LOGARITHMIC BLOCH SPACE AND ITS PREDUAL

Miroslav Pavlović

Abstract. We consider the space $B_{\log}^1$, of analytic functions on the unit disk $D$, defined by the requirement $\int_D |f'(z)|\phi(|z|)\,dA(z) < \infty$, where $\phi(r) = \log^\alpha(1/(1-r))$ and show that it is a predual of the “$\log$-Bloch” space and the dual of the corresponding little Bloch space. We prove that a function $f(z) = \sum_{n=0}^\infty a_n z^n$ with $a_n \downarrow 0$ is in $B_{\log}^1$ iff $\sum_{n=0}^\infty \log^\alpha((n+2)/(n+1)) < \infty$ and apply this to obtain a criterion for membership of the Libera transform of a function with positive coefficients in $B_{\log}^1$. Some properties of the Cesàro and the Libera operator are considered as well.

1. Introduction and some results

Let $H(D)$ denote the space of all functions analytic in the unit disk $D$ of the complex plane. Endowed with the topology of uniform convergence on compact subsets of $D$, the class $H(D)$ becomes a complete locally convex space. In this paper we are concerned with the predual of the space $B_{\log}^\alpha$, $\alpha \in \mathbb{R}$,

$$B_{\log}^\alpha = \{ f \in H(D) : |f'(z)| = O\left((1-|z|)^{-1}\log^\alpha(1-|z|)^2\right) \}.$$

The norm in $B_{\log}^\alpha$ is defined by

$$\|f\|_{B_{\log}^\alpha} = |f(0)| + \sup_{z \in D} |f'(z)|(1-|z|)\log^{-\alpha} \frac{1}{1-|z|}.$$

The subspace, $b_{\log}^\alpha$, of $B_{\log}^\alpha$ is defined by replacing “$O$” with “$o$”. It will be proved:

(A) The dual of $b_{\log}^\alpha$ is isomorphic to $B_{\log}^1$,

(1.1) $\mathcal{B}_{\log}^1 = \{ f : \|f\|_{\mathcal{B}_{\log}^1} = |f(0)| + \int_D |f'(z)|\log^\alpha \frac{2}{1-|z|}\,dA(z) < \infty \}$,

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and the dual of $\mathfrak{B}^1_{\log^\alpha}$ is isomorphic to $\mathfrak{B}_{\log^\alpha}$, in both cases with respect to the bilinear form

$$\langle f, g \rangle = \lim_{r \uparrow 1} \sum_{n=0}^{\infty} \hat{f}(n) \hat{g}(n) r^{2n}. \quad (1.2)$$

(In (1.1) $dA$ stands for the normalized Lebesgue measure on $\mathbb{D}$.) This extends the well-known result on the Bloch space and the little Bloch space $\mathfrak{b} := \mathfrak{B}_{\log^0}$.

These spaces are Banach spaces, and the space $\mathfrak{B}_{\log^\alpha}$ coincides with the closure in $\mathfrak{B}^1_{\log^\alpha}$ of the set of all polynomials. The space $\mathfrak{B}_{\log} := \mathfrak{B}_{\log^1}$ occurs naturally in the study of pointwise multipliers on the usual Bloch space $\mathfrak{B} := \mathfrak{B}_{\log^0}$ (see [3]).

One of interesting properties of $\mathfrak{B}^1_{\log^\alpha}$ is described in the following theorem:

**Theorem 1.1.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $\{a_n\}$ is a nonincreasing sequence, of real numbers, tending to zero. Let $\alpha \geq -1$. Then $f$ belongs to $\mathfrak{B}^1_{\log^\alpha}$ if and only if

$$S_\alpha(f) := \sum_{n=0}^{\infty} a_n \log^{\alpha}(n+2) \frac{n+1}{n+2} < \infty. \quad (1.3)$$

Moreover, there is a constant $C$ independent of $\{a_n\}$ such that $S_\alpha(f)/C \leq \|f\|_{\mathfrak{B}^1_{\log^\alpha}} \leq C S_\alpha(f)$.

**Proof.** See Section 4. \qed

In the case $\alpha = 0$, this assertion is proved in [17]. We can take $a_n$ to be the coefficients of the Libera transform of a function with positive coefficients. Namely, if $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$ and

$$\sum_{n=0}^{\infty} \frac{\hat{g}(n)}{n+1} < \infty, \quad (1.4)$$

then the Libera transform $\mathcal{L}g$ of $g$ is well defined as

$$\mathcal{L}g(z) = \frac{1}{1-z} \int_1^1 f(\zeta) \frac{d\zeta}{\zeta} = \sum_{n=0}^{\infty} z^n \sum_{k=n}^{\infty} \frac{\hat{g}(k)}{k+1}, \quad (1.5)$$

(see, e.g., [12]). If $\hat{g} \geq 0$, then condition (1.4) is also necessary for the existence of the integral in (1.8): take $z = 0$ to conclude that (1.8) implies the convergence of the integral

$$\int_0^1 g(t) \, dt = \sum_{n=0}^{\infty} \frac{\hat{g}(n)}{n+1}. \quad \text{Then, as an application of Theorem 1.1 we get}$$

**Theorem 1.2.** Let $\alpha > -1$, let $g \in H(\mathbb{D})$, and $\hat{g} \geq 0$. Then $\mathcal{L}g$ is in $\mathfrak{B}^1_{\log^\alpha}$ if and only if

$$K_\alpha(g) := \sum_{n=0}^{\infty} \frac{\hat{g}(n) \log^{\alpha+1}(n+2)}{n+1} < \infty.$$

We have $K_\alpha(g)/C \leq \|\mathcal{L}g\|_{\mathfrak{B}^1_{\log^\alpha}} \leq C K_\alpha(g)$. 
Proof. See Section 4.

In the general case, the integral in (1.5) need not exist, but it certainly exists if \( g \in H(\overline{D}) \), which means that \( g \) is analytic in a neighborhood of the closed disk. By using Theorem 1.1 we shall prove that \( \mathcal{L} \) cannot be extended to a bounded operator from \( \mathfrak{B}_{\log^\alpha} \) to \( H(D) \), if \( \alpha < 0 \). In the case \( \alpha \geq 0 \), every function \( g \in \mathfrak{B}_{\log^\alpha} \) satisfies (1.4), whence \( \mathcal{L} \) is well defined, and we will show that \( \mathcal{L} \) maps this space into \( \mathfrak{B}_{\log^{\alpha-1}} \), when \( \alpha > 0 \). If \( \alpha = 0 \) we need a sort of “iterated” logarithmic space.

