THE NORMALIZATION THEOREM FOR EXTENDED NATURAL DEDUCTION

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Abstract. The normalization theorem for the system of extended natural deduction will be proved as a consequence of the cut-elimination theorem, by using the connections between the system of extended natural deduction and a standard system of sequents.

1. Introduction

In [6] Gentzen introduced the natural deduction system, the system $NJ$, and the system of sequents, the system $LJ$, for intuitionistic predicate logic. There are numerous papers (see, for example, [3, 4, 10, 12, 14, 18]) in which natural deduction derivations and sequent derivations were compared, and connections between the normalization procedure from natural deduction systems and the cut-elimination procedure from systems of sequents were studied. Because of the well-known difficulties of the correspondence between reductions which constitute these two procedures (see, for example, [18], part 1.3) in the papers mentioned above modifications of Gentzen’s systems $NJ$ and $LJ$ were considered. In [3] a standard system of sequents, the system $\delta E$, and a new natural deduction system, the system $NE$, were studied, where the most important characteristic of the system $NE$ is that the elimination rules for all connectives and quantifiers are of the same form as the elimination rules of $\lor$ and $\exists$ in Gentzen’s natural deduction system $NJ$. (With regard to introduction rules of the system $NE$, they are the introduction rules from $NJ$.) That system was called the system of extended natural deduction. Namely, natural deduction elimination rules of that kind were introduced in [16], and the natural deduction with these rules was called a natural extension of natural deduction. Moreover, the natural deduction system from [10] (which was also considered in [9]) has the elimination rules of that kind, which were called general elimination rules, and that system was called natural derivation with general elimination rules. In [3] it was showed that these elimination rules make new maximum segments in derivations of the system $NE$, which do not exist in $NJ$, and new conversions (i.e.

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reductions) of normalization procedure in that system. However, normal derivations in $\mathcal{N}\mathcal{E}$ were defined in Gentzen’s and Prawitz’s way: derivations without maximum segments. The map $\psi$ from the set of derivations of $\mathcal{E}$ with upper indices (i.e. the system $\mathcal{SE}$ from [5]), $\text{Der}(\mathcal{SE})$, into the set of derivations of $\mathcal{N}\mathcal{E}$, $\text{Der}(\mathcal{N}\mathcal{E})$, was defined and it was proved the following property: each conversion from the set of conversions of a standard cut-elimination procedure in the system $\mathcal{SE}$ has the corresponding conversion in the set of conversions of a normalization procedure in the system $\mathcal{N}\mathcal{E}$. (That result in the different denotation and with more details was already presented in [1].)

The main goal of this paper is to prove the normalization theorem for a system of extended natural deduction by using the cut-elimination procedure for a standard system of sequents. Namely, the normalization theorem for $\mathcal{N}\mathcal{E}$ can be proved by an induction on the length of one derivation $\pi$ with several cases which depend on the last rule of the derivation $\pi$ (see Prawitz’s proof of Theorem 1 from Section IV in [14]). We note that for natural deduction with general elimination rules from [10] (which is similar to our system $\mathcal{N}\mathcal{E}$) and its normal derivations (see [10] Definition 3.3. in Section 3, p. 549) two proofs of the normalization are presented in [10] and [9]. In [10] the existence of a normal derivation, which corresponds to a non-normal derivation, was shown in the following way (see [10] Section 5): "Given a non-normal derivation, translation to sequent calculus, followed by cut elimination and translation back to natural deduction, will produce a normal derivation"; and in [9] the direct proof was given. Moreover, in [5], normal derivations of the system were defined as in [3] and by using the map $\psi$ and the connections between the conversions of the cut-elimination procedure in $\mathcal{SE}$ and the conversions of the normalization procedure in $\mathcal{N}\mathcal{E}$ from [3] mentioned above, the normalization theorem for $\mathcal{N}\mathcal{E}$ was presented as one consequence of the cut-elimination procedure for $\mathcal{SE}$. Namely, for each derivation $\pi$ from $\mathcal{N}\mathcal{E}$ it was proved that (1) there is one derivation $D$ from $\mathcal{SE}$ such that $\psi D$ is $\pi$; and if $\pi$ is not normal, then (2) there is the sequence of the derivations $D, D_1, \ldots, D_n, n \geq 1$, which are connected by conversions of a cut-elimination procedure, where $D_n$ is cut-free and (3) that sequence makes the sequence of the derivations $\pi, \psi D_1, \psi D_2, \ldots, \psi D_n, 1 \leq i_1 < i_2 < \ldots < n$, which are connected by conversions of normalization procedure, where $\psi D_n$ is one normal derivation. In this paper we want to define the map from derivations of extended natural deduction to derivations of a standard sequent system, which will be used to make the derivation $D$, whose existence was showed in the proof from [5] mentioned above. To define that map we will consider the system $\mathcal{N}\mathcal{E}^o$, which is $\mathcal{N}\mathcal{E}$ with indices i.e. its formulae have indices, and some inference rules and formulae have numbers, and the system $\mathcal{SE}^o$, whose some left rules are different than the left rules of $\mathcal{SE}$.

In Subsections 2.1 and 2.2 the systems $\mathcal{SE}^o$ and $\mathcal{N}\mathcal{E}^o$ will be defined. In Subsection 2.3 the map $\psi$ from the set of derivations of $\mathcal{SE}^o$, $\text{Der}(\mathcal{SE}^o)$, to the set of derivations of $\mathcal{N}\mathcal{E}^o$, $\text{Der}(\mathcal{N}\mathcal{E}^o)$, will be presented (ψ from [3]), and the new map $\phi$ from $\text{Der}(\mathcal{N}\mathcal{E}^o)$ to $\text{Der}(\mathcal{SE}^o)$ will be defined. In Section 3 conversions in the systems $\mathcal{SE}^o$ and $\mathcal{N}\mathcal{E}^o$, which are in fact the conversions of $\mathcal{SE}$ and $\mathcal{N}\mathcal{E}$ from [3], will be presented. Finally, in Section 4 the proof of the cut-elimination theorem
for the system $\mathcal{SE}^\circ$ will be given, and the normalization theorem for the system $\mathcal{NE}^\circ$ will be proved by using that theorem. Namely, it will be shown that for each derivation $\pi$ from $\mathcal{NE}^\circ$ (1) there is one derivation $\phi\pi$ in $\mathcal{SE}^\circ$, and if $\pi$ is not normal, then (2) there is a sequence of the sequent derivations $\phi\pi, D_1, \ldots, D_n, n \geq 1$, which are connected by conversions of the cut-elimination procedure, where $D_n$ is cut-free and (3) that sequence makes the sequence of the derivations $\pi, \pi_1, \ldots, \pi_m, m \leq n$, which are connected by conversions of the normalization procedure in $\mathcal{NE}^\circ$, where $\pi_m$ is a normal derivation.

2. The systems $\mathcal{SE}^\circ$ and $\mathcal{NE}^\circ$

Our language will be the language of the first order predicate logic, i.e. it will have the logical connectives $\land, \lor$ and $\forall$ (i.e. $\Rightarrow$), quantifiers $\forall$ and $\exists$, and the propositional constant $\bot$ (for absurdity). Bound variables will be denoted by $x, y, z, \ldots$, free variables by $a, b, c, \ldots$ and individual terms by $r, s, t, \ldots$. Letters $P, Q, R, \ldots$ will denote atomic formulae and $A, B, C, \ldots$ will denote formulae.

The definition of symbols: (1) each natural number $i, i > 0$, will be a symbol; and (2) if $s_1$ and $s_2$ are symbols, then $(s_1)(s_2)$ and $(\ ) (s_2)$ will be symbols. The symbols will be denoted by $s, t, \ldots$ and length of a symbol $s, \bar{s}$, will be the number of natural numbers in $s$. The symbols of the length 1 will be denoted by $i, j, k, \ldots$. For one symbol $s$ we have the symbol $s^-$: if $s$ is (1) $i$, then $s^-$ does not exist; (2) $(i)(j)$, then $s^-$ is $i$; (3) $(s_1)(s_2)$, where $\pi_1, \pi_2 > 1$, then $s^-$ is $(s_1)(s_2)$; (4) $(\ )(s_2)$, then $s^-$ is $s^\circ_2$. A finite non-empty set of symbols will be called the index and it will be denoted by $a, b, \ldots$. The index of the form $\{s\}$, for a symbol $s$, will be denoted by $s$. The index $\{i\}$ will be called the initial index, and it will be denoted by $i$. The number of the members of an index $a$ will be denoted by $\bar{a}$. There are two operations on indices: (i) the union $a \cup b$ of two indices $a$ and $b$, which is the set-theoretical union; (ii) the product of two indices $a$ and $b$ is $a \times b = \{s, t : s \in a, t \in b\}$, where $\ast$ denotes the concatenation of sequences $(s)$ and $(t)$.

A set of indexed formulae will be denoted by $\Gamma^\circ$, but the index $a$ usually will be omitted. For a set of indexed formulae $\Gamma$ we will make the set $\Gamma^{\times a}$ in the following way $\Gamma^{\times a} = \{C^{\times a} : C \in \Gamma\}$. For a sequent $A^a, A^b, \Gamma \Rightarrow C$ representation such as $A^a, A^b, \Gamma$ implies that $a \neq b$, and $A^c \not\in \Gamma$ and $A^b \not\in \Gamma$, but possibly $A^c \in \Gamma$ for some $c \neq a$ and $c \neq b$.

2.1. The system $\mathcal{SE}^\circ$. A sequent of the system $\mathcal{SE}^\circ$ has the form $\Gamma \Rightarrow A$, where $\Gamma$ is a finite set of formulae with indices and $A$ is one unindexed formula.

Postulates for the system $\mathcal{SE}^\circ$.

Initial sequents

*i-initial sequents: $A^j \Rightarrow A$.*

$\bot$-initial sequents: $\bot \Rightarrow P$, where $P$ is an atomic formula different from $\bot$.

