ON A SUBCLASS OF MULTIVALENT CLOSE TO CONVEX FUNCTIONS

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Abstract. We introduce a new subclass of multivalent close to convex functions related with Janowski functions and study some of their properties: coefficient estimates, inclusion and inverse inclusion, distortion problems and sufficiency criteria to be in these subclasses.

1. Introduction

Let \( \mathcal{A}(p) \) denote the class of functions \( f(z) \) which are analytic and \( p \)-valent in the region \( \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} \) and normalized by the condition

\[
f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad (z \in \mathbb{U}).
\]

We write \( \mathcal{A}(1) = \mathcal{A} \). Robertson introduced in \([9]\) the class \( \mathcal{S}^*(\alpha) \) of starlike functions of order \( \alpha \leq 1 \), which are defined by

\[
\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{A} : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \quad z \in \mathbb{U} \right\}.
\]

By \( \mathcal{S}^* = \mathcal{S}^*(0) \) we denote the subclasses of \( \mathcal{A} \) which consist of univalent starlike functions. An important subclass of analytic functions is the class \( \mathcal{K} \) of close-to-convex functions

\[
\mathcal{K} = \left\{ f \in \mathcal{A} : \exists \beta \in \mathbb{R}, \exists g \in \mathcal{S}^* : \Re \left\{ \frac{zf'(z)}{e^{\beta}g(z)} \right\} > 0, \quad z \in \mathbb{U} \right\}.
\]

Each close-to-convex function is univalent in the unit disc. For two functions \( f(z) \) and \( g(z) \) analytic in \( U \), we say that \( f(z) \) is subordinate to \( g(z) \), denoted by \( f(z) \prec g(z) \), if there is an analytic function \( w(z) \) with \( |w(z)| \leq |z| \) such that \( f(z) = g(w(z)) \). If \( g(z) \) is univalent, then \( f(z) \prec g(z) \) if and only if \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

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In [11] Sakaguchi introduced the class $S^*_a$ of starlike functions with respect to symmetric points; a function $f(z) \in \mathcal{A}$ belongs to the class $S^*_a$, if and only if
\[
\frac{zf'(z)}{f(z) - f(-z)} \prec \frac{1 + z}{1 - z}, \quad (z \in \mathbb{U}).
\]
One can easily obtain that the function $(f(z) - f(-z))/2$ is starlike in $\mathbb{U}$ and therefore the functions in $S^*_a$ are close-to-convex. Motivated from Sakaguchi’s work, Gao and Zhou [3] introduced a class $K_a$. A function $f(z) \in \mathcal{A}$ belongs to the class $K_a$ if it satisfies the subordination
\[
-z^2 f'(z) \prec g(z)g(-z), \quad (z \in \mathbb{U}),
\]
for some $g(z) \in S^*(1/2)$.

In [6] it was introduced the class $K_\gamma(\gamma)$ of functions satisfying
\[
(1.2)
\]
\[
-z^2 f'(z) \prec g(z)g(-z), \quad (z \in \mathbb{U}),
\]
The class $K_\gamma(\gamma)$ has been generalized in several directions, see the references in [7]. Recently, Xu, Srivastava and Li considered in [15] the class $K_\gamma(h)$ of functions satisfying (1.2) with a convex function $h$ instead of $q_\gamma$. Şeker introduced in [12] the class $K_\gamma^a(\gamma)$, $k > 1$, of functions defined by (1.2) with
\[
-z^{2-k} \prod_{\nu=0}^{k-1} e^{-\nu} g(e^{\nu} z), \quad g \in S^*((k-1)/k),
\]
instead of $g(z)g(-z)$. Moreover, Wang, Sun and Xu introduced in [14] the class $\mathcal{M}_K$ of meromorphic functions satisfying (1.2) with $\gamma = 1$. See also the references in [14] for the other papers in this topic.

If $f \in A(p)$, $\alpha < 1$ and $\Re \frac{zf'(z)}{f(z)} > p\alpha$, $z \in \mathbb{U}$, then we say that $f$ is in the class $S^*_p(\alpha)$ of $p$-valent starlike functions of order $\alpha$. Using the techniques of subordination, we now introduce a subclass of $p$-valent analytic functions as follows.