**Cesàro operator.** The dual of \( H(D) \) is equal to \( H(\overline{D}) \), where “\( g \in H(\overline{D}) \)” means that \( g \) is holomorphic in a neighborhood of \( \overline{D} \) (depending on \( g \)). The duality pairing is given
\[
\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n),
\]
where \( f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in H(D) \) and \( g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n \in H(\overline{D}) \), and the series is absolutely convergent (see, e.g., [8]). The Cesàro operator is defined on \( H(D) \) as
\[
\mathcal{C}f(z) = \sum_{n=0}^{\infty} z^n \frac{1}{n+1} \sum_{k=0}^{n} a_k, \quad f \in H(D).
\]
It is easy to verify that the adjoint of \( \mathcal{C} : H(D) \hookrightarrow H(\overline{D}) \) is equal to \( \overline{\mathcal{C}} : H(\overline{D}) \hookrightarrow H(D) \), under the pairing (1.6), and vice versa (see, e.g., [12]).

The operators \( \mathcal{C} \) and \( \overline{\mathcal{C}} \) acting on \( H^p \) spaces were first studied by Siskakis in 1987. In [21] he proved that \( \mathcal{C} \) is bounded on \( H^p \) for \( 1 < p < \infty \), and that \( \overline{\mathcal{C}} \) can be extended to a bounded operator on \( H^p \), \( 1 < p < \infty \), and obtained some results on their spectra and norms. A few years later he proved the boundedness of the Cesàro operator on \( H^1 \) ([22]), while Miao proved its boundedness on \( H^p \) for \( 0 < p < 1 \) ([10]). A short proof of the boundedness of \( \mathcal{C} \) on \( H^p \), \( 0 < p < \infty \), as well as a stronger result, can be also found in Nowak [11]. However, \( H^\infty \) is not mapped into itself by \( \mathcal{C} \) (see [4]). If we write (1.7) as
\[
z\mathcal{C}f(z) = \int_{0}^{z} \frac{f(\zeta)}{1 - \zeta} d\zeta,
\]
and hence
\[
(z\mathcal{C}f(z))' = \frac{f(z)}{1 - z},
\]
we conclude that \( \mathcal{C} \) maps \( H^\infty \) into the Bloch space (see [4]).

On the other hand, by using the inequality
\[
|f(z)| = O\left( \log \frac{2}{1 - |z|} \right), \quad f \in \mathfrak{B},
\]
and the analogous inequality for \( f \in \mathfrak{b} \) (replace “\( O \)” with “\( o \)”), we get:

(B) The operator \( \mathcal{C} \) maps the space \( \mathfrak{B} \) into \( \mathfrak{B}_{\log^\alpha} \), and \( \mathfrak{b} \) into \( \mathfrak{b}_{\log^\alpha} \).
One of our aims is to generalize this assertion to some other values of $\alpha$ and then use assertion (A) together with the duality between $\mathcal{C}$ and $\mathcal{C}$ to obtain an alternative proof of some results on the action of $\mathcal{L}$ from $\mathfrak{B}_{\alpha+1}$ to $\mathfrak{B}_1$, where

$$\mathcal{L}f(z) = \int_0^1 f(t + (1-t)z) \, dt. \quad (1.8)$$

In particular we have:

(C) The operator $\mathcal{L}$ is well defined on $\mathfrak{B}_{\log}^{1}$ and maps it into $\mathfrak{B}^{1}$.

It should be noted that: (a) $\mathfrak{B}^{1} \subset H^{1}$; (b) $\mathcal{L}$ does not map $\mathfrak{B}^{1}$ into $H^{1}$ (see [17]); and (c) $\mathcal{L}$ maps $\mathfrak{B}$ into BMOA [12], which improves an earlier result, namely that $\mathcal{L}$ maps $\mathfrak{B}$ into $\mathfrak{B}$ ([5, 24]).

The formula (1.8) is obtained from (1.5) by integrating over the straight line joining $z$ and 1. A sufficient (not necessary [18]) condition for the possibility of such integration is (1.4) ($g = f$).

In proving some of our results, in particular assertions (A) and (B), we use a sequence of polynomials constructed in [6] (see also [7] and [16]) to decompose the space into a sum which resembles a sum of finite-dimensional spaces (see Section 3).

2. Some more results

Some elementary facts concerning the cases when $\mathcal{L}f$ is well defined are collected in the following theorem, where

$$\ell_{-1}^{1} = \{ g \in H(\mathbb{D}) : \|g\|_{\ell_{-1}^{1}} = \sum_{n=0}^{\infty} \frac{|\hat{g}(n)|}{n+1} < \infty \}. $$

**Theorem 2.1.** Let $\alpha \in \mathbb{R}$. Then:

(a) $\mathfrak{B}_{\log}^{\alpha} \subset \ell_{-1}^{1}$ for all $\alpha$;

(b) $\mathfrak{B}_{\log}^{1} \subset \ell_{-1}^{1}$ if and only if $\alpha \geq 0$;

(c) if $\alpha < 0$, then $\mathcal{L}$ cannot be extended to a continuous operator from $\mathfrak{B}_{\log}^{1}$ to $H(\mathbb{D})$.

**Proof.** See Section 4. \hfill \Box

**Remark 2.1.** The inclusions in (a) and (b) are continuous. Assertion (c) says much more than simply that $\mathfrak{B}_{\log}^{1} \subset \ell_{-1}^{1}$.

In the context of the action of $\mathcal{C}$ and $\mathcal{L}$ some new spaces occur: the space $\mathfrak{B}_{\log}$ is defined by the requirement

$$|f'(z)| = O \left( \log \log \frac{4}{1-|z|} \right),$$

the space $\mathfrak{b}_{\log}$ defined by replacing “$O$” with “$o$”, and the space $\mathfrak{B}_{\log}^{1}$ defined by

$$\int_{\mathbb{D}} |f'(z)| \log \log \frac{4}{1-|z|} \, dA(z) < \infty.$$

Our next result is
Theorem 2.2. (a) If $\alpha > -1$, then $C$ maps the space $B_{\log^\alpha}$, resp. $b_{\log^\alpha}$, into $B_{\log^\alpha+1}$, resp. $b_{\log^\alpha+1}$.
(b) $C$ maps the space $B_{\log^{-1}}$, resp. $b_{\log^{-1}}$, into $B_{\log^1}$, resp. $b_{\log^1}$.

