A NOTE ON THE FEKETE–SZEGÖ PROBLEM
FOR CLOSE-TO-CONVEX FUNCTIONS
WITH RESPECT TO CONVEX FUNCTIONS

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Abstract. We discuss the sharpness of the bound of the Fekete–Szegö functional for close-to-convex functions with respect to convex functions. We also briefly consider other related developments involving the Fekete–Szegö functional $|a_3 - \lambda a_2^2|$ $(0 \leq \lambda \leq 1)$ as well as the corresponding Hankel determinant for the Taylor–Maclaurin coefficients $\{a_n\}_{n\in\mathbb{N}\setminus\{1\}}$ of normalized univalent functions in the open unit disk $\mathbb{D}$, $\mathbb{N}$ being the set of positive integers.

1. Introduction

A classical problem in geometric function theory of complex analysis, which was settled by Fekete and Szegö [4], is to find for each $\lambda \in [0, 1]$ the maximum value of the coefficient functional $\Phi_\lambda(f)$ given by

$$\Phi_\lambda(f) := |a_3 - \lambda a_2^2|$$

over the class $\mathcal{S}$ of univalent functions $f$ in the open unit disk

$$\mathbb{D} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

of the following normalized form (see, for details, [5][22][24]):

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}).$$

By applying the Loewner method, Fekete and Szegö [4] proved that

$$\max_{f \in \mathcal{S}} \Phi_\lambda(f) = \begin{cases} 
1 + 2 \exp \left( -\frac{2\lambda}{1-\lambda} \right) & (0 \leq \lambda < 1) \\
1 & (\lambda = 1).
\end{cases}$$
For various compact subclasses $F$ of the class $A$ of all analytic functions $f$ in $D$ of the form \((1.2)\), as well as with $\lambda$ being an arbitrary real or complex number, many authors computed

\[(1.3) \quad \max_{f \in F} \Phi_\lambda(f)\]

or calculated the upper bound of \((1.3)\) (see, e.g., \([2,8,11,21]\)).

Let $S^*$ denote the class of starlike functions, that is, $f \in S^*$ if

\[f \in A \quad \text{and} \quad \Re \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in D).\]

Given $\delta \in (-\pi/2, \pi/2)$ and $g \in S^*$, let $C_\delta(g)$ denote the class of functions called close-to-convex with argument $\delta$ with respect to $g$, that is, the class of all functions $f \in A$ such that

\[(1.4) \quad \Re \left( e^{i\delta} \frac{zf'(z)}{g(z)} \right) > 0 \quad (z \in D).\]

We also suppose that, given $g \in S^*$, $\mathcal{C}(g) := \bigcup_{g \in S^*} C_\delta(g)$ and that, given $\delta \in (-\pi/2, \pi/2)$, $\mathcal{C}_\delta := \bigcup_{g \in S^*} C_\delta(g)$. Let

\[\mathcal{C} := \bigcup_{\delta \in (-\pi/2, \pi/2)} \mathcal{C}_\delta \bigcup_{g \in S^*} \mathcal{C}(g)\]

denote the class of close-to-convex functions (see, for details, \([20\text{ pp.}\ 184–185}], [6,10]).

For the whole class $\mathcal{C}$, the sharp bound of the Fekete–Szegö coefficient functional $\Phi_\lambda$ for $\lambda \in [0,1]$, given by \([13]\), was calculated by Koeppf \([13]\) who extended the earlier result for the class $\mathcal{C}_0$ and for $\lambda \in \mathbb{R}$ due to Keogh and Merkes \([11]\), namely, it holds

\[\max_{f \in \mathcal{C}} \Phi_\lambda(f) = \max_{f \in \mathcal{C}_0} \Phi_\lambda(f) = \begin{cases} |3 - 4\lambda| & (\lambda \in (-\infty, \frac{1}{4}] \cup [1, \infty)) \\ \frac{1}{4} + \frac{\lambda}{1} & (\lambda \in \left[ \frac{1}{4}, \frac{1}{2} \right]) \\ 1 & (\lambda \in \left( \frac{1}{2}, 1 \right]). \end{cases}\]

For various subclasses of the class of close-to-convex functions, the problem to estimate the coefficient functional $\Phi_\lambda$ is continued in several subsequent works (see, for details, \([9,12,14,16]\)). Some interesting and important subclasses of the class $\mathcal{C}$ are the classes $\mathcal{C}_\delta^c$ and $\mathcal{C}^c$, which are defined below.