**Definition 1.1.** A function $f(z) \in A(p)$ is said to be in the class $\mathcal{W}_p(t, \lambda, A, B)$, $0 < |t| \leq 1$, $-1 < B < A \leq 1$ and $\lambda \in (0,1)$, if it satisfies
\[
(1.3)
\]
\[
\frac{p z^{p+1} f'(z)}{pg(z)g(tz)} \prec q(z) := \frac{1 + Az}{1 + Bz}
\]
for some $g(z) \in S^*_p(1/2)$ and $F_\lambda(z)$ is defined by $F_\lambda(z) = (1 - \lambda)f(z) + \frac{\lambda}{p} z f'(z)$.

In the literature, various interesting subclasses of this class have been studied from a number of different view points. For example: if we set $t = -1$, $p = 1$ and $\xi = 0$ in Definition [11] we get the class $\mathcal{W}_1(-1, \lambda, A, B) \equiv K_\lambda(\lambda, A, B)$ which was studied recently by Wang and Chen [13] and further for $\lambda = 0$, $A = 1 - 2\gamma$ and $B = -1$, we obtain the class $K_\gamma(\gamma)$ introduced in [6]. For more details of the related work see [11, 12, 4, 8, 12, 15].

The main object of the present paper is to introduce a subclass of $p$-valent analytic functions and then investigate some useful results including the coefficient...
estimate, sufficiency criteria to be in a class, distortion problem, radius of convexity and inclusion relationship for the new defined class.

To avoid repetition, we shall assume, unless otherwise stated, that \( \lambda \in (0, 1] \), \(-1 \leq B < A \leq 1 \), and \( 0 < |t| \leq 1 \).

2. Some properties of the class \( W_p(t, \lambda, A, B) \)

**Theorem 2.1.** Let \( g_i(z) \in S^*_{p_i}(\alpha_i) \) with \( \alpha_i < 1 \). Then

\[
G(z) = \frac{g_1(t_1 z)g_2(t_2 z)}{t_1^\gamma t_2^\gamma z^\gamma} \in S^*_p(\gamma),
\]

where \( \gamma = \alpha_1 + \alpha_2 - 1 \) and \( 0 < |t_i| \leq 1, i = 1, 2 \).

**Proof.** Let \( g_i(z) \in S^*_p(\alpha_i) \). Then by definition we have

\[
\text{Re} \left( \frac{t_1 z g_1'(t_1 z)}{g_1(t_1 z)} > p_{\alpha_1}, \quad \text{Re} \left( \frac{t_2 z g_2'(t_2 z)}{g_2(t_2 z)} > p_{\alpha_2}, \quad |z| < 1, \ 0 < |t_i| \leq 1.\right.
\]

By logarithmic differentiating (2.1), we obtain that

\[
\frac{zG'(z)}{G(z)} = \frac{t_1 z g_1'(t_1 z)}{g_1(t_1 z)} + \frac{t_2 z g_2'(t_2 z)}{g_2(t_2 z)} - 1.
\]

It follows that

\[
\text{Re} \left( \frac{zG'(z)}{G(z)} \right) = \text{Re} \left( \frac{t_1 z g_1'(t_1 z)}{g_1(t_1 z)} \right) + \text{Re} \left( \frac{t_2 z g_2'(t_2 z)}{g_2(t_2 z)} \right) - 1 > p_{\alpha_1} + p_{\alpha_2} - p = p\gamma.
\]

It implies that \( G(z) \in S^*_p(\gamma) \) and it completes the proof of the theorem. \( \Box \)

**Corollary 2.1.** If \( g(z) \in S^*_p(1/2) \) and \( 0 < |t| < 1 \), then

\[
\frac{g(z)g(tz)}{t^p z^p} \in S^*_p(0) := S^*_p.
\]

**Theorem 2.2.** If \(-1 < D \), then \( W_p(t, \lambda, A, B) \subset W_p(t, \lambda, C, D) \) if and only if

\[
\left| \frac{1 - CD}{1 - D^2} - \frac{1 - AB}{1 - B^2} \right| \leq \frac{C - D}{1 - D^2} - \frac{A - B}{1 - B^2}.
\]

If \(-1 = D \), then \( W_p(t, \lambda, A, B) \subset W_p(t, \lambda, C, D) \) if and only if

\[
C \geq 1 - \frac{2(1 - A)}{1 - B}.
\]

