PELL NUMBERS WHOSE EULER FUNCTION IS A PELL NUMBER

Bernadette Faye and Florian Luca

Abstract. We show that the only Pell numbers whose Euler function is also a Pell number are 1 and 2.

1. Introduction

Let \( \phi(n) \) be the Euler function of the positive integer \( n \). Recall that if \( n \) has the prime factorization

\[
n = p_1^{a_1} \cdots p_k^{a_k}
\]

with distinct primes \( p_1, \ldots, p_k \) and positive integers \( a_1, \ldots, a_k \), then

\[
\phi(n) = p_1^{a_1-1}(p_1 - 1) \cdots p_k^{a_k-1}(p_k - 1).
\]

There are many papers in the literature dealing with diophantine equations involving the Euler function in members of a binary recurrent sequence. For example, in [11], it is shown that 1, 2, and 3 are the only Fibonacci numbers whose Euler function is also a Fibonacci number, while in [4] it is shown that the Diophantine equation \( \phi(5^n - 1) = 5^m - 1 \) has no positive integer solutions \( (m, n) \). Furthermore, the divisibility relation \( \phi(n) \mid n - 1 \) when \( n \) is a Fibonacci number, or a Lucas number, or a Cullen number (that is, a number of the form \( n2^n + 1 \) for some positive integer \( n \)), or a rep-digit \( (g^m - 1)/(g - 1) \) in some integer base \( g \in [2, 1000] \) have been investigated in [10, 5, 7, 3], respectively.

Here we look for a similar equation with members of the Pell sequence. The Pell sequence \( (P_n)_{n\geq0} \) is given by \( P_0 = 0, P_1 = 1 \) and \( P_{n+1} = 2P_n + P_{n-1} \) for all \( n \geq 0 \). Its first terms are

\[
0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, 470832, \ldots
\]

We have the following result.

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Theorem 1.1. The only solutions in positive integers \((n, m)\) of the equation
\[(1.1) \quad \phi(P_n) = P_m\]
are \((n, m) = (1, 1), (2, 1)\).

For the proof, we begin by following the method from [11], but we add to it some ingredients from [10].

2. Preliminary results

Let \((\alpha, \beta) = (1 + \sqrt{2}, 1 - \sqrt{2})\) be the roots of the characteristic equation \(x^2 - 2x - 1 = 0\) of the Pell sequence \(\{P_n\}_{n \geq 0}\). The Binet formula for \(P_n\) is
\[P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for all} \quad n \geq 0.
\]
This implies easily that the inequalities
\[(2.1) \quad \alpha^{n-2} \leq P_n \leq \alpha^{n-1}\]
hold for all positive integers \(n\).

We let \(\{Q_n\}_{n \geq 0}\) be the companion Lucas sequence of the Pell sequence given by \(Q_0 = 2, Q_1 = 2\) and \(Q_{n+2} = 2Q_{n+1} + Q_n\) for all \(n \geq 0\). Its first few terms are 2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, 16238, 39202, 94642, 228486, 551614, \ldots

The Binet formula for \(Q_n\) is
\[(2.2) \quad Q_n = \alpha^n + \beta^n \quad \text{for all} \quad n \geq 0.
\]
We use the well-known result.

Lemma 2.1. The relations (i) \(P_{2n} = P_n Q_n\) and (ii) \(Q_n^2 - 8P_n^2 = 4(-1)^n\) hold for all \(n \geq 0\).

For a prime \(p\) and a nonzero integer \(m\) let \(\nu_p(m)\) be the exponent with which \(p\) appears in the prime factorization of \(m\). The following result is well known and easy to prove.

Lemma 2.2. The relations (i) \(\nu_2(Q_n) = 1\) and (ii) \(\nu_2(P_n) = \nu_2(n)\) hold for all positive integers \(n\).

The following divisibility relations among the Pell numbers are well known.

Lemma 2.3. Let \(m\) and \(n\) be positive integers. We have:
(i) If \(m \mid n\), then \(P_m \mid P_n\), (ii) \(\gcd(P_m, P_n) = P_{\gcd(m, n)}\).

For each positive integer \(n\), let \(z(n)\) be the smallest positive integer \(k\) such that \(n \mid P_k\). It is known that this exists and \(n \mid P_m\) if and only if \(z(n) \mid m\). This number is referred to as the order of appearance of \(n\) in the Pell sequence. Clearly, \(z(2) = 2\). Further, putting for an odd prime \(p, e_p = \left(\frac{2}{p}\right)\), where the above notation stands for the Legendre symbol of 2 with respect to \(p\), we have that \(z(p) \mid p - e_p\). A prime factor \(p\) of \(P_n\) such that \(z(p) = n\) is called primitive for \(P_n\). It is known that \(P_n\) has a primitive divisor for all \(n \geq 2\) (see [2] or [11]). Write \(P_{z(p)} = p^{e_p}m_p\), where
$m_p$ is coprime to $p$. It is known that if $p^e | P_n$ for some $k > e_p$, then $p_2^e(p) | n$. In particular,
\[(2.3) \quad \nu_p(P_n) \leq e_p \quad \text{whenever} \quad p | n.\]
We need a bound on $e_p$. We have the following result.