Let $S^c$ denote the class of convex functions, that is, $f \in S^c$ if

\[f \in A \quad \text{and} \quad \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in D).\]

Since $S^c \subseteq S^*$, the class $\mathcal{C}_\delta^c := \bigcup_{g \in S^c} C_\delta(g)$ is a proper subclass of the class $\mathcal{C}_\delta$ and the class

\[\mathcal{C}^c := \bigcup_{\delta \in (-\pi/2, \pi/2)} \bigcup_{g \in S^c} C_\delta(g)\]

is a proper subclass of the class $\mathcal{C}$. 

\[\]
The class \( C_{c0} \) was defined by Abdel-Gawad and Thomas [1]. The class \( C_c \) of close-to-convex functions with respect to convex functions was introduced by Srivastava, Mishra and Das [23]. In both of these cited papers, the authors (Abdel-Gawad and Thomas [1] and Srivastava, Mishra and Das [23]) considered the coefficient functional \( \Phi_\lambda \) with \( \lambda \in [0,1] \) also. In fact, in Srivastava, Mishra and Das [23] extended, for the class \( C_c \), the earlier result of Abdel-Gawad and Thomas [1] for the class \( C_{c0} \). However, in each of the above-cited papers, the proof for the sharpness of the bound in (1.3) for \( \lambda \in (\frac{2}{3},1] \) was proposed incorrectly as \( \frac{5}{6} \).

This note is motivated essentially by the earlier papers [1] and [23]. The main purpose of our investigation here is to discuss such sharpness results for the bound in (1.3). We also provide a rather brief consideration of other related developments involving the Fekete–Szegö functional \( |a_3 - \lambda a_2^2| \) (0 \( \leq \) \( \lambda \) \( \leq \) 1) in (1.1) as well as the corresponding Hankel determinant for the Taylor–Maclaurin coefficients \( \{a_n\}_{n \in \mathbb{N} \setminus \{1\}} \) of normalized univalent functions of the form (1.2).

2. Main Observation

As we remarked in Section 1, in both of the afore cited papers [1, 23], the upper bounds of the Fekete–Szegö coefficient functional \( \Phi_\lambda \) (0 \( \leq \) \( \lambda \) \( \leq \) 1) for the classes \( C_{c0} \) and \( C_c \), were computed. In fact, Theorems 5 and 6 of Srivastava, Mishra and Das [23] state that the following sharp inequality (2.1)

\[
\max_{f \in C_c} \Phi_\lambda (f) \leq \frac{5}{6} \quad (\lambda \in \left[\frac{2}{3}, 1\right])
\]

holds true and that this result is the same as in [1] for the class \( C_{c0} \) (a part of Theorem 3). However, the assertion that the extremal function, for which the equality in (2.1) is satisfied when \( \lambda \in (\frac{2}{3},1] \), belongs to \( C_c \) is incorrect. Indeed, here in this section, we note that the above-cited papers [1, 23] contain a statement to the effect that the equality in (2.1) is attained by a function \( f \in A \) given by

\[
zf'(z) \quad h(z) = 1 + \omega(z) \quad (z \in \mathbb{D}),
\]

where \( h \in S_c \) is of the form

\[
h(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathbb{D}; \quad b_2 = b_3 := 1)
\]

and \( \omega \) is a function of the form

\[
\omega(z) = \sum_{n=1}^{\infty} \beta_n z^n \quad (z \in \mathbb{D})
\]

with

\[
\beta_1 := \frac{2 - 3\lambda}{6\lambda} \pm \frac{\sqrt{6\lambda - 4}}{6\lambda} \quad \text{and} \quad \beta_2 := 1 - \beta_1^2.
\]