**Proof.** Condition (1.3) means that for \( z \in U \) the values of the function

\[
H(z) := \frac{t^p z^{p+1} F'(z)}{p g(z) g(tz)}
\]

lie in \( q(U) \) because \( q(z) = (1 + Az)/(1 + Bz) \) is univalent in \( U \). In the case \( B \neq -1 \),

\[
q(U) \text{ is a disc } D(A, B), \text{ with a center } S(A, B) \text{ and a radius } R(A, B)
\]

\[
S(A, B) = \frac{1 - AB}{1 - B^2}, \quad R(A, B) = \frac{A - B}{1 - B^2},
\]
while it is a half-plane for $B = -1$. By simple computation, we can easily obtain that (1.3) is equivalent to

$$\left| \frac{t^p z^{p+1} F'(z)}{pg(z)g(tz)} - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2}, \quad B \neq -1,$$

or

$$(2.4) \quad \text{Re}\left\{ \frac{t^p z^{p+1} F'(z)}{pg(z)g(tz)} \right\} > \frac{1 - A}{2}, \quad B = -1.$$

Therefore, for the case $B \neq -1$ $D \neq -1$, the inclusion relation $W_p(t, \lambda, A, B) \subset W_p(t, \lambda, C, D)$ holds when

$$R(A, B) \leq R(C, D), \quad \text{and} \quad |S(C, D) - S(A, B)| \leq R(C, D) - R(A, B).$$

This is equivalent to

$$\left| \frac{1 - CD}{1 - D^2} - \frac{1 - AB}{1 - B^2} \right| \leq \frac{C - D}{1 - D^2} - \frac{A - B}{1 - B^2}.$$

If $D = -1$, then by (2.4), the inclusion relation $W_p(t, \lambda, A, B) \subset W_p(t, \lambda, C, D)$ holds when $\frac{C - D}{1 - D^2} \leq \frac{A - B}{1 - B^2}$. This is equivalent to (2.2).

**Lemma 2.1.** If $g(z) \in S_p^*(1/2)$ and it has the form (1.1), then

$$|a_{n+p}t^n + a_{n+p-1}a_{p+1}t^{n-1} + \ldots + a_{p+1}a_{n+p-1}t + a_{n+p}| \leq \frac{2p}{n} \prod_{i=1}^{n-1} \left( 1 + \frac{2p}{i} \right).$$

**Proof.** By virtue of Corollary 2.1, we have $\frac{g(z)g(tz)}{t^{p+1}} \in S_p^*(0)$, and if

$$(2.5) \quad G(z) = \frac{g(z)g(tz)}{t^{p+1}} = z^p + \sum_{k=p+1}^{\infty} c_k z^k,$$

then it is well known that

$$(2.6) \quad |c_{p+n}| \leq \frac{2p}{n} \prod_{i=1}^{n-1} \left( 1 + \frac{2p}{i} \right).$$

Substituting the series expansions of $G(z)$ and $g(z)$ in (2.5), we get

$$\left( z^p + \sum_{k=p+1}^{\infty} a_k z^k \right) \left( (tz)^p + \sum_{k=p+1}^{\infty} a_k (tz)^k \right) = z^p + \sum_{k=p+1}^{\infty} c_k z^k,$$

Comparing the coefficients of $z^{n+p}$, we have

$$(2.7) \quad a_{p+n}t^n + a_{p+1}a_{p+n-1}t^{n-1} + a_{p+2}a_{p+n-2}t^{n-2} + \ldots + a_{p+n} = c_{p+n}.$$

Putting the value from (2.6) in (2.7), we get the required result.

**Theorem 2.3.** Let $f(z) \in W_p(t, \lambda, A, B)$ be of the form (1.1). Then

$$|a_{p+n}| \leq (A - B) \left[ 1 + \sum_{i=1}^{n-1} \left( \frac{2p}{i} \right) \prod_{j=1}^{i-1} \left( 1 + \frac{2p}{j} \right) \right] + \frac{2p}{n} \prod_{i=1}^{n-1} \left( \frac{2p}{i} \right).$$
By using (2.10) and (2.6) we have

\[ \frac{zF(z)}{pG(z)} = \frac{1 + Az}{1 + Bz}, \]

where \( G(z) \) is given by (2.8). If we put

\[ q(z) = \frac{zF(z)}{pG(z)}, \]

it follows from (2.8) that

\[ q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n = 1 + \sum_{n=1}^{\infty} A_n z^n, \quad A_n = (A - B)(-B)^{n-1}. \]

The function \((1 + Az)/(1 + Bz)\) is convex univalent, hence applying the well known Rogosinski result \([10]\), we obtain

\[ |q_n| \leq A_1 = A - B, \quad n = 1, 2, \ldots. \]