Inference rules

**structural rules**

\[
\frac{A^a, A^b, \Gamma \Rightarrow C}{A^{\ast a, b}, \Gamma \Rightarrow C} \quad \text{(contraction)}
\]

\[
\frac{\Gamma \Rightarrow A, A^a, \Delta \Rightarrow C}{\Gamma^{\times a}, \Delta \Rightarrow C} \quad \text{(cut)}
\]
rules for connectives

left rules

(\&L) \[ \Gamma \to A \quad /B^b_1, \Delta \to C \]
\[ \Gamma, A \to B^b_1, \Delta \to C \]

(\&L1) \[ /A^b_2, \Gamma \to C \]
\[ A \to B^b_1, \Gamma \to C \]

(\&L2) \[ /B^b_1, \Gamma \to C \]
\[ A \to B^b_1, \Gamma \to C \]

(\&R) \[ /A^a, \Gamma \to B \]
\[ \Gamma \to A \to B \]

\[ \Gamma \to A \\Delta \to B \]
\[ \Gamma, \Delta \to A \wedge B \]

\[ \Gamma \to A \to B \]
\[ \Gamma \to A \to B \]

right rules

(\lor R) \[ /A^a, \Gamma \to C \]
\[ \forall x A x \quad \Gamma \to C \]

(\lor R) \[ /A^a, \Gamma \to C \]
\[ \exists x A x \quad \Gamma \to C \]

(\lor L) \[ /A^a, \Gamma \to C \]
\[ \forall x A x \quad \Gamma \to C \]

(\lor L) \[ /A^a, \Gamma \to C \]
\[ \exists x A x \quad \Gamma \to C \]

The indices \( j \) from the initial sequents and the left rules are called initial indices, and they have to satisfy the restrictions on indices: in any derivation, all initial indices have to be distinct. In the rules (\lor R) and (\exists L) the variable \( a \) is called the proper variable of these rules, and, as usual, has to satisfy the restrictions on variables: if \( \forall R \): a does not appear in \( \Gamma \cup \{ \forall x A x \} \); if \( \exists L \): a does not appear in \( \Gamma \cup \{ \exists x A x, C \} \). The notation \( /C^c, \Theta \to D \) for a sequent is used to indicate the possibility that \( c \) is empty (and hence not strictly an index by the definition above, but we may still call an index for convenience). So, \( /C^c, \Theta \to D \) is interpreted as \( C^c, \Theta \to D \), when \( c \neq \emptyset \); and as \( \Theta \to D \), when \( c = \emptyset \).

\( D, E, F, D', D_1 \ldots \) will denote derivations in the system \( SE^\circ \). A derivation with the end sequent \( \Gamma \to A \) will be denoted by

\[ \frac{D}{\Gamma \to A} \quad \text{while} \quad \frac{\Delta}{\Gamma \to A} \]

will denote a derivation \( F \) with the last rule \( R \) and the end sequent \( \Gamma \to A \). All formulae making up sequents in a derivation \( D \) of \( SE^\circ \) will be called d-formulae of the derivation \( D \). A derivation \( D \) of \( SE^\circ \) has the proper variable property (PVP) if every occurrence in \( D \) of a proper variable of an inference of (\lor R) or (\exists L) is above the lower sequent of that inference.

2.2. The system \( NSE^\circ \). The system \( NSE^\circ \) is an extended natural deduction system. \( \pi, \bar{\pi}, \pi_1, \pi_2, \ldots \) will denote derivations of the system \( NSE^\circ \). \( \Gamma, \Delta, \ldots \) will denote finite sets of a-classes (see the definition below) in derivations of the system \( NSE^\circ \). One derivation of the formula \( C \) from the set of a-classes \( \Gamma \) will be denoted by

\[ \Gamma \to C \]

\[ \pi, \bar{\pi}, \pi_1, \pi_2, \pi_3 \quad \text{or} \quad \pi \wedge \Delta \]

will denote one derivation of \( C \) whose last inference is \( R \). Some inference rules in a derivation \( \pi \) will have numbers and the largest of them will be the number of the derivation \( \pi \) in \( NSE^\circ \). All formulae making up a derivation \( \pi \) of the system \( NSE^\circ \)
will be called the d-formulae of the derivation $\pi$. In $\Gamma \vdash \pi$ the d-formulae from the a-classes of $\Gamma$ are the top-formulae of that derivation and their full-forms, can be:

\[
\begin{align*}
&\text{(T1) } A^t, \\
&\text{(T2) } A^t, \quad t \geq 1, \quad \text{(T3) } ([\ldots ((A^t)^{n_1} \ldots)^{n_s})^t]^{[t]}, \quad t \geq 1, \quad n \geq 1, \\
&\text{(T4) } A^t \left[\ldots [\left(\ldots ((A^t)^{n_1} \ldots)^{n_s})^t\right)^{[t]}\right]^{[t]}, \quad t \geq 1, \quad n \geq 0, \quad l \geq 0,
\end{align*}
\]

where $i$ and $t$ are indices of these formulae and $N_1, \ldots, N_5, L_1, \ldots, L_4$ and $l$ are numbers of these formulae and some inference rules.

**Example 2.1.** Trivial derivations (see the definition below), for example, $A^t$, $A \land B^t$ have the top-formulae of the form (T1). The top-formulae of $\pi_1$: $A^t \vdash B^t \land \Gamma \vdash$ is one inference rule (see below). The operations on derivations, substitutions and contractions (see below) make top-formulae of the forms (T2)–(T4). The result of the substitution of the derivation $\pi_1$ for the d-formula $A \land B^t$ (which is one a-class, see below) of the trivial derivation $A \land B^t$ is $\pi_2$: $A^t \vdash B^t \land \Gamma \vdash$, whose top-formulae are of the form (T2). $\pi_3$: $A \land B^t \vdash \Gamma \vdash$, $\Gamma \vdash$. $\pi_4$ is the result of the substitution of the derivation $A^t$ for the a-class $A^t$ of $\pi_1$ and its top-formula $(A^t)^{[1]}$ is of the form (T3). In $\pi_4$:

\[
\frac{[A^t]^4}{A \land B} \vdash \frac{B^t}{A \land B \vdash A} \land \Gamma \vdash, \quad (A \land B) \land A \vdash (A \land B)^4
\]

we have the contraction of the formulae (the a-classes) $A^t$ and $A^t$ after the second $\land \Gamma \vdash$ and the top-formulae $[A^t]^4$ and $[A^t]^4$ are of the form (T4), where $n = l = 0$ and $L$ is 4. The result of the substitution of the derivation $A^t$ for the a-class $[A^t, A^t]^4$ of $\pi_5$ (the d-formulae $[A^t]^4$ and $[A^t]^4$ is $\pi_5$:

\[
\frac{[A^t(5(1)]^4}{A \land B} \vdash \frac{B^t}{A \land B \vdash A} \land \Gamma \vdash, \quad (A \land B) \land A \vdash (A \land B)^4
\]

where $n = 1, l = 0, N_1$ is 6 and $L$ is 4.

The part $A^t$ of a top-formula will be called the core of that top-formula. If from a top-formula its core and the first [and the last] with its number (when they exist) are deleted, then the bark of that top-formula will be obtained, and it will be denoted by $\langle \rangle$. For a top-formula of the form (T4), its core is $A^t$ and its bark is $\ldots [\ldots ((\ldots)^{n_1} \ldots)^{n_s})^t\ldots]^{[t]}$ (it is empty, when $n = l = 0$), and that top-formula will be written $[(A^t)^{[t]}$, but $\langle \rangle$ and $[\ldots]^{[t]}$ will usually be omitted. The top-formulae of the forms (T1) and (T2) are equal to their cores and their barks are empty.

The d-formula $C$ is the end-formula of $\Gamma \vdash \pi$ and its full-form can be:

\[
\begin{align*}
&\text{(E1) } C^i, \\
&\text{(E2) } C, \\
&\text{(E3) } (C^s)^{N_1} \ldots, \quad n \geq 1,
\end{align*}
\]

where $i$ and $s$ are indices of these formulae and $N_1, \ldots, N_s$ are numbers of these formulae and some inference rules. The part $C^s$ of each end-formula (where the index $s$ does not exist, when its form is (E2)) will be called the core of that end-formula. If from one end-formula its core is deleted, then the bark of that
end-formula will be obtained and it will be denoted by || (but, || will usually be omitted). For an end-formula of the form (E3), its core is $C^n$ and $(\ldots (\; s_1 \ldots )^m$ is its bark. The end-formulae of the forms (E1) and (E2) are equal to their cores and their barks are empty.

**Example 2.2.** Trivial derivations have the end-formulae of the form (E1). In Example 2.1 the end-formulae of $\pi_1, \pi_3, \pi_4$, and $\pi_5$ are of the form (E2) and the end-formula of $\pi_2$ is of the form (E3), where $s$ is $(4), n = 1$ and $N_1$ is 5.

If a derivation $\pi$ has one top-formula of the full-form (T4), then it has $k$ top-formulae $(k > 1)$ of the full-form (T4) with the same form (i.e. one predicate formula $A$) and the same number 1. The union of the indices (they have one symbol) of all these top-formulae is the index $a$ and all top-formulae whose indices belong to $a$ will make the a-class (i.e. the assumption class) in that derivation $\pi$ and it will be denoted by $[A^a]^k$ or $A^a$. Specially, each top-formula of the form (T1)–(T3) is one a-class $A^t$ (in (T1) $t$ is $i$) in that derivation $\pi$. If we write $[A^a]^1 \ldots [A^a]^m: \pi^t$, where $a = \{s_1, \ldots, s_m\}$, then it means that the d-formulae $[A^a]^{s_1} \ldots [A^a]^{s_m}$ make the a-class $[A^a]^k$ and in $\pi$ there is not any d-formula from that a-class.

Postulates in the system $\mathcal{N}\mathcal{E}^0$.

A derivation $\pi$ in the system $\mathcal{N}\mathcal{E}^0$ is either one trivial derivation $A^t$ or one derivation which is made by the following operations on derivations and the inference rules.

1. Operations on derivations

   If the derivation $\pi_1$ is $[A^a]^k \ldots [A^a]^m: \pi^t$, $B$, where $a = \{s_1, \ldots, s_m\}$, $b = \{t_1, \ldots, t_n\}, m, n \geq 1$ ($m$ (t) does not exist when $m$ (n) is 1), then the derivation $\pi$ $[\pi_{1}^t]^k \ldots \pi_{1}^m: \pi^t$, $B$ $[\pi_{2}^t]^k \ldots \pi_{2}^m: \pi^t$, $B$ is the result of the contraction of the a-classes $[A^a]^k$ and $[A^b]^m$ and it has the a-class $[A^{a\cup b}]^N$, where $N$ is one number larger than $N\pi_1$ when $N\pi_1$ exists and an arbitrary number, otherwise; and $N$ is the number of all formulae of that a-class. The number $N$ is also one number of the last inference rule (see below) of $\pi$ (that rule can have several numbers) and $N\pi = N$.