Proof. See Section 5.

Remark 2.2. If $f \in B_{\log^\alpha}$ and $\alpha < -1$, then, as it can easily be shown, $f \in A(D)$, where $A(D)$ is the disk-algebra, i.e., the subset of $H^\infty$ consisting of those $f$ which have a continuous extension to the closed disk. Moreover, the modulus of continuity of the boundary function $f_\ast(\zeta)$, $\zeta \in \partial D$, satisfies the condition
$$\omega(f_\ast, t) = O\left(\left(\log^\alpha\frac{2}{t}\right)^{\alpha+1}\right), \quad t \downarrow 0.$$ 
This follows from the inequality
$$\omega(f_\ast, t) \leq C \int_{1-t}^1 M_\infty(r, f') \, dr,$$
see [15, Theorem 2.2]. It should be noted that the modulus of continuity of $f_\ast$ is “proportional” to that of $f(z)$, $z \in D$, see [23, 19].

Concerning the Libera operator we shall prove, besides Theorem 2.1(c), the following facts.

Theorem 2.3. (a) If $\alpha > 0$, then $L$ is well defined on $B_{\log^\alpha}$ and maps this space to $B_{\log^{\alpha-1}}$.
(b) $L$ is well defined on $B_{\log^1}$ and maps this space into $B_{\log^{\alpha-1}}$.
(c) $L$ is well defined on $B_1$ and maps it into $B_1^\alpha$ for all $\alpha < -1$.

Proof. See Section 5.

Theorem 2.4. Let $\alpha \in \mathbb{R}$. Then the dual of $b_{\log^\alpha}$, resp. $B_{\log^\alpha}$, is isomorphic to $B_{\log^\alpha}$, resp. $B_{\log^\alpha}$ under the pairing (1.2). Similarly, the dual of $b_{\log^1}$, resp. $B_{\log^1}$, is isomorphic to $B_{\log^1}$, resp. $B_{\log^1}$, under the same pairing.

Proof. See Section 6.

Remark 2.3. The phrase “the dual of $X$ is isomorphic to $Y$ under the pairing (1.2)” means that if $f \in X$ and $g \in Y$, then the limit in (1.2) exists and the functional $\Phi(f) = \langle f, g \rangle$ is bounded on $X$; and on the other hand, if $\Phi \in X^*$, then there exists $g \in Y$ such that $\Phi(f) = \langle f, g \rangle$, and moreover, there exists a constant $C$ independent of $g$ such that $\|g\|_Y/C \leq \|\Phi\| \leq C\|g\|_Y$.

As an application of Theorems 2.2, 2.3, and 2.4, one can prove the following fact.

Theorem 2.5. Let $\alpha > 0$. Then the adjoint (with respect to (1.2)) of the operator $L : B_{\log^\alpha} \mapsto B_{\log^{\alpha-1}}$ is equal to $C : B_{\log^{\alpha-1}} \mapsto B_{\log^\alpha}$. The adjoint of the operator $C : b_{\log^{\alpha-1}} \mapsto b_{\log^\alpha}$ is equal to $L : B_{\log^\alpha} \mapsto B_{\log^{\alpha-1}}$. The analogous assertions hold in the case when $\alpha = 0$. 
3. Decompositions

In [6], a sequence \( \{V_n\}_{n=0}^{\infty} \) was constructed in the following way. Let \( \omega \) be a \( C^\infty \)-function on \( \mathbb{R} \) such that

1. \( \omega(t) = 1 \) for \( t \leq 1 \),
2. \( \omega(t) = 0 \) for \( t \geq 2 \),
3. \( \omega \) is decreasing and positive on the interval \((1, 2)\).

Let \( \varphi(t) = \omega(t/2) - \omega(t) \), and let \( V_0(z) = 1 + z \), and, for \( n \geq 1 \),

\[
V_n(z) = \sum_{k=0}^{\infty} \varphi(k/2^n) z^k = \sum_{k=2^{n-1}}^{2^n-1} \varphi(k/2^n) z^k.
\]

The polynomials \( V_n \) have the following properties:

(3.1) \( g(z) = \sum_{n=0}^{\infty} V_n * g(z) \), for \( g \in H(D) \);
(3.2) \( \|V_n * g\|_p \leq C \|g\|_p \), for \( g \in H^p \), \( p > 0 \);
(3.3) \( \|V_n\|_p \asymp 2^n(1-1/p) \), for all \( p > 0 \),

where \(*\) denotes the Hadamard product. Here \( \|h\|_p \) denotes the norm in the \( p \)-Hardy space \( H^p \),

\[
\|h\|_p = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|d\theta \right)^{1/p} = \sup_{0 < r < 1} M_p(r, g).
\]

We need additional properties.

**Lemma 3.1.** Let \( P(z) = \sum_{k=m}^{j} a_k z^k \), \( m < j \). Then

\[
r_j \|P\|_p \leq M_p(r, P) \leq r^m \|P\|_p, \quad 0 < r < 1.
\]

When applied to the polynomial \( P = V_n * g' \), this gives:

(3.4) \( r^{2^n+1-1} \|V_n * g'\|_p \leq M_p(r, V_n * g') \leq r^{2^n-1-1} \|V_n * g'\|_p \) for \( n \geq 1 \).

Another inequality will be used (see [16, Exercise 7.3.5]):

(3.5) \( 2^n-1 \|V_n * g'\|_p / C \leq \|V_n * g'\|_p \leq C 2^{n+1} \|V_n * g\|_p \) for \( n \geq 1 \),

where \( C \) is a constant independent of \( n \) and \( g \).