Now by putting the series expansions of \( f(z) \), \( G(z) \) and \( q(z) \) in (2.9) and then comparing the coefficients of \( z^n+p \), we obtain

\[ \frac{1}{p}(p + n\lambda)(p + n)a_{p+n} = c_{p+n} + q_{p+1}c_{p+n-1} + \cdots + q_{p+n-1}c_{p+1} + q_{p+n}. \]

By using (2.10) and (2.6) we have

\[ \frac{1}{p}(p + n\lambda)(p + n)|a_{p+n}| \leq (A - B)(|c_{p+n-1}| + \cdots + |c_{p+1}| + 1) + |c_{p+n}| \]

\[ \leq (A - B) \left[ 1 + \sum_{i=1}^{n-1} |c_{p+i}| \right] + \left( \frac{2p}{n} \right) \prod_{i=1}^{n-1} \left( 1 + \frac{2p}{i} \right) \]

\[ \leq (A - B) \left[ 1 + \sum_{i=1}^{n-1} \left( \frac{2p}{i} \right)^{i-1} \prod_{j=1}^{i-1} \left( 1 + \frac{2p}{j} \right) + \left( \frac{2p}{n} \right) \prod_{i=1}^{n-1} \left( 1 + \frac{2p}{i} \right) \right]. \]

This completes the proof. \( \square \)

**Theorem 2.4.** If \( g(z) \in \mathcal{S}_p^+(1/2) \) and \( f(z) \in A(p) \) is of the form (1.4), and if it satisfies the condition

\[ \frac{1 + A}{2} - \sum_{n=1}^{\infty} \frac{(1 + n\lambda/p)(p + n)}{p} |a_{p+n}| - \frac{1 - A}{2} \sum_{n=1}^{\infty} |c_{p+n}| > 0, \]

where \( c_{p+n} \) is given by

\[ G(z) = \frac{g(z)g(tz)}{trzp} = z^p + \sum_{n=1}^{\infty} c_{p+n} z^{p+n}, \]

then \( f(z) \in \mathcal{W}_p(t, \lambda, A, -1). \) Moreover, if it satisfies the condition

\[ \frac{A - B}{1 + B} \sum_{n=1}^{\infty} \left\{ \frac{(1 + n\lambda/p)(p + n)}{p} |a_{p+n}| + \left( \frac{A - B}{1 - B^2} + \frac{1 - AB}{1 - B^2} \right) |c_{p+n}| \right\} > 0, \]

then \( f(z) \in \mathcal{W}_p(t, \lambda, A, B), B > -1. \)
Proof. To prove that \( f(z) \in W_p(t, \lambda, A, -1) \), it is enough to show that
\[
\left| \frac{zF'_1(z)}{pG(z)} \right| > \frac{1 - A}{2}, \quad z \in \mathbb{U}
\]
or equivalently to show that
\[
\left| \frac{F'_1(z)}{pz^{p-1}} - \frac{(1 - A)G(z)}{2z^p} \right| > 0, \quad z \in \mathbb{U}.
\]
We have
\[
\left| \frac{F'_1(z)}{pz^{p-1}} - \frac{(1 - A)G(z)}{2z^p} \right|
\]
\[
= \left| 1 + \sum_{n=1}^{\infty} \frac{(1 + n\lambda/p)(p + n)}{p} a_{p+n}z^n \right| - \frac{(1 - A)}{2} \left(1 + \sum_{n=1}^{\infty} c_{p+n}z^n\right)
\]
\[
\geq 1 - \sum_{n=1}^{\infty} \frac{(1 + n\lambda/p)(p + n)}{p} a_{p+n}z^n - \frac{(1 - A)}{2} \left(1 + \sum_{n=1}^{\infty} c_{p+n}z^n\right)
\]
\[
= \frac{1 + A}{2} - \sum_{n=1}^{\infty} \frac{(1 + n\lambda/p)(p + n)}{p} a_{p+n}z^n - \frac{1 - A}{2} \sum_{n=1}^{\infty} c_{p+n}z^n \geq 0
\]
by (2.11). For the case \( B > -1 \) it suffices to show that
\[
\left| \frac{zF'_1(z) - S(A, B)}{pG(z)} \right| < R(A, B),
\]
where \( S(A, B) \) and \( R(A, B) \) are given in (2.3). This is equivalent to
\[
\left| \frac{F'_1(z)}{pz^{p-1}} - \frac{G(z)S(A, B)}{z^p} \right| < \left| \frac{G(z)R(A, B)}{z^p} \right|, \quad z \in \mathbb{U}.
\]
We have
\[
\left| \frac{G(z)R(A, B)}{z^p} \right|
\]
\[
= \left| \frac{F'_1(z)}{pz^{p-1}} - \frac{G(z)S(A, B)}{z^p} \right|
\]
\[
= \left| R(A, B) \left(1 + \sum_{n=1}^{\infty} c_{p+n}z^n\right) \right|
\]
\[
= \left| 1 + \sum_{n=1}^{\infty} \frac{(1 + n\lambda/p)(p + n)}{p} a_{p+n}z^n - S(A, B)1 + \sum_{n=1}^{\infty} c_{p+n}z^n \right|
\]
\[
= \left| R(A, B) \left(1 + \sum_{n=1}^{\infty} c_{p+n}z^n\right) \right|
\]
\[
= \left| \frac{B(A - B)}{1 - B^2} + \sum_{n=1}^{\infty} \left\{ \frac{(1 + n\lambda/p)(p + n)}{p} a_{p+n} - S(A, B)c_k \right\} z^n \right|
\]
shown in Corollary 2.1 that $G\ w$ where $\Box$ by (2.12).