**Example 2.3.** In $\pi_4$ from Example 2.1 the contraction of the a-classes $[A^1]$ and $[A^3]$ make the a-class $[A^{1\cup 3}]^4$, 4 is the number of the last inference rule of $\pi_4$ and $N\pi_4 = 4$. The result of the contraction of the a-classes $[A^{1,3}]^4$ and $[A^{5,6}]^7$ in $\pi_4$ $[A^7]^7 \text{ IIE } [A^7]^7 \text{ IIE } [A^7]^7 \text{ IIE } [A^7]^7 \text{ IIE } [A^7]^7 \text{ IIE } [A^7]^7 \text{ IIE } [A^7]^7 \text{ IIE }$ is $[\pi_{1}^t]^k \ldots \pi_{1}^m: \pi^t$, $B$ $[\pi_{2}^t]^k \ldots \pi_{2}^m: \pi^t$, $B$ $[\pi_{3}^t]^k \ldots \pi_{3}^m: \pi^t$, $B$ $[\pi_{4}^t]^k \ldots \pi_{4}^m: \pi^t$, $B$ $[\pi_{5}^t]^k \ldots \pi_{5}^m: \pi^t$, $B$ $[\pi_{6}^t]^k \ldots \pi_{6}^m: \pi^t$, $B$ $[\pi_{7}^t]^k \ldots \pi_{7}^m: \pi^t$, $B$ $[\pi_{8}^t]^k \ldots \pi_{8}^m: \pi^t$, $B$.
If the derivation $\pi_1$ is $\Gamma \vdash \Delta$ and the derivation $\pi_2$ is
$$\frac{[(A^1)]^\delta \ldots [(A^n)]^\delta \Delta}{\pi''} B,$$
where d-formulae (inference rules) of $\pi_1$ and $\pi_2$ have different indices (numbers), $b$ is $\{t_1, \ldots, t_n\}, n \geq 1$ (if does not exist when $n$ is 1), then the result of the substitution of the derivation $\pi_1$ for the a-class $[A^b]^c$ in the derivation $\pi_2$ is the derivation $\pi$
$$\frac{[\Gamma \times 1]^\delta \ldots [\Gamma \times n]^\delta \Delta}{\pi''} (A^c(b)) \Delta \quad \text{(denoted by } \pi' \text{)}$$

where $N$ is one number larger than $N\pi_1$ and $N\pi_2$, when $N\pi_1$ or $N\pi_2$ exists and an arbitrary number, otherwise; in each d-formula $[\{A^c(t_i)\}]^b$, $1 \leq i \leq n$, which is one sb-formula of $\pi$, $\{\}$ is the bark of the top-formula $[\{A^c\}]^b$ of $\pi_2$ and $\{\}$ is the bark of the end-formula of $\pi_1$, and $N$ is its number and $k$ is its c-number.

If $\pi_1$ is one d-formula and $L$ exists, then the sb-formulae are $[\{\{A^c(t_i)\}\}]^b$, $1 \leq i \leq n$. Each a-class $[C^c]^b$ from the set of a-classes $\Gamma$ of $\pi_1$, $c = \{r_1, \ldots, r_m\}$, $m \geq 1$, has its corresponding a-class $[C^c]^b$ in $\pi$, i.e. each d-formula $[\{C^c\}]^b$ from $[C^c]^b$ has $n$ corresponding d-formulae $[\{\{C^c(t_i)\}\}]^b$, $1 \leq i \leq n$. If one of $\pi_1$ and $\pi_2$ is not one d-formula, then the number $N$ is the number of the last inference rule of $\pi$ which can have several numbers. If $\pi_1$ and $\pi_2$ are d-formulae, $\ldots (A^1)^n \ldots$ and $\ldots (A^l)^i \ldots$, respectively, where $n, l \geq 0$, then $\pi$ is the d-formula $\ldots (\ldots (A^c(t_i) \ldots)^n \ldots)^i \ldots$. In all cases $N\pi = N$.

**Example 2.4.** In $\pi_2$ from Example 2.1 $(A \land B)((4))$ is the sb-formula of one substitution where $5$ is its number and the number of the last inference rule of $\pi_2$, $\land I E$. The substitution of the derivation $A \land B^9$ for the a-class $[A \land B^9]^7$ in
$$\frac{|A \land B^6|^8 |A \land B|^8}{(A \land B) \land (A \land B)} \land I E; 8$$
is the derivation $\pi_6$: 
$$\frac{|(A \land B)^{(9)(6)}|^8 |(A \land B)^{(9)(7)}|^8}{(A \land B) \land (A \land B)} \land I E; 8, 10$$

and the substitution of $\pi_2$ for the a-class $[A \land B^9(6),(9)(7)]^8$ in $\pi_6$ is
$$\frac{|A|^8 |B|^8}{((A \land B)((4)(9)(6)))^5} \land I E \quad \frac{|A|^8 |B|^8}{((A \land B)((4)(9)(7)))^5} \land I E \quad \land I E; 8, 10, 11,$$
where $s_1 = \{(1)(4)((9)(6))\}$, $t_1 = \{(2)(4)((9)(6))\}$, $s_2 = \{(1)(4)((9)(7))\}$,
$t_2 = \{(2)(4)((9)(7))\}$.
(2) Logical inference rules

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<th>Introduction rules</th>
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<tr>
<td>$A \supset B \quad \pi_2$</td>
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<td>$A \supset C \quad \supset \mathcal{E}$</td>
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</table>

$\bot$-rule: $\frac{1}{\bot} \bot \supset \mathcal{E}$, where $P$ is any atomic formula different from $\bot$.

In the rules $(\forall \mathcal{E})$ and $(\exists \mathcal{E})$ the variable $\alpha$ is the proper variable of these rules, and it has to satisfy the well known restrictions on variables, which is similar to the restrictions on variables in the system $\mathcal{SE}^\alpha$ (see also [18] 2.3.8.(b)). In $\mathcal{NE}^\alpha$ (by using the notions above) we can define the proper variable property (PVP) of a derivation $\pi$ (see [18] 2.5.1.(c) or [14] p. 28) which is very similar to PVP in $\mathcal{SE}^\alpha$.

In the rule $(\supset \mathcal{E})$ and all elimination rules in the brackets $(\supset)$ there is the a-class which is discharged by that rule if its index is not $0$, and if it is $0$, then nothing is discharged by that rule. Moreover, the other a-classes of the same formula (like the one discharged) may exist, and they are not discharged by that rule. We note that in one rule and the discharged a-class by that rule one number can be written, for example $\supset \mathcal{E}1$ and $/A^\alpha/1$, where $1$ is not the number of that rule.

**Example 2.5.** In the system $\mathcal{NE}^\alpha$ we consider the following derivation $\pi$

\[
\frac{[B^{(8)(7)}]^{\mathcal{E}} \quad C^{(9)(7)}}{(B \land C^{(1)(7)})^{\mathcal{E}}} \quad \frac{[B^{(5)(4)}]^{\mathcal{E}} \quad C^{(6)(4)} \quad A^{1}}{(B \land C^{(1)(4)})^{1}} \quad \frac{A^{1} \quad /B^{2}/^{2} \quad \supset \mathcal{E}}{A \land B \quad \supset \mathcal{E}} \quad \frac{\supset \mathcal{E}1 \quad /C^{3}/^{1}}{\supset \mathcal{E}} \\
\frac{(A \land B) \land C \quad (A \land B) \land C \quad \supset \mathcal{E}}{\supset \mathcal{E}21 \quad 1 \quad 2}$

where in $\land \supset \mathcal{E}21$; $1$: 1 denotes that the discharged a-class by that rule $\land \supset \mathcal{E}1$ is $C^3$, which is denoted by $/C^3/^{1}$; and $1$ denotes that the operation on derivations whose d-formulae have the number $1$ is made after that rule, i.e. the substitution with the sb-formula $(B \land C^{(1)(4)})^{1}$ and $1$ is the number of that inference rule $\land \supset \mathcal{E}21 ; 1$. In $\land \supset \mathcal{E}21 ; 2, 3$; 2 denotes that the discharged a-class by that rule is $/B^{2}/^{2}$; $2$ denotes that the substitution with the sb-formula $(B \land C^{(1)(7)})^{2}$ is made after that rule; $3$
denotes that the contraction of the a-classes $B^{(5)}(4)$ and $B^{(8)}(7)$ is made after the substitution with number 2; and 2 and 3 are the numbers of that inference rule.

**Remark 2.1.** If in the system $\mathcal{NE}^\circ$ we have the derivations which are of the same form, their indexed d-formulae have different indices, but there is a bijection between the indices of the corresponding d-formulae, then we do not make distinction between these derivations.

In the system $\mathcal{NE}^\circ$ for elimination rules of all connectives and quantifiers we have the notions of *minor* and *major premisses* which are defined analogously to these notions in [14]. For example, $A \lor B$ is the *major premiss* and the d-formulae $C$ are *minor premisses* of the rule $(\lor \mathcal{E})$.

### 2.3. The maps which connect derivations of $\mathcal{SE}^\circ$ and $\mathcal{NE}^\circ$.

We will present two maps which connect the set of derivations of the systems $\mathcal{SE}^\circ$ and $\mathcal{NE}^\circ$, the sets $\text{Der}(\mathcal{SE}^\circ)$ and $\text{Der}(\mathcal{NE}^\circ)$, respectively. Both maps will be defined by induction on the length of the derivation, where the *lengths of derivations* $\mathcal{D}$ and $\pi$ will be defined in the usual way, as the number of all rules in these derivations.

The map $\psi$: $\text{Der}(\mathcal{SE}^\circ) \rightarrow \text{Der}(\mathcal{NE}^\circ)$ is in fact the map $\psi$ from [3] and it has the property that the image of a derivation $\mathcal{D}$ with the end sequent $\Gamma \rightarrow C$ is the derivation $\psi \mathcal{D}$ of the formula $C$ from the set of a-classes $\Gamma$:

$$\psi \left( \mathcal{D} \Gamma \rightarrow C \right) = \psi \mathcal{D} C \Gamma$$

where $\Gamma$ from $\Gamma \rightarrow C$ is the set of indexed formulae and $\Gamma$ from $\psi \mathcal{D}$ is the set of a-classes and each d-formula $D^d$ from $\Gamma$ of $\Gamma \rightarrow C$ has the corresponding a-class $D^d$ of d-formulae from $\Gamma$ of the derivation $\psi \mathcal{D}$. There are several cases which depend on the last rule of the derivation $\mathcal{D}$, $r\mathcal{D}$.

- **$\mathcal{D}$**
  - $C^i \rightarrow C$
  - $\bot^i \rightarrow P$
  - $D'$
  - $\Gamma \rightarrow A^a, \Delta \rightarrow C$
  - $\Gamma^a, \Delta \rightarrow C$ **cut**

- **$D'$**
  - $A^a, A^b, \Gamma \rightarrow C$
  - $A^a \rightarrow B, \Gamma \rightarrow C$ **contraction**

- **$\mathcal{D}'$**
  - $\Gamma^a \rightarrow B \rightarrow R$

- **$\psi \mathcal{D}$**
  - $C^i$
  - $\bot^i \rightarrow P$
  - $\Gamma^a \rightarrow C$

- **$\psi \mathcal{D}'$**
  - $[A^a]^i \rightarrow \Gamma$
  - $\psi \mathcal{D}'$

- **$\psi \mathcal{D''}$**
  - $[A^a]^i \rightarrow \Gamma$
  - $\psi \mathcal{D''}$

where $s > \max(N \psi \mathcal{D}' \cap N \psi \mathcal{D''})$.
The cases when \( rD \) is \( \land L_2, \forall L \) and \( \exists L \) are similar to the case when \( rD \) is \( \land L_1 \).