**Theorem 3.1.** Let \( \alpha \in \mathbb{R} \), and \( f \in H(D) \). Then:

(i) \( f \in B_{\log}^\alpha \) if and only if \( \sup_{n \geq 0} (n+1)^{-\alpha} \|V_n * f\|_\infty < \infty \).
(ii) \( f \in B_{\log}^{-\alpha} \) if and only if \( \lim_{n \to \infty} (n+1)^{-\alpha} \|V_n * f\|_\infty = 0 \).
(iii) \( f \in B_{\log}^\alpha \) if and only if \( \sum_{n=0}^{\infty} (n+1)^{\alpha} \|V_n * f\|_1 < \infty \).

Moreover, the inequality

\[
C^{-1} \|f\|_{B_{\log}^{\alpha}} \leq \sup_{n \geq 0} (n+1)^{-\alpha} \|V_n * g\|_\infty \leq C \|f\|_{B_{\log}^{\alpha}}
\]

holds, where \( C \) is independent of \( f \). The analogous inequality holds in the case of (iii) as well.
For the proof we need the following reformulation of [9, Proposition 4.1].

**Lemma 3.2.** Let \( \varphi \) be a continuous function on the interval \((0, 1)\) such that \( \varphi(x)/x^\gamma \) \((0 < x < 1)\) is nonincreasing, and \( \varphi(x)/x^\beta \) \((0 < x < 1)\) is nondecreasing, where \( \beta \) and \( \gamma \) are positive constants independent of \( x \). \(^1\) Let

\[
F_1(r) = (1 - r)^{-1/q} \varphi(1 - r) \sup_{n \geq 1} \lambda_n r^{2^n + 1 - 1},
\]

\[
F_2(r) = (1 - r)^{1/q} \varphi(1 - r) \sum_{n=0}^{\infty} \lambda_n r^{2^n - 1 - 1},
\]

where \( \lambda_n \geq 0 \), \( 0 < q \leq \infty \). If \( F = F_1 \) or \( F = F_2 \), then

\[
C^{-1} \|F\|_{L^q(0, 1)} \leq \|\{\varphi(2^{-n})\lambda_n\}\|_{\ell^q} \leq C \|F\|_{L^q(0, 1)}.
\]

**Proof of Theorem 3.1.** Case (i). Let \( \varphi(x) = x \log \alpha \left(\frac{2}{x}\right) \), and \( q = \infty \). That \( \varphi \) is normal follows from the relation

\[
\lim_{x \downarrow 0} x \varphi'(x) \varphi(x) = 1.
\]

Let \( \lambda_n = 2^n \|V_n \ast f\|_{\infty} \). By (3.1), (3.2), (3.4), and (3.5), we have

\[
C^{-1} |\hat{f}(1)| + C^{-1} \sup_{n \geq 1} \lambda_n r^{2^n + 1 - 1} \leq M_{\infty} \|r, f'\| \leq C |\hat{f}(1)| + C \sum_{n=1}^{\infty} \lambda_n r^{2^n - 1 - 1}.
\]

Hence, by Lemma 3.2, we obtain the desired result.

Case (ii). In this case we can proceed in two ways:

1° Modify the proof of Lemma 3.2 to get the inequalities

\[
C^{-1} \|F\|_{C_0[0, 1]} \leq \|\{\varphi(2^{-n})\lambda_n\}\|_{\ell^0} \leq C \|F\|_{C_0[0, 1]},
\]

where \( C_0[0, 1] = \{u \in C[0, 1] : u(1) = 0\} \) and \( \ell^0 \) is the set of the sequences tending to zero.

2° Consider the spaces \( b_{\log^\alpha} \subset B_{\log^\alpha} \) and \( X = \{f : \|V_n \ast f\| = o((n + 1)\alpha)\} \), which is, by assertion (i) and its proof, a subspace of a space \( Y \) isomorphic to \( B_{\log^\alpha} \). It is not hard to show that the polynomials are dense in both \( b_{\log^\alpha} \) and \( X \). This proves (ii).

Case (iii). In this case we use the function \( \varphi(x) = x \log \alpha(2/x) \), let \( q = 1 \), and then proceed as in the proof of (i). The details are omitted. This concludes the proof of the theorem. \( \square \)

**Remark 3.1.** By choosing \( \phi(x) = x \log \log \left(\frac{2}{x}\right) \), then we can conclude that Theorem 3.1 remains true if \( \log^\alpha \), resp. \((n + 1)^\alpha \), are replaced with \( \log \log \), resp. \( \log(n + 2) \).

\(^1\)Following Shields and Williams [20], we call such a function normal.
4. Functions with decreasing coefficients

Proof of Theorem 1.1. Assuming that (1.3) holds, we want to prove that

$$
\| f \|_{B_1^{\log}} \leq C a_0 + C \sum_{n=1}^{\infty} a_{2^{n-1}} (n + 1)^{\alpha}.
$$

According to Theorem 3.1 and its proof, we have

$$
C^{-1} \| f \|_{B_1^{\log}} \leq a_0 + \sum_{n=1}^{\infty} (n + 1)^{\alpha} \| V_n * f \|_1 \leq C \| f \|_{B_1^{\log}}.
$$

Let $n \geq 1$, $m = 2^{n-1}$, and $Q_k = \sum_{j=\frac{k}{m}}^{\frac{k+1}{m}} \varphi(j/m) e_j$. Since $Q_{4m-1} = V_n$, we have

$$
V_n * f = \sum_{k=m}^{4m-1} \varphi(k/m) a_k e_k = \sum_{k=m}^{4m-1} (a_k - a_{k+1}) Q_k + a_{4m} Q_{4m-1}
$$

$$
= \sum_{k=m}^{4m-1} (a_k - a_{k+1}) Q_k + a_{4m} V_n.
$$

On the other hand, $Q_k = V_n * \Delta_{n,k}$, where

$$
\Delta_{n,k} = \sum_{j=2^{n-1}}^{k} z^k, \quad 2^{n-1} \leq k \leq 2^{n+1}.
$$

By (3.2), with $g = \Delta_{n,k}$, we have

$$
\| Q_k \|_1 \leq C \| \Delta_{n,k} \|_1 \leq C \log(k + 1 - 2^{n-1}) \leq C(n + 1).
$$