**Lemma 2.4.** The inequality
\[(2.4) \quad e_p \leq \frac{(p + 1) \log \alpha}{2 \log p}\]
holds for all primes $p$.

**Proof.** Since $e_2 = 1$, the inequality holds for the prime 2. Assume that $p$ is odd. Then $z(p) | p + \varepsilon$ for some $\varepsilon \in \{\pm 1\}$. Furthermore, by Lemmas 2.1 and 2.3 we have $p^e(p) | P_{z(p)} = P_{p+\varepsilon}Q_{(p+\varepsilon)/2}$. By Lemma 2.1, it follows easily that $p$ cannot divide both $P_n$ and $Q_n$ for $n = (p + \varepsilon)/2$ since otherwise $p$ will also divide $Q_{n}^2 - 8P_n^2 = \pm 4$, a contradiction since $p$ is odd. Hence, $p^e(p)$ divides one of $P_{(p+\varepsilon)/2}$ or $Q_{(p+\varepsilon)/2}$. If $p^e(p)$ divides $P_{(p+\varepsilon)/2}$, we have, by (2.1), that $p^e(p) \leq P_{(p+\varepsilon)/2} \leq P_{(p+1)/2} < \alpha^{(p+1)/2}$, which leads to the desired inequality (2.4) upon taking logarithms of both sides. In case $p^e(p)$ divides $Q_{(p+\varepsilon)/2}$, we use the fact that $Q_{(p+\varepsilon)/2}$ is even by Lemma 2.2 (i). Hence, $p^e(p)$ divides $Q_{(p+\varepsilon)/2}/2$, therefore, by formula (2.2), we have
\[p^e(p) \leq \frac{Q_{(p+\varepsilon)/2}}{2} \leq \frac{Q_{(p+1)/2}}{2} < \alpha^{(p+1)/2} < \alpha^{(p+1)/2},\]
which leads again to the desired conclusion by taking logarithms of both sides. \qed

For a positive real number $x$ we use $\log x$ for the natural logarithm of $x$. We need some inequalities from the prime number theory. For a positive integer $n$ we write $\omega(n)$ for the number of distinct prime factors of $n$. The following inequalities (i), (ii) and (iii) are inequalities (3.13), (3.29) and (3.41) in [15], while (iv) is Théorème 13 from [6].

**Lemma 2.5.** Let $p_1 < p_2 < \cdots$ be the sequence of all prime numbers. We have:
\begin{enumerate}
\item [(i)] The inequality $p_n < n(\log n + \log \log n)$ holds for all $n \geq 6$.
\item [(ii)] The inequality
\[\prod_{p \leq x} \left(1 + \frac{1}{p - 1}\right) < 1.79 \log x \left(1 + \frac{1}{2 \log x^2}\right)\]
holds for all $x \geq 286$.
\item [(iii)] The inequality
\[\phi(n) > \frac{n}{1.79 \log \log n + 2.5/\log \log n}\]
holds for all $n \geq 3$.
\end{enumerate}
(iv) The inequality

$$\omega(n) < \frac{\log n}{\log \log n - 1.1714}$$

holds for all \( n \geq 26 \).

For a positive integer \( n \), we put \( \mathcal{P}_n = \{ p : z(p) = n \} \). We need the following result.

**Lemma 2.6.** Put \( S_n := \sum_{p \in \mathcal{P}_n} \frac{1}{p-1} \). For \( n > 2 \), we have

$$S_n < \min \left\{ \frac{2 \log n}{n}, \frac{4 + 4 \log \log n}{\phi(n)} \right\}.$$  

**Proof.** Since \( n > 2 \), it follows that every prime factor \( p \in \mathcal{P}_n \) is odd and satisfies the congruence \( p \equiv \pm 1 \pmod{n} \). Further, putting \( \ell_n := \# \mathcal{P}_n \), we have

$$(n-1)^{\ell_n} \leq \prod_{p \in \mathcal{P}_n} p \leq \alpha^{n-1}$$

(by inequality (2.1)), giving

$$\ell_n \leq \frac{(n-1) \log \alpha}{\log(n-1)}.$$  

Thus, the inequality

$$\ell_n < \frac{n \log \alpha}{\log n}$$

holds for all \( n \geq 3 \), since it follows from (2.7) for \( n \geq 4 \) via the fact that the function \( x \mapsto x/\log x \) is increasing for \( x \geq 3 \), while for \( n = 3 \) it can be checked directly. To prove the first bound, we use (2.7) to deduce that

$$S_n \leq \sum_{1 \leq \ell \leq \ell_n} \left( \frac{1}{n\ell - 2} + \frac{1}{n\ell} \right) \leq \frac{2}{n} \sum_{1 \leq \ell \leq \ell_n} \frac{1}{\ell} + \sum_{m \geq n} \left( \frac{1}{m-2} - \frac{1}{m} \right)$$

$$\leq \frac{2}{n} \left( \int_1^n \frac{dt}{t} + 1 \right) + \frac{1}{n-2} + \frac{1}{n-1} \leq \frac{2}{n} \left( \log \ell_n + 1 + \frac{n}{n-2} \right)$$

$$\leq \frac{2}{n} \log \left( \frac{(\log \alpha) e^{2+2/(n-2)}}{\log n} \right).$$