The cases when \( rD \) is \( \lor R_2, \forall R \) and \( \exists R \) are similar to the case when \( rD \) is \( \lor R_1 \).

Now we define the map \( \phi : \text{Der}(\mathcal{NE}^\circ) \rightarrow \text{Der}(\mathcal{SE}^\circ) \), which has the property that the image of a derivation of \( C \) from the set of \( a \)-classes \( \Gamma \) is the derivation with the end sequent \( \Gamma \rightarrow C \):

\[
\phi \left( \begin{array}{c}
\Gamma \\
\pi \\
C
\end{array} \right) = \phi \pi \\
\Gamma \rightarrow C
\]

where for each \( a \)-class \( D^d \) from \( \Gamma \) of the derivation \( \frac{\Gamma}{C} \) from \( \mathcal{NE}^\circ \) there is the corresponding \( d \)-formula \( D'^d \) in \( \pi \) from its \( \phi \)-image, \( \frac{\phi \pi}{\Gamma} \), and there is a bijection from \( d \) to \( d' \), i.e. each symbol \( s \) from \( d \) has the corresponding symbol \( s' \) in \( d' \), where \( s' \) is \( \ldots ((s)(t_1)) \ldots (t_m), m \geq 0 \) (see the definition below when \( \pi \) ends with an elimination rule). There are several cases which depend on the last rule of \( \pi, r\pi, \) where the last rule of \( \pi \) is its last inference rule when it does not have any number; or the contraction or the substitution whose \( d \)-formulae have the largest number of the last inference rule of \( \pi, i.e. N\pi. \)

If the last inference rule of \( \pi \) does not have any number, then \( \pi \) and \( \phi \pi \) are

\[
\begin{array}{c}
\pi \\
\phi \pi \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'
\end{array}
\]

and

\[
\begin{array}{c}
\pi' \\
\phi \pi' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi''''
\end{array}
\]

and

\[
\begin{array}{c}
\phi \pi'''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]

\[
\begin{array}{c}
\phi \pi'''''''''''' \\
C' \\
\downarrow \\
P \\
\downarrow \\
\pi'''''''''''
\end{array}
\]
The case when $r\pi$ is $\land E\bar{E}_2$ is similar to the case when $r\pi$ is $\land E\bar{E}_1$.

The case when $r\pi$ is $\lor I\bar{E}_2$ is similar to the case when $r\pi$ is $\lor I\bar{E}_1$.

The case when $r\pi$ is $\lor I\bar{E}$ is similar to the case when $r\pi$ is $\lor I\bar{E}_1$.

The case when $r\pi$ is $\exists I\bar{E}$ is similar to the case when $r\pi$ is $\exists I\bar{E}_1$.

If the last inference rule of $\pi$ has numbers, $N$ is the largest of its numbers, and

- $N$ is the number of the d-formulae of one contraction, i.e. $\pi$ is the result of a contraction in a derivation $\pi'$, then $\pi$ and $\phi\pi$ are

$$\Gamma[A^\pi]\vdash[A^\bar{A}]^\pi\quad\text{and}\quad A^\prime\pi^\prime,\Gamma\vdash C$$

- $N$ is the number of the d-formulae of one substitution, i.e. $\pi$ is the result of a substitution of $a^\pi$ in $A^\pi$, then $\pi$ and $\phi\pi$ are
Remark 2.2. We consider a derivation \( \mathcal{D} \) of the system \( \mathcal{SE}^\omega \) and its \( \psi \)-image, the derivation \( \psi \mathcal{D} = \pi \), in the system \( \mathcal{NX}^\omega \). By the definition of the map \( \psi \), if the major premiss of one elimination rule of \( \pi \) is the sb-formula \( (\langle\langle A^{(s)(\ell')}\rangle\rangle)^{n}\rangle \), then its full-form is \([\ldots[[\ldots((\ldots((A^{(s)(\ell')}))^{k_1})^{k_2})^{n_1})^{n_2})^{[1]}\ldots[l]]\ldots]\) \((\ast)\)

where \( n, k, \ell \geq 0 \), \( s \) may not exist and \( N=\max(K_1, \ldots, K_k, N_1, \ldots, N_n, L_1, \ldots, L_l) \), when one of \( n, k \) and \( l \) is not 0.

Now, for a derivation \( \pi \) of the system \( \mathcal{NX}^\omega \) we define the derivation \( \pi^- \). If \( \pi \) is a trivial derivation \( \pi^\omega \), then \( \pi^- = \pi \). If \( \pi \) is

\[
\Gamma \ (A^\delta)^{\pi''} \quad \text{and} \quad \Delta \to A, A^\alpha', \Gamma \to C
\]

then \( \pi^- \) is \( \pi \). If \( \pi \) is \( \pi^\omega \) from Remark 2.2, and its number \( N \) is the largest number of \( \mathcal{R} \), then \( \pi^- \) is

\[
\Gamma \quad \Delta
\]

\[
A^- \quad C
\]

\[
\pi_1 \quad \pi_2
\]

\[
\pi_3
\]

\[
\pi_4
\]

\[
A \quad B \quad C
\]

\[
\mathcal{R}
\]

\[
\Gamma \quad \Delta \quad \Lambda
\]

\[
A^- \quad B \quad C
\]

\[
\pi_1 \quad \pi_2 \quad \pi_3
\]

\[
\pi_4
\]

\[
\mathcal{R}
\]

where \( A^- \) is either \( (\langle\langle A^{(s)(\ell')}\rangle\rangle)^{n}\rangle \) without \( (\ast) \), when \( n > 0 \) (or \( (\langle\langle A^{(s)(\ell')}\rangle\rangle)^{n}\rangle \)), when \( n = 0 \), and in the rule whose number is \( N_1 \) (or \( N \) that number is deleted, \( t \) is replaced by \( \ell^- \) in all d-formulae of the subderivations of \( \pi^- \); (1.2) one contraction, one substitution which not the substitution from (1.1) or the rule \( \mathcal{R} \), then \( \pi^- \) is

\[
\Gamma \quad \Delta
\]

\[
A^- \quad C
\]

\[
\pi_1 \quad \pi_2 \quad C
\]

\[
\pi_3
\]

\[
\pi_4
\]

\[
A \quad B \quad C
\]

\[
\mathcal{R}
\]

\[
\Gamma \quad \Delta \quad \Lambda
\]

\[
A^- \quad B \quad \mathcal{C}
\]

\[
\pi_1 \quad \pi_2 \quad \pi_3
\]

\[
\pi_4
\]

\[
\mathcal{R}
\]

respectively.

Theorem 2.1. In \( \mathcal{NX}^\omega \) for each derivation \( \pi \) and its \( \psi \circ \phi \)-image \( \pi_1 \); \( \pi_1^- \) is \( \pi \).

Proof. By an induction on the length of a derivation \( \pi \) from the system \( \mathcal{NX}^\omega \). There are several cases according to the last rule of \( \pi \). We only consider the case when the last rule of \( \pi \) is \( \supsetee \mathcal{E} \mathcal{E} \) without numbers, and the other cases are similar. In that case \( \pi \) is

\[
\Gamma' \quad \Delta' \quad \psi_1 \quad \phi_1 \quad \phi_2 \quad \phi_3
\]

\[
A \supset B \quad A \supset C \quad \mathcal{E} \mathcal{E} 1
\]

its \( \psi \)-image is

\[
\Gamma \to A \supset B
\]

\[
\Delta, A \supset B, \Lambda \to C
\]

\[
\mathcal{L}
\]

\[
\Gamma \supset, \Delta, \Lambda \to C
\]

\[
\mathcal{R}
\]

where for each a-class \( D^\delta \) from \( \Gamma, \Delta, \Lambda \) in the derivation \( \pi \) and the corresponding d-formula \( D^\delta \) from \( \Gamma^{\omega_1}, \Delta, \Lambda \) in the last sequent of \( \phi \pi \) the following holds: there
is a bijection between symbols from $d$ and $d'$, i.e. each symbol $s$ from $d$ has the corresponding symbol $s'$ in $d'$, where $s' = (s(t_1))\ldots(t_m)$, $m \geq 0$.

Next, the $\psi$-image of the derivation $\phi\pi$ is

$$
\begin{array}{c@{\hspace{1em}}c@{\hspace{1em}}c@{\hspace{1em}}c@{\hspace{1em}}c}
\Gamma^x_i & \Delta & \psi\phi\pi'' & /B''/3A & \psi\phi\pi'''' \\
\Phi & \psi\phi\pi'' & A & C & \supset EE1; N,
\end{array}
$$

where $n > \max\{N\psi\phi\pi', N\psi\phi\pi''', N\psi\phi\pi''''\}$. By the induction hypothesis, $(\psi\phi\pi')^-$ is $\pi'$, $(\psi\phi\pi'')^-$ is $\pi''$ and $(\psi\phi\pi''')^-$ is $\pi'''$. The derivation $(\psi\phi\pi)^-$ is obtained from $\psi\phi\pi$ when, $\psi\phi\pi'$, $\psi\phi\pi''$, $\psi\phi\pi'''$ are replaced by $(\psi\phi\pi')^-$, $(\psi\phi\pi'')^-$, $(\psi\phi\pi''')^-$, respectively. $(A \triangleright B)\d$ is replaced by $A \triangleright B$, $i$ is deleted in the d-formulae from $\Gamma^x_i$ and $N$ is deleted in $EE1; N$. Finally, by the Remark 2.1, $(\psi\phi\pi)^- = \pi$. □

3. Conversions in the systems $\mathcal{SE}^o$ and $\mathcal{NE}^o$

3.1. Conversions in the system $\mathcal{SE}^o$. The conversions of derivations in the system $\mathcal{SE}^o$, the pm$^2$-conversions, are in fact conversions of derivations of the system $\mathcal{SE}$ from [3]. Almost all of them are the transformations from Gentzen’s proof of Hauptsatz in [6]. The characteristic of the pm$^2$-conversions is that they can be applied on a derivation $\mathcal{D}$ of the following form

$$
\begin{array}{c@{\hspace{1em}}c}
\mathcal{D}' & \mathcal{D}'' \\
\Gamma \rightarrow A & A^x, \Delta \rightarrow B \\
\end{array}
$$

where subderivations $\mathcal{D}'$ and $\mathcal{D}''$ do not contain any cut rule. The degree $d$, the left rank $lr$, the right rank $rr$ and the rank $r$ of the derivation $\mathcal{D}$ above are the well-known Gentzen’s notions (see, for example, [3]).