Combining these inequalities we get

$$
\| V_n * f \|_1 (n + 1)^{\alpha} \leq C \sum_{k=m}^{4m-1} (a_k - a_{k+1})(n + 1)^{\alpha+1} + C a_{4m} \| V_n \|_1 (n + 1)^{\alpha}
$$

$$
\leq C(n + 1)^{\alpha+1} (a_m - a_{4m}) + C a_{4m} (n + 1)^{\alpha}
$$

$$
= C(n + 1)^{\alpha+1} (a_{2^{n-1}} - a_{2^n}) + C(n + 1)^{\alpha} a_{2^{n+1}}.
$$

Here we have used the relation $\| V_n \|_1 \leq C$ (see (3.3))! Thus

$$
(n + 1)^{\alpha} \| V_n * f \|_1 \leq C(n + 1)^{\alpha+1} (a_{2^{n-1}} - a_{2^n})
$$

$$
+ C(n + 1)^{\alpha+1} (a_{2^n} - a_{2^{n+1}}) + C(n + 1)^{\alpha} a_{2^{n+1}},
$$

and therefore it remains to estimate the sums

$$
S_1 = \sum_{n=1}^{\infty} (n + 1)^{\alpha+1} (a_{2^{n-1}} - a_{2^n}) \text{ and } S_2 = \sum_{n=1}^{\infty} (n + 1)^{\alpha+1} (a_{2^n} - a_{2^{n+1}}).
$$

If $\alpha > -1$, then

$$
(n + 1)^{\alpha+1} \leq C \sum_{k=1}^{n} (k + 1)^{\alpha},
$$

and hence
\[ S_1 \leq C \sum_{n=1}^{\infty} (a_{2n-1} - a_{2n}) \sum_{k=1}^{n} (k+1)^{\alpha} = C \sum_{k=1}^{\infty} (k+1)^{\alpha} \sum_{n=k}^{\infty} (a_{2n-1} - a_{2n}) = C \sum_{k=1}^{\infty} (k+1)^{\alpha} 2_{2k-1}. \]

In the case of \( S_2 \) we get
\[ S_2 \leq C \sum_{k=1}^{\infty} (k+1)^{\alpha} a_{2k}, \]
which completes the proof of “if” part of the theorem in the case \( \alpha > -1 \). If \( \alpha = -1 \), then
\[ \|V_n * f\|_1 (n+1)^{-1} \leq C (a_{2n-1} - a_{2n+1}) + C(n+1)^{-1} a_{2n+1}, \]
from which we get the desired result in the case \( \alpha = -1 \).

To prove “only if” part we use Hardy’s inequality in the form
\[ \pi M_1(r, g) \geq \sum_{n=0}^{\infty} \frac{|\hat{g}(n)|}{n+1} r^n. \]

It follows that
\[ \int_{\mathbb{D}} |f'(z)| \log^{\alpha} \frac{2}{1-|z|} dA(z) = 2 \int_{0}^{1} M_1(r, f') \log^{\alpha} \frac{2}{1-r} dr \geq 2 \pi \sum_{n=1}^{\infty} a_n \frac{n}{n+1} \int_{0}^{1} \log^{\alpha} \frac{2}{1-r} r^n dr \]
Now the desired result follows from the inequality
\[ \int_{0}^{1} \varphi\left(\frac{1-r}{1-r}\right) r^n dr \geq c \varphi\left(\frac{1}{n+1}\right) \quad (c = \text{const.} > 0), \]
valid for any normal function \( \varphi \) (see [9, Lemma 4.1]).

Before proving Theorem 1.2, some remarks are in order. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), \( a_n \geq 0 \). In order that \( Lf \) be well defined by (1.8) it is necessary and sufficient that
\[ \sum_{n=0}^{\infty} \frac{a_n}{n+1} < \infty. \]
We already have mentioned in Introduction that this condition implies the existence of the integral in (1.8). In fact, this integral converges uniformly on compact subsets of \( \mathbb{D} \), which means that the limit
\[ \lim_{x \uparrow 1} \int_{0}^{x} f(t + (1-t)z) dt \]
exists and is uniform in \( |z| < \rho \), for every \( \rho < 1 \). This guarantees that \( Lf \) is analytic. On the other hand, if the integral in (1.8) exists, then we take \( z = 0 \) to conclude that (4.1) holds.
Proof of Theorem 1.2. The Taylor coefficients of $L f$ are

$$b_n = \sum_{k=n}^{\infty} \frac{a_n}{n+1}.$$ 

The sequence $\{b_n\}$ is nonincreasing so we can apply Theorem 1.1 to conclude that $L f \in \mathcal{B}_{\log^{-1}}^1$ if and only if

$$\sum_{n=0}^{\infty} \frac{\log^\alpha(n+2)}{n+1} \sum_{k=n}^{\infty} \frac{a_k}{k+1} < \infty.$$ 

Now the desired result follows from the estimate

$$C^{-1} \log^{\alpha+1}(k+2) \leq \sum_{n=0}^{k} \frac{\log^\alpha(n+2)}{n+1} \leq C \log^{\alpha+1}(k+2),$$

which holds because $\alpha > -1$. □

Remark 4.1. The above proof shows that $L f$ belongs to $\mathcal{B}_{\log^{-1}}^1$ if and only if

$$\sum_{n=0}^{\infty} \frac{a_n}{\log \log(n+4)} < \infty.$$ 

Now we pass to the proof of Theorem 2.1.

Proof of Theorem 1.1(c). Since $\mathcal{B}_{\log^{\alpha}} \subset \mathcal{B}_{\log^{\beta}}$ for $\beta < \alpha$, we may assume that $-1 < \alpha < 0$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \log^{-\varepsilon-\alpha}(n+2),$$

where $\varepsilon > 1$. Condition (4.1) holds because $\varepsilon > 1$. For every $r \in (0,1)$ the function $f_r(z) = f(rz)$ belongs to $H(D)$ and, by Theorem 1.1 and its proof, the set $\{f_r : 0 < r < 1\}$ is bounded in $\mathcal{B}_{\log^{\alpha}}^1$. On the other hand,

$$\overline{\mathcal{L}}(f_r)(0) = \sum_{k=0}^{\infty} \frac{r^k}{(k+1)\log(k+2)}.$$

Now choose $\varepsilon = 1 - \alpha > 1$ (because $\alpha < 0$) to get

$$\overline{\mathcal{L}}(f_r)(0) = \sum_{k=0}^{\infty} \frac{r^k}{(k+1)\log(k+2)} \rightarrow \infty \ (r \uparrow 1).$$