Unfortunately, however, \( \omega \) is not a Schwarz function for \( \lambda \in (\frac{2}{3},1] \). We recall here that a Schwarz function means an analytic self-mapping of \( \mathbb{D} \) with \( \omega(0) := 0 \). Let us
denote the class of Schwarz functions by \( \mathcal{B}_0 \). In order to see that \( \omega \notin \mathcal{B}_0 \), we verify (by straightforward computation) that, for \( \lambda \in \left( \frac{2}{3}, 1 \right] \), the following inequality:

\[
|\beta_2| \leq 1 - |\beta_1|^2
\]

is false, so a necessary condition for \( \omega \) to be in \( \mathcal{B}_0 \) (see, for example, [5] Vol. II, p. 78) does not hold true. Alternatively, in order to get a contradiction, we suppose that \( \omega \) with its coefficients in (2.5) is a Schwarz function. Thus, clearly, (2.6) holds true. Hence we find from (2.5) that \( 1 - |\beta_1|^2 \geq |\beta_2| = |1 - \beta_1^2| \geq 1 - |\beta_1|^2 \). Thus we have \( |1 - \beta_1^2| = 1 - |\beta_1|^2 \) and, therefore, \( \beta_1 = |\beta_1| \) or \( \beta_1 = -|\beta_1| \). This means that \( \beta_1 \) is a real number, which by (2.5) is possible only for \( \lambda = \frac{2}{3} \). Consequently, for \( \lambda \in \left( \frac{2}{3}, 1 \right] \), the function \( \omega \) with its coefficients in (2.5) does not belong to \( \mathcal{B}_0 \). So, in light of (2.2), it does not follow that \( f \) is in \( \mathcal{C}^c \) or in \( \mathcal{C}_0^c \).

Equivalently, let

\[
p(z) := \frac{1 + \omega(z)}{1 - \omega(z)} \quad (z \in \mathbb{D}),
\]

where \( \omega \) is as given above. Then

\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}),
\]

where, in view of (2.7), (2.4) and (2.5), we have \( c_1 = 2 \beta_1 \) and \( c_2 = 2(\beta_2 + \beta_1^2) = 2 \). We observe further that, for \( \lambda \in \left( \frac{2}{3}, 1 \right] \), the function \( p \) does not belong to the Carathéodory class. We recall here that the Carathéodory class, denoted as \( \mathcal{P} \), contains analytic functions \( p \) of the form (2.8) with a positive real part. In order to see that \( p \notin \mathcal{P} \), we verify for \( \lambda \in \left( \frac{2}{3}, 1 \right] \) that the inequality \( |c_2 - c_1^2/2| \leq 2 - |c_1|^2/2 \), is false, which happens to be a necessary condition for \( p \) to be in the class \( \mathcal{P} \) (see, for example, [22] p. 166).

3. Concluding remarks and further developments

By means of Theorem 3 of Abdel-Gawad and Thomas [1], Theorems 1 to 4 of Srivastava, Mishra and Das [23], and in light of our observation in Section 2, we arrive at the following result.

**Theorem 1.** Each of the following assertions holds true:

\[
\max_{f \in \mathcal{C}^c} \Phi_\lambda(f) = \max_{f \in \mathcal{C}_0^c} \Phi_\lambda(f) = \begin{cases} \frac{5}{8} - \frac{\lambda}{4} & (\lambda \in \left[ 0, \frac{2}{3} \right) ) \\ \frac{5}{8} + \frac{\lambda}{4} & (\lambda \in \left[ \frac{2}{3}, \frac{8}{5} \right) ) \end{cases}
\]

\[
\max_{f \in \mathcal{C}^c} \Phi_\lambda(f) \leq \frac{5}{8} \quad (\lambda \in \left( \frac{2}{3}, 1 \right) ).
\]

**Remark 1.** The sharpness of the inequality in (3.2) for the classes \( \mathcal{C}^c \) and \( \mathcal{C}_0^c \) is an open problem.

We now note that, by Loewner Theorem (see, for example, [5] Vol. I, p. 1127)), the function \( h \in \mathcal{S}^c \) of the form (2.3) (with \( b_2 = b_3 := 1 \)) is uniquely determined, that is, \( h(z) = \frac{1}{1 - z} = \sum_{n=1}^{\infty} z^n \) (\( z \in \mathbb{D} \)). Then (1.3) with \( g := h \) is of the form

\[
\text{Re}(e^{it}(1 - z)f'(z)) > 0 \quad (z \in \mathbb{D})
\]
and defines the class $C_q(h)$, and further the class $C(h)$. For the first time, the inequality in (3.3), treated as the univalence criterion, was distinguished explicitly in [20] p.185. For the class $C(h)$, the upper bound of the Fekete–Szegö coefficient functional $\Phi_\lambda$ for $\lambda \in \mathbb{R}$ was recently obtained in [14], where the following result was proven.