There are three kinds of pm$^2$-conversions: pm$^2$-conversions, mf$^2$-conversions and ms$^2$-conversions. In each conversion below its redex will be the derivation $\mathcal{D}$ and its contractum will be the derivation $\mathcal{C}$. If we have, for example, that the conversion $(1 < lr \triangleright rr - c_a > 1)^c$ is applied on the derivation $\mathcal{D}$ above, then $1 < lr$ means that the left rank of $\mathcal{D}$ is greater than 1, thus the last rule of $\mathcal{D}'$ is either a contraction or a left rule; $rr - c_a > 1$ means that $\mathcal{D}''$ ends with several contractions of the cut formula $A$ and the principal formula of the first rule above these contractions, i.e. the rule $R$, is not any of the contracted formulae $(rr - c = 1$, otherwise); $^c$ means that by that conversion the cut and the contractions are permuted with that rule $R$. If a contractum $\mathcal{C}$ has two (or three) subderivations of the same form we will suppose that the indices of their formulae are different. More precisely, the end sequents of two of these subderivations of $\mathcal{C}$ will be denoted, for example, by $\Gamma^c \rightarrow C$ and $\Gamma'^c \rightarrow C$ which will mean that there is a bijection between $\Gamma^c$ and $\Gamma'^c$ (i.e. $\Gamma^c \simeq \Gamma'^c$). (See [3] Section 5.1 for details.) $A^{c^n}$ will denote the sequence $A^{a_1}, \ldots, A^{a_m}$, $m \geq 1$. The rule $c\ldots$ with $A^{c^n}$ in its upper sequent and $A^a$ in its lower sequent will mean $m - 1$ contractions of formulae from $A^{c^n}$, where $a$ is $a_1 \cup \ldots \cup a_m$ ($m \geq 1$); otherwise it will mean several contractions of formulae which are not emphasized. Finally, $\Gamma^x a$ and $\Gamma^x a$ will denote the same set of indexed formulae.
**p²-conversions.** The p²-conversions consist of the following four kinds of conversions:

\[(Is)^2, (1 ≤ lr ∨ rr_k > 1)^2, (1 ≤ lr ∨ rr - r_k > 1)^2 \text{ and } (1 ≤ lr ∨ rr - c_k = 1)^2,\]

where the rule \(\bigstar\) cannot be a cut rule.

\[\Rightarrow (Is)^2\] Conversions \((Is)^2\) will be conversions \((Is)\) from [3], when at list one of the subderivation \(D'\) or \(D''\) of \(D\) is an initial sequent (i.e. Zucker’s trivial cut, see [18 p. 40]).

\[\Rightarrow (1 ≤ lr ∨ rr > 1)^2\] In the redex the last rule of the subderivation \(D''\) is either a right rule; or a left rule or a contraction whose principal formula is not the cut formula \(A^a\).

We present, as one example, the case when \(rD''\) is \(\forall L\).

\[\Rightarrow (1 ≤ lr ∨ rr ∨ L > 1)^2\] The derivations \(D\) and \(C\) are

\[
\begin{array}{c}
\frac{D'_1 \quad D'_2}{C \lor D', A^a, A^s, \Delta' \lor B} \quad \frac{\lor L \quad \lor L}{\Gamma \lor A}
\end{array}
\]

\[\Rightarrow (1 ≤ lr ∨ rr ∨ L > 1)^2\] In the redex the subderivation \(D''\) ends with \(n - 1\) contractions of the formulae \(A (n ≥ 2)\) and the principal formula of the first rule above these contractions is not any of the contracted formulae. We only present the case when the rule above the contractions is \(\lor L\).

\[\Rightarrow (1 ≤ lr ∨ rr ∨ L > 1)^2\] The derivation \(D\) is

\[
\begin{array}{c}
\frac{D'_1 \quad D'_2}{C \lor D', A^a, \Delta' \lor B} \quad \frac{\lor L \quad \lor L}{\Gamma \lor A}
\end{array}
\]

where \(a = a_1 ∪ \ldots ∪ a_n\) and the end sequents of \(D'_1\) and \(D'_2\) contain the formulae from \(A^a\), i.e. \(1 ≤ k ≤ n - 1\) in \(D\). The derivation \(C\) is

\[
\begin{array}{c}
\frac{D'_1 \quad D'_2}{C \lor D', A^a, \Delta' \lor B} \quad \frac{\lor L \quad \lor L}{\Gamma \lor A}
\end{array}
\]

where \(a' = a_1 ∪ \ldots ∪ a_k\), \(a'' = a_{k+1} ∪ \ldots ∪ a_n\) and \(I^e\) is \(I^e \lor x = x \lor x' = x' \lor x'' = x'' \lor x\) (see [18 Note on p. 40]). If the end sequent of \(D'_1\) or \(D'_2\) does not contain any formula from \(A^a\), then the contractum is the derivation \(C\) above, where its subderivation which ends with the sequent \(I^e \lor x = x' = x'' \lor x\) is replaced by \(D'_1\) or its subderivation which ends with \(I^e \lor x = x' = x'' \lor x\) is replaced by \(D'_2\) respectively.
In the conversions below the indexed formulae of the end sequent of the contractum \( C \) will not be explained in details as in the case above. We will assume that these formulae are obtained by using (oi): operations on indices.

\( \Rightarrow \text{IV} \langle 1 \leq lr \not\supset rr - c = 1 \rangle \) In the redex \( D'' \) ends with \( n - 1 \) contractions of \( A \) \((n \geq 2)\) and the principal formula of the first rule above these contractions is one of the contracted formulae. We present the case when that rule is \( \wedge L_1 \).

\( 1 \leq lr \not\supset rr - c \leq L_1 = 1 \) The derivation \( D \) is

\[
\begin{array}{c}
\text{D}' \\
\Gamma \rightarrow C \land D \\
/ \text{C}^n /, C \land D, \Delta \rightarrow B \\
\hline
\bigwedge L_1 \\
\hline
\end{array}
\]

where the principal formula of \( \wedge L_1 \) has, for example, the index \( a_i \), i.e. \( a_i \) is \( a_1 \) (but, we note that each \( a_i, 1 \leq l \leq n \), can be the index \( i \)), and \( a \) is \( a_1 \cup \ldots \cup a_n \). The derivation \( C \) (when \( b \not\in \emptyset \)) is

\[
\begin{array}{c}
\text{D}'' \\
\Gamma \rightarrow C \land D \\
/ \text{C}^n /, C \land D, \Delta \rightarrow B \\
\hline
\bigwedge L_1 \\
\hline
\end{array}
\]

where \( a' \) is \( a_2 \cup \ldots \cup a_n \) and \( \Gamma, \Delta \rightarrow B \) is obtained by using (oi). If \( b \not\in \emptyset \), then the contractum is only the subderivation of \( C \) above, which ends with the upper cut.

\textbf{mf}^\varepsilon \text{-conversions} consists of \( \text{mf}^\varepsilon \text{-conversions and ms}^\varepsilon \text{-conversions.}

\textbf{mf}^\varepsilon \text{-conversions}. The \( \text{mf}^\varepsilon \text{-conversions consist of two kinds of conversions:} \)

\( 1 = lr \not\leftrightarrow rr = 1 \) and \( 1 = lr \not\leftrightarrow rr - c = 1 \). In all \( \text{mf}^\varepsilon \text{-conversions of the first kind the left and the right rank of the redex are 1.}

\( \Rightarrow V \langle 1 = lr \leftrightarrow rr = 1 \rangle \) In the redex the subderivations \( D' \) and \( D'' \) end with rules whose principal formulae are the cut formulae. We present the case when the cut formula when \( D \) is \( C \supset D \).

\( 1 = lr \leftrightarrow rr = 1 \) The derivations \( D \) and \( C \) (when \( c \) and \( d \) are not \( \emptyset \)) are

\[
\begin{array}{c}
\text{D}'_1 \\
\Gamma \rightarrow C \supset D \\
/ \text{C}^n /, \Gamma \rightarrow D \\
\hline
\text{R} \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{D}'_2 \\
\Gamma \rightarrow C \supset D \\
/ \text{C}^n /, \Gamma \rightarrow D \\
\hline
\text{L} \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{D}'_1 \\
\Delta^j \rightarrow C \\
/ \text{C}^n /, \Gamma \rightarrow D \\
\hline
\text{D}'_2 \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{D}'_1 \\
\Delta^j \rightarrow C \\
/ \text{C}^n /, \Gamma \rightarrow D \\
\hline
\text{D}'_2 \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{D}'_1 \\
\Delta^j \rightarrow C \\
/ \text{C}^n /, \Gamma \rightarrow D \\
\hline
\text{D}'_2 \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{D}'_1 \\
\Delta^j \rightarrow C \\
/ \text{C}^n /, \Gamma \rightarrow D \\
\hline
\text{D}'_2 \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{D}'_1 \\
\Delta^j \rightarrow C \\
/ \text{C}^n /, \Gamma \rightarrow D \\
\hline
\text{D}'_2 \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{D}'_1 \\
\Delta^j \rightarrow C \\
/ \text{C}^n /, \Gamma \rightarrow D \\
\hline
\text{D}'_2 \\
\hline
\end{array}
\]
If $c = \emptyset$ and $d \neq \emptyset$, then $C$ is $D'_f$ and $D''_f$, cut, and similarly when $c \neq \emptyset$ and $d = \emptyset$. If $c = d = \emptyset$, then $C = D''_f$.

\(\textbf{VI}_5\) \((1 = lr \leftrightarrow rr - c = 1)^2\) In the redex the subderivation $\mathcal{D}'$ ends with a right rule i.e. its principal formula is the cut formula and the derivation $\mathcal{D}''$ ends with $n - 1$ contractions of the formulae $A$ ($n \geq 2$) where the principal formula of the first rule above these contractions is one of the contracted formulae. We present the case when the cut formula is $C \supset D$.

\((1 = lr \leftrightarrow rr - c = 1)^2\) The derivation $\mathcal{D}$ is

\[
\begin{align*}
\mathcal{D}'_f & \quad \quad \quad \mathcal{D}''_f \\
\frac{C'' \cup, \Gamma \rightarrow D}{\Gamma \rightarrow C \supset D \supset R} & \quad \quad \quad \frac{C \supset D^{c_{k-1}}, \Delta' \rightarrow C \supset D^{a_{k}}/\overline{C}, C \supset D^{k+1}, \Delta'' \rightarrow B \supset L}{C \supset D', C \supset D^{k+1}, \Delta', \Delta'' \rightarrow B} \\
\frac{\Gamma \supset C \supset D \supset R}{\Gamma \supset C \supset D} & \quad \quad \quad \frac{\Gamma^{a_{k}}, \Delta', \Delta'' \rightarrow B}{C \supset D''}
\end{align*}
\]

where the principal formula of the rule $\supset L$ has, for example, the index $a_k$, i.e. $a$ is $a_k$, but, we note that each $a_i$, $1 \leq l \leq n$, can be the index $i$ and the end sequents of $\mathcal{D}'_f$ and $\mathcal{D}''_f$ contain formulae from $A^j$, i.e. $2 \leq k \leq n - 1$, and $a$ is $a_1 \cup \ldots \cup a_n$.