This contradicts the fact that if a set $X \subset \mathcal{B}_{\log^{\alpha}}^1$ is bounded and $\overline{\mathcal{L}}$ is bounded on $\mathcal{B}_{\log^{\alpha}}^1$, then the set $\{\overline{\mathcal{L}}f(0) : f \in X\}$ is bounded because the functional $h \mapsto h(0)$ is continuous on $H(D)$. This completes the proof. □

Proof of Theorem 2.1(a). Let $g \in \mathcal{B}_{\log^{\alpha}}$. Then

$$M_2(r, g') \leq C(1-r)^{-1} \log \frac{2}{1-r}.$$
It follows that

\[ 2^n \left( \sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)|^2 \right)^{1/2} r^{n+1} \leq C(1-r)^{-1} \log^\alpha \frac{2}{1-r}. \]

Taking \( r = 1 - 2^{-n} \), \( n \geq 1 \), we get

\[ \left( \sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)|^2 \right)^{1/2} \leq C \log^\alpha (n+1). \]

Hence,

\[ 2^{-n} \sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)| \leq \left( 2^{-n} \sum_{k=2^n}^{2^{n+1}-1} |\hat{g}(k)|^2 \right)^{1/2} \leq 2^{-n/2} \log^\alpha (n+1). \]

This gives the result. \( \square \)

**Proof of Theorem 2.1(b).** In this case we use Hardy’s inequality as in the proof of Theorem 1.1 to get

\[ \|g\|_{\mathcal{B}_{\log^\alpha}^1} \geq c \sum_{n=0}^{\infty} \frac{|\hat{g}(n)| \log^\alpha (n+2)}{n+1}. \]

This proves the result because \( \alpha \geq 0 \). \( \square \)

### 5. Proofs of Theorem 2.2 and 2.3

Define the operator \( \mathcal{R} : H(D) \mapsto H(D) \) by

\[ \mathcal{R}f(z) = \sum_{n=0}^{\infty} (n+1)\hat{f}(n)z^n = \frac{d}{dz}(zf(z)). \]

By using Theorem 3.1 and the relation

(5.1) \( C^{-1}2^n \|V_n * f\|_p \leq \|V_n * \mathcal{R}f\|_p \leq C2^n \|V_n * f\|_p \quad (n \geq 0), \)

one proves that the norm in \( \mathfrak{B}_{\log^\alpha} \) is equivalent to

\[ \sup_{z \in \mathbb{D}} (1-|z|) \log^{-\alpha} \frac{2}{1-|z|} |\mathcal{R}f(z)|. \]

**Proof of Theorem 2.2(a).** Let \( \alpha > -1 \) and \( f \in \mathfrak{B}_{\log^\alpha} \). Then, by integration,

\[ |f(z)| \leq \log^{\alpha+1} \frac{1}{1-|z|}. \]

Since

\[ \mathcal{R}f(z) = \frac{f(z)}{1-z}, \]

we see that

\[ |\mathcal{R}f(z)| \leq C(1-|z|)^{-1} \log^{\alpha+1} \frac{1}{1-|z|}. \]

The result follows. \( \square \)
Proof of Theorem 2.2(b). The function \( φ(x) = x \log \log(4/x) \) is normal because \( \lim_{x \to 0} xφ'(x)/φ(x) = 1 \). Hence, arguing as in the proof of Theorem 3.1 we conclude that \( f \in \mathcal{B}_{\log} \) if and only if \( \sup_{n \geq 0} \| V_n * f \|_\infty / \log(n + 2) < \infty \). Then using (5.1) we find that \( g \in \mathcal{B}_{\log} \) if and only if

\[
\left| R g(z) \right| \leq C(1 - |z|)^{-1} \log \log \frac{4}{1 - |z|},
\]

The rest of the proof is the same as in the case of (a).

□

Remark 5.1. In the case of the small spaces the proofs are similar and therefore omitted.

For the proof of Theorem 2.3 we need the following lemma [18]:

Lemma 5.1. If \( f \in \ell^1_\infty \), then \( \mathcal{L} f \) is well defined by (1.8) and the inequality

\[
\left. r M_1(r, (\mathcal{L} f)' ) \right| \leq 2(1 - r)^{-1} \int_r^1 M_1(s, f') \, ds, \quad 0 < r < 1,
\]

holds.

Before passing to the proof observe that \( \mathcal{B}_{\log}^n \subset \mathcal{B}^1 \) and \( \mathcal{B}_{\log}^1 \subset \mathcal{B}_1 \), and, since \( \mathcal{B}_1 \subset H^1 \), we see that in all cases of Theorem 2.3 the operator \( \mathcal{L} \) is well defined.

Proof of Theorem 2.3(a). We have, by (5.2),

\[
\int_D |(\mathcal{L} f)'(z)| \log^{a-1} \frac{2}{1 - |z|} \, dA(z) = 2 \int_0^1 M_1(r, (\mathcal{L} f)' ) \log^{a-1} \frac{2}{1 - r} \, dr \\
\leq 4 \int_0^1 (1 - r)^{-1} \log^{a-1} \frac{2}{1 - r} \, dr \int_r^1 M_1(s, f') \, ds \\
= 4 \int_0^1 M_1(s, f') \, ds \int_0^s (1 - r)^{-1} \log^{a-1} \frac{2}{1 - r} \, dr \\
\leq C \int_0^1 M_1(s, f') \log^a \frac{2}{1 - s} \, ds.
\]

A standard application of the maximum modulus principle shows that the inequality remains valid if we replace \( ds \) with \( s \, ds \). This gives the result. □

The proofs of Theorem 2.3, (b) and (c), are similar and we omit them.