**Theorem 2.** It is asserted that

$$\max_{f \in C(h)} \Phi_\lambda(f) \leq \begin{cases} \left| \frac{1}{2} - \frac{1}{3\lambda} \right| + \frac{2}{3} |2 - 3\lambda| \quad (\lambda \in \left( -\infty, -\frac{2}{3}\right] \cup \left[ \frac{10}{9}, \infty \right)) \\ \left(\frac{1}{12} \cdot \frac{(2-3\lambda)^2}{2-3\lambda} + \frac{1}{3} - \frac{1}{\lambda} \right) + \frac{2}{3} \quad (\lambda \in \left[ \frac{2}{9}, \frac{10}{9}\right]). \end{cases}$$

For each $\lambda \in \left( -\infty, -\frac{2}{3}\right] \cup \left[ \frac{4}{9}, \infty \right)$, the inequality is sharp and the equality in (2) is attained by a function in $C_0(h)$.

**Remark 2.** For $\lambda \in \left( -\infty, -\frac{2}{3}\right] \cup \left[ \frac{4}{9}, \infty \right)$, we can rewrite (3.4) as the following corollary.

**Corollary 1.** The following assertion holds true:

$$\max_{f \in C(h)} \Phi_\lambda(f) = \begin{cases} \left| \frac{1}{2} - \frac{2\lambda}{3}\right| \quad (\lambda \in \left( -\infty, -\frac{2}{3}\right] \cup \left[ \frac{10}{9}, \infty \right)) \\ \left(\frac{1}{4} + \frac{1}{\lambda}\right) \quad (\lambda \in \left[ \frac{2}{9}, \frac{4}{9}\right]). \end{cases}$$

**Remark 3.** For $\lambda \in \left[ 0, \frac{2}{3}\right]$, the result (3.5) asserted by Corollary 3 coincides with (3.1). Thus, naturally, Theorem 1 and Theorem 2 yield Corollary 2 below.

**Corollary 2.** Each of the following assertions holds true:

$$\max_{f \in C(h)} \Phi_\lambda(f) = \max_{f \in C_q(h)} \Phi_\lambda(f) = \max_{f \in C(h)} \Phi_\lambda(f) \quad (\lambda \in \left[ 0, \frac{2}{3}\right]),$$

$$\max_{f \in C(h)} \Phi_\lambda(f) \leq \frac{9\lambda^2 - 30\lambda + 26}{6(4 - 3\lambda)} \leq \frac{5}{6} \quad (\lambda \in \left( \frac{2}{3}, 1\right)).$$

**Remark 4.** The maximum of $\Phi_\lambda$ for $\lambda \in \left[ 0, \frac{2}{3}\right]$, over the class $C^c$ of close-to-convex functions with respect to convex functions and over its subclass $C(h)$ of close-to-convex functions with respect to convex function $h$, are identical.

**Remark 5.** The sharpness of the inequality in (3.3) for $\lambda \in \left( \frac{2}{3}, \frac{4}{3}\right)$ is an open problem.

**Remark 6.** We reiterate the fact that the Fekete–Szegö coefficient functional $\left| a_3 - \lambda a_2^2 \right|$ is well known for its rich history in geometric function theory. Its origin was in the disproof by Fekete and Szegö [4] of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see, for details, [4]). The $\lambda$-generalized Fekete–Szegö coefficient functional $\left| a_3 - \lambda a_2^2 \right|$ has since received great attention, particularly in connection with many subclasses of the class $S$ of normalized analytic and univalent functions. On the other hand, in the year 1976, Noonan and Thomas [17] defined the $q$th Hankel determinant of
the function $f$ in (1.2) by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (n, q \in \mathbb{N}; a_1 := 1).$$

The determinant $H_q(n)$ has also been considered by several other authors. For example, Noor [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions $f$ given by (1.1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained in the recent works [7,18] for different classes of functions. We note, in particular, that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad \text{and} \quad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$ 

The Hankel determinant $H_2(1) = a_3 - a_2^2$ is the classical Fekete–Szegö coefficient functional. The upper bounds of $H_2(2)$ for some specific analytic function classes were discussed quite recently by Deniz et al. [3] (see also [19]).

References

8. Z. J. Jakubowski, Sur le maximum de la fonctionnelle $|A_3 - \alpha A_2^2|$ ($0 \leq \alpha < 1$) dans la famille de fonctions $F_M$, Bull. Soc. Sci. Lett. Łódź 13(1) (1962), 1–19.