The derivation $\mathcal{C}$ (when $c \neq \emptyset$ and $d \neq \emptyset$) is

\[
\begin{align*}
\mathcal{C}'_f & \quad \quad \quad \mathcal{C}''_f \\
\frac{C'' \cup, \Gamma \rightarrow D}{\Gamma \rightarrow C \supset D \supset R} & \quad \quad \quad \frac{C \supset D^{c_{k-1}}, \Delta' \rightarrow C \supset D^{a_{k}}/\overline{C}, C \supset D^{k+1}, \Delta'' \rightarrow B \supset L}{C \supset D', C \supset D^{k+1}, \Delta', \Delta'' \rightarrow B} \\
\frac{\Gamma^{a_{k}}, \Delta', \Delta'' \rightarrow B}{C \supset D''}
\end{align*}
\]

where $a'$ is $a_1 \cup \ldots \cup a_{k-1}$, $a''$ is $a_{k+1} \cup \ldots \cup a_n$ and $\Gamma^k$, $\Delta'$, $\Delta'' \rightarrow B$ is obtained by using (01). If the end sequent of $\mathcal{D}'_f$ or $\mathcal{D}''_f$ does not contain any formula from $C \supset D^c$, then the contractum is $\mathcal{C}$ above, where its subderivation which ends with $\Gamma_\rightarrow D$, $\Delta' \rightarrow C$ is replaced by $\mathcal{D}'_f$, or its subderivation which ends with $D^d/\overline{C}$, $\Gamma_\rightarrow D$, $\Delta'' \rightarrow B$ is replaced by $\mathcal{D}''_f$, respectively. If $c$ is $\emptyset$, then the contractum is the derivation $\mathcal{C}$ above, where its subderivation which ends with $\Gamma_\rightarrow D$, $\Delta' \rightarrow C$, $\Delta'' \rightarrow B$ is replaced by $\mathcal{D}'_f$. If $d$ is $\emptyset$, then the contractum is the subderivation of the derivation $\mathcal{C}$ above, which ends with $\Gamma_\rightarrow D$, $\Gamma_\rightarrow D$, $\Gamma' \rightarrow D$ is replaced by $\mathcal{D}''_f$.

\textbf{ms}$_\mathcal{E}$-conversions. The ms$_\mathcal{E}$-conversions are \((1 < lr \wedge \neg (rr) = 1)^2\) conversions. In all ms$_\mathcal{E}$-conversions the right rank of the redex is 1 and its left rank is greater than 1.

\(\textbf{VI}_7\) \((1 < lr \wedge \neg rr = 1)^2\) In the redex the last rule of derivation $\mathcal{D}'$ is a left rule or a contraction, and the last rule of $\mathcal{D}''$ is a left rule whose principal formula is the cut formula. We only present the case when the rule $r\mathcal{D}'$ is $\exists L$ and the cut formula $A$ is $C \supset D$. 
The derivation $D$ is

$$
\frac{\lnot Fx^i, \Gamma' 
\vdots \exists x Fx^i, \Gamma' \rightarrow C \land D 
\exists x Fx^i, \Gamma' \rightarrow C \land D}{\exists x Fx^i, \Gamma', \Delta \rightarrow B} \text{ cut}
$$

and the derivation $C$ is

$$
\frac{\lnot Fx^i, \Gamma' \rightarrow C \land D}{\exists x Fx^i, \Gamma', \Delta \rightarrow B} \text{ cut}
$$

An arbitrary derivation $F$, whose one subderivation is the redex of one conversion above, will be converted into a derivation $F'$ by replacing $D$ (in $F$) with the contractum of that conversion, the derivation $C$, by using the operation $\text{pruning}^\ell$ defined in [3], which is completely analogous to the definition of the operation pruning in [18] 3.1.5 and 7.8.3. Then we define

$F \text{ p}^\ell-\text{conv} F'$, $F \text{ m}^\ell-\text{conv} F'$ or $F \text{ ms}^\ell-\text{conv} F'$ iff $F'$ is obtained from $F$ by replacing one its subderivation $D$, which is the redex of a $p^\ell$-conversion, $m^\ell$-conversion or $ms^\ell$-conversion respectively, with the contractum of that conversion, and the corresponding pruning is applied on the part of $F$ below $D$. Moreover, the right cut formula of the last cut in $D$ will be the $d$-formula of that conversion.

A derivation $F$ $\text{ pm}^\ell-\text{converts into}$ a derivation $F'$ iff there is a sequence of derivations, $\mathcal{F}_0, \ldots, \mathcal{F}_n, n \geq 0$, such that $\mathcal{F}_0 = F$, $\mathcal{F}_n = F'$, and for all $i < n$ (when $n > 0$) either $\mathcal{F}_i \text{ p}^\ell-\text{conv} \mathcal{F}_{i+1}$ or $\mathcal{F}_i \text{ m}^\ell-\text{conv} \mathcal{F}_{i+1}$.

If a derivation $F$ does not have any cut rule, then $F$ will be called a cut-free derivation in the system $\mathcal{S}^\ell$.  

### 3.2. Conversions in the system $\mathcal{N}^\ell$.

We will define the conversions in the system $\mathcal{N}^\ell$, which are the conversions in the system $\mathcal{N}^\ell$ presented in [3] with indices of formulae, and numbers of some formulae and inference rules. In the system $\mathcal{N}^\ell$ all elimination rules make maximum segments of several formulae i.e. there are special conversions of derivations from $\mathcal{N}^\ell$ which do not exist for derivations from Gentzen’s system $NJ$. Each conversion will have the redex, the derivation $\pi$, and the contractum, the derivation $\overline{\pi}$. In fact, we will consider the derivations with subderivations $\pi$ and $\overline{\pi}$, respectively (see (cn) below).

- $\lor (\mathcal{E} \text{ maxf-conversions})$ These conversions are used to eliminate a maximum formula in a derivation of the system $\mathcal{N}^\ell$.

  $(\mathcal{E} \lor_1-\text{conv})$ The redex, the derivation $\pi$, is

  $$
  \frac{\Gamma \overset{\pi_1}{C} \lor_1 \overset{\pi_2}{(C \lor D) \overset{\pi_3}{E}}}{}
  $$

$1 < \text{lr}_3 \land \text{rr}_3 = 1)^\ell$
and $C \lor D$ is a maximum formula of $\pi$, where $k$, $l$ and $m$ may not exist. We present $(E \lor i$-convn) in the full form. The derivation $\pi$ is the subderivation of $\pi'$

\[\frac{[E^{n_1}]^\pi / F^{b_1}/ \ast \Gamma_1}{(C \lor D)^\pi \lor I \mathcal{E}_1} \quad \frac{[E^{n_1}]^\pi / F^{b_2}/ \ast \Delta_1^i}{(B)^l \ast \pi_2} \quad \frac{[E^{n_2}]^\pi / F^{b_3}/ \ast \Delta_1''}{(B)^l \ast \pi_4} \lor \mathcal{E} \mathcal{E} \quad [E^{n_4}]^\pi / F^{b_4}/ \ast \Phi}\]

$\Gamma$ is $[E^{n_1}]^\pi / F^{b_1}/ \ast \Gamma_1$; $\Delta'$ is $[E^{n_2}]^\pi / F^{b_2}/ \ast \Delta_1^i$; $\Delta''$ is $[E^{n_3}]^\pi / F^{b_3}/ \ast \Delta_1''$, where in $\pi_1$, $1 \leq i \leq 4$: $[E^{n_1}]^\pi$ denotes the part of one a-class which is made by the contraction whose number is $l$ and ... denotes the parts of all a-classes which are made by all contractions of that kind; $/ F^{b_1}/ \ast$ denotes the part of one a-class which is the discharged a-class by one rule from $\pi_4$ and ... denotes the parts of all a-classes of that kind. The rule $\lor I \mathcal{E}_1$ can have several numbers, which appear: 1-1 only in $\pi_1$; the rule $\lor \mathcal{E} \mathcal{E}$ can have several numbers which appear: 2-1 only in $\pi_1$, 2-2 only in $\pi_2$, 2-3 only in $\pi_3$, 2-12 in $\pi_1$ and $\pi_2$, 2-13 in $\pi_1$ and $\pi_3$, 2-23 in $\pi_2$ and $\pi_3$, 3-123 in $\pi_1$, $\pi_2$ and $\pi_3$; the numbers $k$, $l$ and $m$, when they exist, can be the numbers either of $\lor \mathcal{E} \mathcal{E}$ or of one rule from $\pi_4$; and $i$ can be the number of one rule from $\pi_4$.

The contractum $\pi$ has different forms in the cases when $c \neq \emptyset$ and $c = \emptyset$. If the number $k$ ($l$) exists and the number of the sb-formulae with that number is greater than 1, then $\pi_1$ ($\pi_2$) below is $\pi_1$ ($\pi_2$) without the contraction whose number is the c-number of the sb-formula whose number is $k$ ($l$) and without each contraction whose number is greater than that number.

If $c \neq \emptyset$, then $\pi$ is the subderivation

\[\frac{\Gamma^\times c}{\pi_1'} \frac{[C^{n_2}]^\pi / F^{b_2}/ \ast \Delta_1''}{|B| \pi_2} \quad \frac{\pi_4^\times c}{[E^{n_4}]^\pi / F^{b_4}/ \ast \Phi} \quad \ast \Delta_1''}

in $\pi''$:

\[\frac{[E^{n_1}]^\pi / F^{b_1}/ \ast \Gamma_1}{(C^{n_2})^\pi / F^{b_2}/ \ast \Delta_1''} \quad \frac{\pi_1'}{(B)^n \ast \pi_2} \quad \frac{\pi_4'}{(B)^m \ast \pi_2} \quad \frac{[E^{n_4}]^\pi / F^{b_4}/ \ast \Phi} \quad \ast \Delta_1''}]

where: (1) $[E^{n_1}]^\pi / F^{b_1}/ \ast \Gamma_1$, $[E^{n_2}]^\pi / F^{b_2}/ \ast \Delta_1''$ and $[E^{n_4}]^\pi / F^{b_4}/ \ast \Phi$ denote the parts of one a-class which is made by the contraction whose number is the number of one rule in $\pi_4$ and ... denotes the a-classes which are made by all contractions of that kind; (2) $/ F^{b_1}/ \ast$ and $/ F^{b_2}/ \ast$ and $/ F^{b_3}/ \ast$ denote the parts of one a-class which is the discharged a-class by one rule from the derivation $\pi_4$, and ... denotes all a-classes of that kind; (3) $N_1$ is $j + 1$ and $N_2$ is $j + 2$, where $j$ is the number of a rule $\rho$ from $\pi_4$ in $\pi'$ which is the smallest of all numbers of the rules from $\pi_4$ in $\pi'$, and $N_1$ and now $N_2$ are the numbers of the rule from $\pi_4$ in $\pi''$, (3.1) in $\pi''$ there are $j$, $j \geq 0$, contractions of top-formulae from $\pi_1$, $\pi_2$ and $\pi_4$ whose corresponding top-formulae from the subderivations $\pi_1$, $\pi_2$
and \( \pi_4 \) of \( \pi' \) have the c-number of the sub-formulae whose numbers are K and L, and their numbers are the numbers of \( \pi \), so if \( j'' \) is the corresponding number of a number \( j' \) from \( \pi_4 \) in \( \pi' \) and (3.1.1) \( j' \geq 3 \), then \( j'' = j' + 2 \); (3.1.2) \( j' < 3 \), then \( j'' = j' \).