Now by using (2.14) in (2.13), we obtain the required result. $\Box$

(2.14)

Applying [5] Theorem 2.5. Let $f(z) \in W_p(t, 0, A, B)$. Then
\[
\frac{1 - Ar}{1 - Br} \leq \frac{(1 - r)^{2p}}{(1 + r^2)^n} \leq |f'(z)| \leq \frac{1 + Ar}{1 + Br} \frac{(1 + r)^{2p}}{r^{2p} + 1}.
\]

Proof. Suppose that $f(z) \in W_p(t, 0, A, B)$. Then by using definition of subordination between analytic functions, we can write
\[
(2.13) \quad \frac{1 - Ar}{1 - Br} \leq \frac{1 - A|w(z)|}{1 - B|w(z)|} \leq \frac{|z f'(z)|}{pG(z)} \leq \frac{1 + A|w(z)|}{1 + B|w(z)|} \leq \frac{1 + Ar}{1 + Br}.
\]

where $w(z)$ is the Schwarz function with $w(0) = 0$, $|w(z)| < |z| = r$. Now it is shown in Corollary 2.1 that $G(z) = \frac{g(z)(t(z))}{r^{2p} + 1} \in S^*_p(0)$, thus we have
\[
(2.14) \quad \frac{r^p}{(1 + r)^{2p}} \leq |G(z)| \leq \frac{r^p}{(1 - r)^{2p}}.
\]

Now by using (2.14) in (2.13), we obtain the required result. $\Box$

Theorem 2.6. Let $f(z) \in W_p(t, 0, 1, B)$. Then
\[
1 + \frac{zf''(z)}{f'(z)} > 0, \quad |z| < r_0,
\]

where $r_0$ is the smallest positive root of the equation
\[
(2.15) \quad p(1 - r)^2(1 - Br) - r(1 + r)(1 - B) = 0.
\]

Proof. Suppose $f(z) \in W_p(t, 0, 1, B)$ and let
\[
\frac{zf'(z)}{pG(z)} = q(w(z)) := s(z), \quad |w(z)| < |z|.
\]

where $q(z)$ is given in (1.3). Logarithmic differentiation gives us
\[
(2.16) \quad 1 + \frac{zf''(z)}{f'(z)} = \frac{zG'(z)}{G(z)} + \frac{zs'(z)}{s(z)}
\]

Applying [5] Theorem 3] we have
\[
(2.17) \quad \Re \frac{zs'(z)}{s(z)} > -\frac{(1 - B)r}{(1 - r)(1 - Br)}, \quad |z| < r.
\]
Moreover, $G(z) \in S^*_p(0)$, so

$$\text{Re} \left\{ \frac{zG''(z)}{pG(z)} \right\} > \frac{1 - r^p}{1 + r^p}, \quad |z| < r. \quad \text{(2.18)}$$

Applying (2.17) and (2.18) in (2.16), we obtain

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > p \frac{1 - r^p}{1 + r^p} \frac{(1 - B)r}{(1 - r)(1 - Br)}$$

$$= \frac{p(1 - r)^2(1 - Br) - r(1 + r)(1 - B)}{(1 - r^2)(1 - Br)}$$

and this is positive for $r < r_0$, where $r_0$ is described in (2.15). \qed

References