6. Proof of Theorem 2.4

We consider a more general situation. Let \( X \subset H(D) \) (with continuous inclusion) be a Banach space such that the functions \( f_w(z) = f(wz) \), \( |w| \leq 1 \), belong to \( X \) whenever \( f \in X \), and \( \sup_{|w| \leq 1} \| f_w \|_X \leq \| f \|_X \). Such a space is said to be homogeneous (see [2]). A homogeneous space satisfies the condition

\[
\| V_n * f \|_X \leq C \| f \|_X, \quad f \in X,
\]

where \( C \) is independent of \( n \) and \( f \).
If in addition
\[
\lim_{r \downarrow 1} \|f - f_r\|_X = 0, \quad f \in X,
\]
then the dual of $X$ can be identified with the space, $X'$, of those $g \in H(\mathbb{D})$ for which limit (1.2) exists for all $f \in X$ (see [1, 2]). Also, the dual of a homogeneous space $X$ satisfying (6.2) can be realized as the space of coefficient multipliers, $(X, A(\mathbb{D}))$, from $X$ to $A(\mathbb{D})$; in this case we have $(X, A(\mathbb{D})) = (X, H^\infty) =: X^*$ (see [2]). The norm in $X^*$ is introduced as
\[
\|g\|_{X^*} = \sup\{\|f \ast g\|_\infty : f \in X, \|f\|_X \leq 1\},
\]
and, if $X$ is homogeneous and satisfies (6.2), it is equal to
\[
\|g\|_{X^*} = \sup\{\|f \ast g\| : f \in X, \|f\|_X \leq 1\}.
\]
There is another way to express $(f, r)$, when $f \in X$, $X$ satisfies (6.2), and $g \in X'$; namely, in this case, the function $f \ast g$ belongs to $A(\mathbb{D})$, and we have $(f, g) = (f \ast g)(1)$ (see [13, 2]).

We fix a sequence $\lambda = \{\lambda_n\}_{0}^\infty$ of positive real numbers such that
\[
0 < \inf_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_n}, \quad \sup_{n \geq 0} \frac{\lambda_{n+1}}{\lambda_n} < \infty.
\]

It is clear that the spaces $H^p$ ($0 < p \leq \infty$), $A(\mathbb{D})$, $B_{\log^*}$, $b_{\log^*}$, and $B_{\log^*}^1$ are homogeneous. Among them only $H^\infty$ and $B_{\log^*}$ do not satisfy condition (6.2).

Consider the following three spaces of sequences $\{f_n\}_{0}^\infty$, $f_n \in H(\mathbb{D})$:

(a) $\mathcal{E}_0(\lambda, X) = \{f_n : \lim_{n \to \infty} \|f_n\|_X / \lambda_n = 0\}$;

(b) $\ell^\infty(\lambda, X) = \{f_n : \sup_{n \geq 0} \lambda_n \|f_n \ast f\|_X < \infty\}$;

(c) $\ell^1(\lambda, X) = \{f_n : \sum_{n=0}^{\infty} \|f_n\|_X / \lambda_n < \infty\}$.

We also define the spaces $v_0(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$ (as subsets of $H(\mathbb{D})$) by replacing $f_n$ with $V_n \ast f$ in (a), (b), and (c), respectively. The proof of the following lemma is rather easy, and is therefore left to the reader.

**Lemma 6.1.** If $X$ is a homogeneous space, then so are $v(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$. The spaces $v_0(\lambda, X)$ and $V^1(\lambda, X)$ satisfy (6.2). The space $v_0(\lambda, X)$ is equal to the closure in $V^\infty(\lambda, X)$ of the sets of all polynomials.

Theorem 2.4 will be deduced from Theorem 3.1 and the following.

**Proposition 6.1.** If $X$ is a homogeneous space satisfying (6.2), then the dual of $v_0(\lambda, X)$, resp. $V^1(\lambda, X)$, is isomorphic to $V^1(\lambda, X')$, resp. $V^\infty(\lambda, X')$, with respect to (1.2).

In proving we use ideas from [13, 14, 7]. For the proof we need the following lemma.

**Lemma 6.2.** The operator $T(\{f_n\}) = \sum_{n=0}^{\infty} V_n \ast f_n$ acts as a bounded operator from $Y$ to $Z$, where $Y$ is one of the spaces $v_0(\lambda, X)$, $\ell^\infty(\lambda, X)$, and $\ell^1(\lambda, X)$, while $Z$ is $v_0(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$, respectively.
Proof. We have $V_n \ast V_j = 0$ for $|j - n| \geq 2$ and hence
\[ V_n \ast T(\{f_j\}) = \sum_{j=0}^{n+1} V_n \ast V_j \ast f_j, \quad n \geq 0, \]
where, by definition, $w_j = f_j = 0$ for $j < 0$. It follows that
\[ \|V_n \ast T(\{f_j\})\|_X \leq C \sum_{j=n-1}^{n+1} \|f_j\|_X, \]
where we have used (6.1). Now the proof is easily completed by using (6.3).

Lemma 6.3. Let $g \in (v_0(\lambda, X))^\prime$, resp. $g \in (V^1(\lambda, X))^\prime$, and define the operator $S$ on $c_0(\lambda, X)$, resp. $\ell^1(\lambda, X)$, by
\[ S(\{f_n\}) = T(\{f_n\}) \ast g = \sum_{k=0}^{\infty} f_k \ast V_k \ast g. \]
Then $S$ maps $c_0(\lambda, X)$, resp. $\ell^1(\lambda, X)$, into $H^\infty$ and $\|S\| \leq C\|g\|_{(v_0(\lambda, X))^\prime}$, resp. $\|S\| \leq C\|g\|_{(V^1(\lambda, X))^\prime}$.

Proof. By the preceding lemma, we have
\[ \|S(\{f_n\})\|_{\infty} \leq \|T(\{f_n\})\|_{(v_0(\lambda, X))} \|g\|_{(v_0(\lambda, X))^\prime} \leq C\|\{f_n\}\|_{c_0(\lambda, X)} \|g\|_{(v_0(\lambda, X))^\prime}. \]
This proves the result in one case. In the other case the proof is the same.