If \( c = \emptyset \), then \( \bar{\pi} \) is the subderivation \( [E^{e_1} \ldots / F^{b_2} / \ast \ldots \Delta_1] \) in the derivation \( \pi'' \):

\[
\begin{align*}
\Gamma & \quad \Delta \\
\pi_1 & \quad \pi_2 \\
C & \quad D \\
P & \quad B \\
\square & \quad \bar{\pi} \\
\bar{\pi}_4 & \quad \pi_4 \\
(\Delta^\pi)^{\pi_1} & \quad \Lambda \\
(\Delta^\pi)^{\pi_3} & \quad \Phi
\end{align*}
\]

where \( N_1 \) and numbers in \( \pi_4 \) from \( \pi'' \) are obtained as in the case when \( c \neq \emptyset \).

(CN): If in the derivation \( \pi \) there are some a-classes which are contracted (they may be discharged a-classes of some rules), and they appear in \( \bar{\pi} \), then these a-classes are contracted in the derivation \( \bar{\pi} \), too.

(\( \mathcal{E} \lor_2 \)-convn) Similarly to (\( \mathcal{E} \lor_1 \)-convn).

The other \( \mathcal{E} \)-maxf-conversions and all \( \mathcal{E} \)-maxx-conversions below will not be presented in the full form, but we will assume that in each of them (CN) holds and the derivation which contains its contractum will be obtained similarly as in the case (\( \mathcal{E} \lor_1 \)-convn).

(\( \mathcal{E} \land_1 \)-convn) The redex, the derivation \( \pi \), is

\[
\begin{align*}
\Gamma & \quad \Delta \\
\pi_1 & \quad \pi_2 \\
C & \quad D \\
P & \quad B \\
\square & \quad \bar{\pi} \\
\bar{\pi}_3 & \quad \pi_3 \\
(\Delta^\pi)^{\pi_1} & \quad \Lambda \\
(\Delta^\pi)^{\pi_3} & \quad \Phi
\end{align*}
\]

where \( C \land D \) is a maximum formula of \( \pi \). The contractum \( \bar{\pi} \) has two forms in the cases when \( c \neq \emptyset \) and \( c = \emptyset \):

\[
\begin{align*}
\Gamma \times c & \quad \Delta \\
\pi_1 & \quad \pi_2 \\
C & \quad D \\
P & \quad B \\
\square & \quad \bar{\pi} \\
\bar{\pi}_3 & \quad \pi_3 \\
(\Delta^\pi)^{\pi_1} & \quad \Lambda \\
(\Delta^\pi)^{\pi_3} & \quad \Phi
\end{align*}
\]

(\( \mathcal{E} \land_2 \)-convn) Similarly to (\( \mathcal{E} \land_1 \)-convn).

(\( \mathcal{E} \triangleright \)-convn) The redex, the derivation \( \pi \), is

\[
\begin{align*}
\Gamma & \quad \Delta \\
\pi_1 & \quad \pi_2 \\
C & \quad D \\
P & \quad B \\
\square & \quad \bar{\pi} \\
\bar{\pi}_3 & \quad \pi_3 \\
(\Delta^\pi)^{\pi_1} & \quad \Lambda \\
(\Delta^\pi)^{\pi_3} & \quad \Phi
\end{align*}
\]

where \( C \triangleright D \) is a maximum formula of \( \pi \). The contractum \( \bar{\pi} \) has three forms in the cases when \( c \neq \emptyset \), \( d \neq \emptyset \); \( c = \emptyset \), \( d \neq \emptyset \); and \( d = c = \emptyset \):
Similarly to \((\mathcal{E}∨\text{convn})\).

Similarly to \((\mathcal{E}∧\text{convn})\).

\(\mathcal{E}(\text{maxs-conversions})\). The \(\mathcal{E}\text{maxs}\)-conversions are used to eliminate maximum segments (for details see [3] Section 5.3).

\(\mathcal{E}\text{maxs}_n\). The redex \(\pi\) and the contractum \(\bar{\pi}\) are

\[
\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta[\pi_n]} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\Gamma \vdash \Delta[\pi_n] \wedge \mathcal{E}_1} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\Gamma \vdash \Delta[\pi_n] \wedge \mathcal{E}_1 \text{ and } \mathcal{E}_n} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\mathcal{E}_1 \wedge \mathcal{E}_1} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\mathcal{E}_1 \wedge \mathcal{E}_1} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\mathcal{E}_1 \wedge \mathcal{E}_1}
\]

where \((C \wedge D)_n\) is the last d-formula of one maximum segment, which is defined in the usual way: it is one a sequence of d-formulae \(F_1, \ldots, F_n\) which are of the same form, \(F_{i+1}\) is immediately below \(F_i\), \(1 \leq i \leq n - 1\), \(F_1\) is the consequence of one introduction rule and \(F_n\) is the major premiss of one elimination rule. In \(\pi\): \(n, k, l\) and \(m\) may not exist; the upper rule \(\wedge \mathcal{E}_1\) can have several numbers, which appear: 1-1 only in \(\pi_1\), 1-2 only in \(\pi_2\) or 1-12 in \(\pi_1\) and \(\pi_2\), and the numbers \(N\) and \(\mathcal{L}\) when they exist; the lower rule \(\wedge \mathcal{E}_1\) can have several numbers which appear: 2-1 only in \(\pi_1\), 2-2 only in \(\pi_2\), 2-3 only in \(\pi_3\), 2-12 in \(\pi_1\) and \(\pi_2\), 2-13 in \(\pi_1\) and \(\pi_3\), 2-23 in \(\pi_2\) and \(\pi_3\), 3-123 in \(\pi_1\), \(\pi_2\) and \(\pi_3\), and the numbers \(N\) and \(\mathcal{L}\) when they exist and they are not numbers of the upper rule \(\wedge \mathcal{E}_1\), and \(\kappa\) and \(m\) when they exist. In \(\bar{\pi}\): the upper rule \(\wedge \mathcal{E}_1\) can have several numbers from 1-2, 2-2 and 2-23 above, the number \(\kappa''\) and \(\mathcal{L}''\); the lower rule \(\wedge \mathcal{E}_1\) have all numbers of two last rules \(\wedge \mathcal{E}_1\) from \(\pi\) which are not numbers of the upper rule \(\wedge \mathcal{E}_1\) and \(N''\), and it can have the numbers \(\kappa''\) and \(\mathcal{L}''\) when they are not number of the upper rule \(\wedge \mathcal{E}_1\).

In all conversions below the connections between numbers of \(\pi\) and \(\bar{\pi}\) are as in this conversion.

\(\mathcal{E}\text{maxs}^\wedge_2\) Similarly to \((\mathcal{E}\text{maxs}^\wedge_1)\).

\(\mathcal{E}\text{maxs}^\wedge_3\) The redex \(\pi\) and the contractum \(\bar{\pi}\) are

\[
\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta[\pi_n]} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\Gamma \vdash \Delta[\pi_n] \wedge \mathcal{E}_1} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\Gamma \vdash \Delta[\pi_n] \wedge \mathcal{E}_1 \text{ and } \mathcal{E}_n} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\mathcal{E}_1 \wedge \mathcal{E}_1} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\mathcal{E}_1 \wedge \mathcal{E}_1} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\mathcal{E}_1 \wedge \mathcal{E}_1}
\]

\(\mathcal{E}\text{maxs}^\wedge_4\) Similarly to \((\mathcal{E}\text{maxs}^\wedge_1)\).

\(\mathcal{E}\text{maxs}^\wedge_5\) and \((\mathcal{E}\text{maxs}^\wedge_4)\) are defined similarly to \((\mathcal{E}\text{maxs}^\wedge_1)\).

The conversions \((\mathcal{E}\text{maxs}^\wedge_3)\) are completely analogous to the conversions \((\mathcal{E}\text{maxs}^\wedge_1)\), where \(\bar{\pi}\) is an arbitrary elimination rule.

\(\mathcal{E}\text{maxs}^\wedge_6\) The redex \(\pi\) and the contractum \(\bar{\pi}\) are

\[
\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta[\pi_n]} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\Gamma \vdash \Delta[\pi_n] \wedge \mathcal{E}_1} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\Gamma \vdash \Delta[\pi_n] \wedge \mathcal{E}_1 \text{ and } \mathcal{E}_n} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\mathcal{E}_1 \wedge \mathcal{E}_1} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\mathcal{E}_1 \wedge \mathcal{E}_1} \quad \frac{\Gamma \vdash \Delta[\pi_n]}{\mathcal{E}_1 \wedge \mathcal{E}_1}
\]
(\(\mathcal{E}_{\text{max}}^{\pi_0}\)) Similarly to (\(\mathcal{E}_{\text{max}}^{\pi_1}\)).

(\(\mathcal{E}_{\text{max}}^{\pi_2}\)) The redex \(\pi\) and the contractum \(\check{\pi}\) are

\[
\begin{array}{c}
\frac{\Gamma \vdash \Delta / A \vdash B / \Theta_{1} / C \vdash D / G}{\vdash EE} \\
A \supset B \quad C \supset D
\end{array}
\]

(\(\mathcal{E}_{\text{max}}^{\pi_3}\)) Similarly to (\(\mathcal{E}_{\text{max}}^{\pi_2}\)).

(\(\mathcal{E}_{\text{max}}^{\pi_2}\)) and (\(\mathcal{E}_{\text{max}}^{\pi_5}\)) are defined similarly to (\(\mathcal{E}_{\text{max}}^{\pi_1}\)).

The conversions (\(\mathcal{E}_{\text{max}}^{\pi_2}\)) are completely analogous to the conversions (\(\mathcal{E}_{\text{max}}^{\pi_0}\)), where \(\pi\) is an arbitrary elimination rule.

