{n-j} (j+1)|a_{j+1}| \right\}, \]

are simple.

Proof of Theorem 1.2 We have

\[ p'(z) = \sum_{j=0}^{n-1} (j+1)a_{j+1}z^j = \sum_{j=0}^{n-1} b_jz^j, \text{ say.} \]

Now

\[ ta_{j+1} \geq a_j, \quad j = 1, 2, \ldots, n-1, \quad (\text{by } (1.2)). \]

Further \( \frac{n-1}{n} \geq \frac{1}{j+1}, \quad j = 1, 2, \ldots, n-1, \) which, by (3.4), makes it possible to write \( t^{n-1}(j+1)a_{j+1} \geq ja_j, \quad j = 1, 2, \ldots, n-1, \) i.e.,

\[ \text{Re } b_j \geq \frac{n}{t(n-1)}\text{Re } b_{j-1} \geq 0, \quad j = 1, 2, \ldots, n-1, \quad (\text{by } (3.3)). \]

As \( \text{Re } b_{n-1} \geq 0 \), by (3.3) and (1.2), we can say by using (3.5) that the polynomial \( p'(z) \) satisfies the hypotheses of Lemma 2.4 and therefore has all its zeros in

\[ |z| \leq \frac{t(n-1)}{n}\left\{ 1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left( \frac{n}{t(n-1)} \right)^{n-j} (j+1)|\beta_{j+1}| \right\}. \]

Accordingly all the zeros of \( p(z) \), with moduli greater than

\[ \frac{t(n-1)}{n}\left\{ 1 + \frac{2}{n\alpha_n} \sum_{j=0}^{n-1} \left( \frac{n}{t(n-1)} \right)^{n-j} (j+1)|\beta_{j+1}| \right\}, \]

are simple.
Proof of Theorem 1.3. It is similar to the proof of Theorem 1.1 with one change: Lemma 2.5 instead of Lemma 2.2. □

Proof of Theorem 1.4. With \( w = ae^{i\alpha} \), let us consider the polynomial

\[
F(z) = p(wz/n + w) = p(w) + z(w/n)p'(w) + z^2/(2n)(w/n)^2p''(w) + \cdots + z^n/(n!)^n p^n(w).
\]

Then the polynomial

\[
G(z) = z^nF(1/z) = \sum_{k=0}^{n} \frac{w/n}{k!}^k z^{n-k},
\]

will have all its zeros, by using Lemma 2.6 with \( r = a \), in the disc

\[
|z| \leq \max \left\{ a, \sum_{k=0}^{n-1} \frac{|w/n|^{n-k}}{(n-k)!|p(w)|a^{n-k-1}} \right\},
\]

and as

\[
|p(w)| \geq \frac{a^k}{n(n-1)\cdots(n-k+1)}|p(k)(w)|, \quad k = 1, 2, \ldots, n,
\]

by (1.4) and Lemma 2.9 we can say that \( G(z) \) has all its zeros in the disc

\[
|z| \leq \max \left\{ a, \sum_{k=0}^{n-1} \frac{1}{a^{n-k-1}} \right\} = \frac{a}{nR_a}, \quad \text{(by (1.5))}
\]

Therefore by (3.7), we can say that \( F(z) \) has all its zeros in \( |z| \geq nR_a/a \), i.e., \( F(z) \) does not vanish in \( |z| < nR_a/a \) and accordingly, by using (3.6), we can say that \( p(z) \) does not vanish in the disc \( |z - w| < R_a \). □

Proof of Theorem 1.5. Using (1.6) we get

\[
|\alpha_k| \leq r^{k-m}|\alpha_m|, \quad m \leq k \leq n.
\]

Now for \( |z| > r \), we have

\[
|p(z)| \geq |a_0||z|^n \left\{ 1 - \frac{1}{|a_0|} \sum_{k=m}^{n} \frac{|\alpha_k|}{|z|^k} \right\}, \quad \text{(by (3.8))}
\]

\[
\geq |a_0||z|^n \left\{ 1 - \frac{|a_m| r^{-m}}{|a_0|} \sum_{k=m}^{n} \left( \frac{r}{|z|} \right)^k \right\}, \quad \text{(by (3.9))}
\]

\[
> |a_0||z|^n \left\{ 1 - \frac{|a_m| r^{-m}}{|a_0|} \sum_{k=m}^{\infty} \left( \frac{r}{|z|} \right)^k \right\}.
\]

\[
= |a_0||z|^n \left\{ \frac{|z|^m - r|z|^{m-1} - (|a_m|/|a_0|)}{(|z|^m - r|z|^{m-1})} \right\}.
\]
Further if \( g(x) = x^m - rx^{m-1} - \frac{|a_m|}{|a_0|} \), then
\[
g(r) = -\frac{|a_m|}{|a_0|} < 0,
\]
and as \( \xi \) is unique positive root of the equation (1.7), we can say that
\[
x^m - rx^{m-1} - \frac{|a_m|}{|a_0|} \geq 0 \quad \text{for} \quad x \geq \xi,
\]
with
\[
\xi > r, \quad \text{(by (3.10))}.
\]
Now by (3.9), (3.11) and (3.12) we can say that \( |p(z)| > 0 \) for \( |z| \geq \xi \). □

References

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