Proof of Proposition 6.1. Define the polynomials $P_n$ $(n \geq 0)$ by
\[ P_n = V_{n-1} + V_n + V_{n+1}. \]
Hence
\[ V_n = \sum_{j=0}^{\infty} V_j \ast V_n = (V_{n-1} + V_n + V_{n+1}) \ast V_n = P_n \ast V_n. \]
Let $f \in v_0(\lambda, X)$ and $g \in V^1(\lambda, X')$. It is easily verified that, when $0 < r < 1$,
\[ (f \ast g)(z) = \sum_{n=0}^{\infty} (f \ast V_n \ast g)(z) = \sum_{n=0}^{\infty} (P_n \ast f \ast V_n \ast g)(z), \quad z \in \mathbb{D}. \]
the series being absolutely convergent. Since
\[ \|P_n \ast f \ast V_n \ast g\|_{\infty} \leq \|P_n \ast f\|_X \|V_n \ast g\|_{X^\prime}, \]
we have
\[ \|f \ast g\|_{\infty} \leq \sum_{n=0}^{\infty} \|P_n \ast f\|_X \|V_n \ast g\|_{X^\prime} \leq C \sum_{n=0}^{\infty} \|P_n \ast f\|_X \|V_n \ast g\|_{X^\prime}
\]
\[ = C \sum_{n=0}^{\infty} (\|P_n \ast f\|_X / \lambda_n) (\lambda_n \|V_n \ast g\|_{X^\prime}) \leq C \|f\|_{v_0(\lambda, X)} \|g\|_{V^1(\lambda, X')}. \]
This proves the inclusion $V^1(\lambda, X') \subset (v_0(\lambda, X))^\prime$. 

To prove the converse, let \( g \in (v_0(\lambda, X))' \). Let \( S \) denote the operator defined in Lemma 6.3. By Lemma 6.3, \( S \) acts as a bounded operator from \( c_0(\lambda, X) \) into \( H^\infty \) and we have \( \|S\| \leq C\|g\|_{v_0(\lambda, X)} \). Now it suffices to prove that
\[
\|S\| \geq (1/2)\text{sup}\{\|g_n\|_{C^1}\} = (1/2)\|g\|_{v^1(\lambda, X)}.
\]
For each \( n \geq 0 \) choose \( f_n \in X \) so that \( \|f_n\|_X = 1 \) and \( (f_n, g_n) \) is a real number such that \( \langle f_n, g_n \rangle \geq (1/2)\|g_n\|_{X'} \). If \( \{a_n\} \) is a finite sequence of nonnegative real numbers, then
\[
S(\{a_n f_n\}) = \sum_{n=0}^{\infty} a_n (f_n, g_n) \geq (1/2) \sum_{n=0}^{\infty} a_n \|g_n\|_{X'} = (1/2) \sum_{n=0}^{\infty} (a_n/\lambda_n) \lambda_n \|g_n\|_{X'}.
\]
Hence, by taking the supremum over all \( \{a_n\} \) such that \( 0(a_n) \lambda_n \), we get \( S(\lambda_n f_n) \geq (1/2) \sum_{n=0}^{\infty} \lambda_n \|g_n\|_{X'} \). Since \( \|\{a_n f_n\}\|_{\ell^0(\lambda, X)} \leq 1 \), where \( a_n = \lambda_n \) for \( 0 \leq n \leq N \) \((N \in \mathbb{N})\) and \( a_n = 0 \) for \( n > N \) we see that \( \|S\| \geq (1/2)\|g\|_{v^1(\lambda, X')} \), as desired. This completes the proof that \( v_0(\lambda, X)' = V^1(\lambda, X') \). In a similar way one proves that \( V^1(\lambda, X)' = V^\infty(\lambda, X') \), which is all what has to be proved.

**Proof of Theorem 2.4.** First we prove that \((b_{log^n})' = B_{log^n}^1\). By Theorem 3.1, we have \( b_{log^n} = v_0(\lambda, A(\mathbb{D})) \), where \( \lambda_n = (n+1)^n \). Hence, by Proposition 6.1, the dual of \( b_{log^n} \) is isomorphic to \( V^1(\lambda, A(\mathbb{D})') \). In order to estimate \( \|V_n * g\|_{A(\mathbb{D})'} \), first observe that \( H^1 \subset A(\mathbb{D})' \) and moreover \( \|V_n * g\|_{A(\mathbb{D})'} \leq \|V_n * g\|_1 \). On the other hand, let \( \Phi \) be a bounded linear functional on \( A(\mathbb{D}) \), let \( \Phi_0 \) be the Hahn/Banach extension of \( \Phi \) to \( hC(\mathbb{D}) \), and choose \( g \in A(\mathbb{D})^a \) so that \( \Phi(f) = \langle f, g \rangle \) for all \( f \in A(\mathbb{D}) \). By the Riesz representation theorem, we have
\[
\Phi_0(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{-i\theta}) d\mu(e^{i\theta}) = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{-i\theta})g(re^{i\theta}) d\theta = \lim_{r \to 1^-} \sum_{n=0}^{\infty} f(n)g(n)r^{2n},
\]
and \( \|\mu\| = \|\Phi\| = \|\Phi_0\| \). In particular, taking \( f(w) = (1 - zw)^{-1} \), where \( z \in \mathbb{D} \) is fixed, we get \( \frac{1}{2\pi} \int_0^{2\pi} (1 - e^{-i\theta}z)^{-1} d\mu(e^{i\theta}) = g(z) \). Hence
\[
\mathcal{R}^1 g(z) = \frac{1}{2\pi} \int_0^{2\pi} (1 - e^{-i\theta}z)^{-2} d\mu(e^{i\theta}),
\]
and hence, by integration, \( M_1(r, \mathcal{R}^1 f) \leq \|\mu\| (1 - r^2)^{-1} = \|g\|_{A(\mathbb{D})'} (1 - r^2)^{-1} \). Now we proceed as in the proof of Theorem 3.1 to conclude that \( \|V_n * g\|_1 \leq C\|V_n * g\|_{A(\mathbb{D})'} \). It follows that \( g \in (b_{log^n}') \) if and only if \( g \in V^1(\lambda, H^1) \), i.e., by Theorem 3.1, \( g \in B_{log^n}^1 \).

In proving that \((B_{log^n})' \) is isomorphic to \( B_{log^n}^1 \), we use the inclusions \( H^\infty \subset (H^1)' \subset B \), and then proceed as above. \( \square \)

**Remark 6.1.** The above proof of Theorem 2.4 certainly is not the simplest one. However, it can be applied to prove some general duality and multipliers theorems (see [13, 14, 7]). For instance, the dual of \( b_{log^n} \) is isomorphic to \( B_{log^n}^1 \), and the dual of \( B_{log^n}^1 \) is isomorphic to \( B_{log}^1 \).
References


Faculty of Mathematics, University of Belgrade (Received 24 12 2015)
Belgrade, Serbia
pavlovic@matf.bg.ac.rs