The conversions (\(\mathcal{E}_{\text{max}}^{\pi_2}\)) and (\(\mathcal{E}_{\text{max}}^{\pi_5}\)) are completely analogous to the conversions (\(\mathcal{E}_{\text{max}}^{\pi_0}\)), where \(\pi\) is an arbitrary elimination rule.

\(\mathcal{E}_{\text{max}}^{\pi_2}\)-conversions and \(\mathcal{E}_{\text{max}}^{\pi_2}\)-conversions make the set of \(\mathcal{E}_{\text{max}}^{\text{conversions}}\).

In a derivation \(\pi\) its maximum formulae (i.e. maximum segments of one d-formula) and formulae from its maximum segments will be called the max-formulae of \(\pi\).

\(\pi' \ E_{\text{max}} \ \pi'' \ (\pi' \ E_{\text{max}} \ \pi'')\) iff the derivation \(\pi''\) is obtained from the derivation \(\pi'\) by replacing one of its subderivation \(\pi\), which has the form of the redex of an \(\mathcal{E}_{\text{max}}\)-conversion (\(\mathcal{E}_{\text{max}}\)-conversion), with the contractum of that \(\mathcal{E}_{\text{max}}\)-conversion (\(\mathcal{E}_{\text{max}}\)-conversion).

\(\pi' \ E_{\text{max}} \ \pi''\) iff either \(\pi' \ E_{\text{max}} \ \pi''\) or \(\pi' \ E_{\text{max}} \ \pi''\).

If for \(\pi' \ E_{\text{max}} \ \pi''\), we want to note the last formula \(A^*\) of the maximum segment of the redex \(\pi\), then we will write \(\pi' \ E_{\text{max}} \ \pi''\) by the formula \(A^*\).

\(\pi' \ E_{\text{max}} > \ \pi''\) iff there is a sequence \(\pi_0, \ldots, \pi_n, n > 0\), such that \(\pi_0 = \pi'\), \(\pi_n = \pi''\), and for each \(i, i < n, \pi_i \ E_{\text{max}} \ \pi_{i+1}\).

\(\pi' \ E_{\text{max}}\)-converts into \(\pi''\) iff either \(\pi' \ E_{\text{max}} > \ \pi''\) or \(\pi'\) is \(\pi''\).

If a derivation \(\pi\) does not have any subderivation which is the redex of a \(\mathcal{E}_{\text{max}}\)-conversion with a max-formula, then the derivation \(\pi\) will be called a normal derivation in \(\mathcal{N}\).

3.3. Connections between conversions from \(\mathcal{SE}^\circ\) and \(\mathcal{NE}^\circ\). We will present the connections between the conversions of derivations from the systems \(\mathcal{SE}^\circ\) and \(\mathcal{NE}^\circ\) by using the connections between the conversions of derivations from the systems \(\delta\mathcal{E}\) and \(\mathcal{NE}\) from [3, Section 6].

**Theorem 3.1.** Let \(D\) and \(C\) be derivations in the system \(\mathcal{SE}^\circ\). If \(D \ p^-\text{-conv} \ C\), then

1. \(\psi D = \psi C\) in the system \(\mathcal{NE}^\circ\);
2. \((\psi D)^- = (\psi C)^-\) in the system \(\mathcal{NE}^\circ\).

**Proof.**

1. See [3, Theorem 6.1].
2. By the definition of \(\pi^-\) for a derivation \(\pi\) and the part (1). \(\Box\)

If a symbol is of the form: – either \(i\) or \(j\), then the symbol \(i\) is its part; – \(i(j\), then \(i\) and \(j\) are its parts; – \(s(t\), then the symbols \(s\) and \(t\) and all their parts are its parts. The set which contains each symbol from an index \(a\) whose part
is the index of the principal formula of one rule for connectives will be called the m-subset of the index a.

**Theorem 3.2.** If \( D \) m\(^{-}\)-conv \( C \), then \( \psi D \) \( E \)\( \max \)\( \psi C \) in the system \( NE^o \). Moreover, if \( D \) m\(^{-}\)-conv \( C \), \( A^o \) is its formula and \( b \) is the m-subset of \( a \), then there is a sequence of derivations \( \psi D \equiv \pi_1, \ldots, \pi_{l+1} \equiv \psi C \), such that \( 1 \leq i \leq l, \pi_i \) \( E \)\( \max \)\( \pi_{i+1} \) by some \( A_s, s \in b \).


**Theorem 3.3.** In the system \( NE^o \), if \( \pi \) \( E \)max \( \pi' \), then \( \pi^- \)\( E \)\( max \)\( \pi^- \).

**Proof.** By the definition of \( \pi^- \) for a derivation \( \pi \). \( \Box \)

**Theorem 3.4.** If \( D \) is a cut-free derivation in the system \( SE^o \), then \( \psi D \) and \( (\psi D)^- \) are normal derivations in the system \( NE^o \).

**Proof.** See [3] Corollary 6.5 and Theorem 3.3. \( \Box \)

4. Cut elimination and normalization

4.1. The cut-elimination theorem for the system \( SE^o \). We will prove the cut-elimination theorem for the system \( SE^o \).

**Theorem 4.1.** In the system \( SE^o \) each derivation \( F \) pm\(^{-}\)-converts into a cut-free derivation \( F_{cf} \).

It is well known that to prove this theorem it is sufficient to prove the following lemma.

**Lemma 4.1.** Each derivation \( D \) of the form \( \frac{D'}{F} \frac{A}{\Delta} \frac{D''}{\Delta} \frac{B}{\text{cut}} \), where \( D' \) and \( D'' \) are cut-free derivations, pm\(^{-}\)-converts into a cut-free derivation \( F \).

**Proof.** In the usual way, by an induction in the pair \( \langle d, r \rangle \) where \( d \) is the degree of \( D \) and \( r \) is its rank. See the proof of Cut-lemma in [4]. We note that pm\(^{-}\)-conversions and m\(^{-}\)-conversions are pm-conversions and m-conversions without conversions \( (1 < lr \land rr - c = 1) \) from [4]. But, it is easy to see that in the proof of Cut-lemma from [4] the conversions \( (1 \leq lr \land rr - c = 1) \) can be used instead of conversions \( (1 < lr \land rr - c = 1) \). \( \Box \)

4.2. The normalization theorem for the system \( NE^o \). We will prove the normalization theorem for the system \( NE^o \) by using the cut-elimination theorem for the system \( SE^o \).

**Theorem 4.2.** In the system \( NE^o \) each derivation \( \pi \) \( E \)\( max \)-converts into a normal derivation \( \pi_N \).

**Proof.** In the system \( NE^o \) we consider a derivation \( \pi \). If \( \pi \) is a normal derivation, then the derivation \( \pi_N \) is \( \pi \). If \( \pi \) is not a normal derivation, then in the system \( SE^o \) we consider the \( \phi \)-image of the derivation \( \pi \), the derivation \( F = \phi \pi \). In the system \( NE^o \) there is the derivation \( \psi F \), i.e. \( \psi \phi \pi \), and by Theorem 2.1 \( \psi F^- \) is \( \pi \). By Theorem 4.1 the derivation \( F \) pm\(^{-}\)-converts into a cut-free derivation \( F_{cf} \),
i.e. there is a sequence of derivations \( F_1, \ldots, F_n \), such that \( n > 1 \) (by \( \psi F^- = \pi \) and \( \text{Theorem 3.1} \)), \( F_1 \) is \( \mathcal{F} \), and for all \( i < n \): either \( F_i \) \( p^\mathcal{F}\text{-conv} F_{i+1} \) or \( F_i \) \( m^\mathcal{F}\text{-conv} F_{i+1} \).

So, in \( \mathcal{N}E^o \) there is the sequence \( \psi F_1, \ldots, \psi F_n, n > 1 \), such that:

1. \( \psi F_1 = \psi \phi \pi \);

2. for each \( i, 1 \leq i \leq n - 1 \),
   (2.1) if \( F_i \) \( p^\mathcal{F}\text{-conv} F_{i+1} \), then \( \psi F_i \equiv \psi F_{i+1} \) (by Theorem \( 5.1(1) \));
   (2.2) if \( F_i \) \( m^\mathcal{F}\text{-conv} F_{i+1} \), then \( \psi F_i \mathcal{E}_{\max} > \psi F_{i+1} \) (by Theorem \( 3.2 \));

3. \( \psi F_n = \psi F_{cf} \) is a normal derivation (by Theorem \( 3.4 \)).

Thus, in the system \( \mathcal{N}E^o \) we have a sequence of different derivations from the sequence \( \psi F_1, \ldots, \psi F_n \), the sequence \( \psi F_{i_1}, \psi F_{i_2}, \ldots, \psi F_{i_k}, 1 = i_1 < \cdots < i_k \leq n \), \( k \leq n \). \( \psi F_{i_k} \) is the normal derivation \( \psi F_{cf} \) and for all \( i_j \) and \( i_{j+1} \), \( 1 \leq j \leq k - 1 \):

\[ \psi F_{i_j}, \mathcal{E}_{\max} > \psi F_{i_{j+1}}. \]

Thus, there is a sequence of derivations \( \pi^1, \pi^2, \ldots, \pi^m \), \( m_j > 1 \), such that \( \psi F_{i_1} \equiv \pi^1_1 \mathcal{E}_{\max} \pi^1_2 \mathcal{E}_{\max} \ldots \mathcal{E}_{\max} \pi^m_{m_1} \equiv \psi F_{i_{j+1}}. \)

By (1) and \( \text{Theorem 2.4} \) \( \psi F_i^- \equiv \psi F_{i_k}^- \) is \( \pi \) and by \( \text{Theorem 3.3} \) \( \psi F_{i_k}^- \) is a normal derivation. Next, in \( \mathcal{N}E^o \) for each derivation \( \pi' \), there is the derivation \( \pi'^- \), so for each \( j, 1 \leq j \leq k - 1 \), we have the sequence of derivations \( \pi^j_1, \pi^j_2, \ldots, \pi^j_{m_j} \), \( m_j > 1 \), and by \( \text{Theorem 3.3} \) \( \psi F_{i_j}^- \equiv \pi^j_1 \mathcal{E}_{\max} \pi^j_2 \mathcal{E}_{\max} \ldots \mathcal{E}_{\max} \pi^j_{m_j} \equiv \psi F_{i_{j+1}}^- \).

Thus, there is the following sequence of conversions:

\[ \pi \equiv \pi^1_1 \mathcal{E}_{\max} \ldots \mathcal{E}_{\max} \pi^m_{m_1} \equiv \psi F_{i_2}^- \equiv \pi^1_2 \mathcal{E}_{\max} \ldots \mathcal{E}_{\max} \pi^k_{m_k-1} \equiv \psi F_{i_k}, \]

i.e. the derivation \( \pi \mathcal{E}_{\max}\text{-converts into a normal derivation, the derivation } \psi F_{i_k}' \).

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