Since the inequality \( \log n > (\log \alpha) e^{2+2/(n-2)} \) holds for all \( n \geq 800 \), (2.8) implies that \( S_n < \frac{2}{n} \log n \) for \( n \geq 800 \). The remaining range for \( n \) can be checked on an individual basis. For the second bound on \( S_n \), we follow the argument from [10] and split the primes in \( \mathcal{P}_n \) in three groups:

(i) \( p < 3n \); (ii) \( p \in (3n, n^2) \); (iii) \( p > n^2 \);

We have

$$T_1 = \sum_{\substack{p \in \mathcal{P}_n \atop p < 3n}} \frac{1}{p-1} \leq \left( \frac{1}{n-2} + \frac{1}{n} + \frac{1}{2n-2} + \frac{1}{2n} + \frac{1}{3n-2} \right) < \frac{10.1}{3n}, \quad \text{n even},$$

$$< \frac{10.1}{4n}, \quad \text{n odd},$$
where the last inequalities above hold for all $n \geq 84$. For the remaining primes in $P_n$, we have

\[
(2.10) \quad \sum_{p \in P_n, p > 3n} \frac{1}{p - 1} < \sum_{p \in P_n, p > 3n} \frac{1}{p} + \sum_{m \geq 3n + 1} \left( \frac{1}{m - 1} - \frac{1}{m} \right) = T_2 + T_3 + \frac{1}{3n},
\]

where $T_2$ and $T_3$ denote the sums of the reciprocals of the primes in $P_n$ satisfying (ii) and (iii), respectively. The sum $T_2$ was estimated in [10] using the large sieve inequality of Montgomery and Vaughan [13] (see also page 397 in [11]), and the bound on it is

\[
(2.11) \quad T_2 = \sum_{3n < p < n^2} \frac{1}{p} < \sum_{p \mid n} \frac{4}{\phi(n) \log n} < \frac{1}{\phi(n)} + \frac{4 \log \log n}{\phi(n)},
\]

where the last inequality holds for $n \geq 55$. Finally, for $T_3$, we use estimate (2.7) on $\ell_n$ to deduce that

\[
(2.12) \quad T_3 < \frac{\ell_n}{n^2} < \frac{\log \alpha}{n \log n} < \frac{0.9}{3n},
\]

where the last bound holds for all $n \geq 19$. To summarize, for $n \geq 84$, we have, by (2.9), (2.10), (2.11) and (2.12),

\[
S_n < \frac{10.1}{3n} + \frac{1}{3n} + \frac{0.9}{\phi(n)} + \frac{1}{\phi(n)} + \frac{4 \log \log n}{\phi(n)} = \frac{4}{n} + \frac{1}{\phi(n)} + \frac{4 \log \log n}{\phi(n)} \leq 3 + \frac{4 \log \log n}{\phi(n)}
\]

for $n$ even, which is stronger than the desired inequality. Here, we used that $\phi(n) \leq n/2$ for even $n$. For odd $n$, we use the same argument except that the first fraction $10.1/(3n)$ on the right-hand side above gets replaced by $7.1/(3n)$ (by (2.9)), and we only have $\phi(n) \leq n$ for odd $n$. This was for $n \geq 84$. For $n \in [3, 83]$, the desired inequality can be checked on an individual basis.

The next lemma from [9] gives an upper bound on the sum appearing in the right-hand side of (2.5).

**Lemma 2.7.** We have

\[
\sum_{d \mid n} \frac{\log d}{d} < \left( \sum_{p \mid n} \frac{\log p}{p - 1} \right) \frac{n}{\phi(n)}.
\]

Throughout the rest of this paper we use $p$, $q$, $r$ with or without subscripts to denote prime numbers.

### 3. Proof of the Theorem

**3.1. A bird’s-eye view of the proof of the Theorem.** In this section, we explain the plan of attack for the proof of the Theorem. We assume $n > 2$. We put $k$ for the number of distinct prime factors of $P_n$ and $\ell = n - m$. We first show that $2^k \mid m$ and that any putative solution must be large. This only uses the fact that $p - 1 \mid \phi(P_n) = P_m$ for all prime factors $p$ of $P_n$, and all such primes with at most one exception are odd. We show that $k \geq 416$ and $n > m \geq 2^{416}$. This is Lemma 3.1. We next bound $\ell$ in terms of $n$ by showing that $\ell < \log \log \log n / \log \alpha + 1.1$
Next we show that \( k \) is large, by proving that \( 3^k > n/6 \) (Lemma 3.3). When \( n \) is odd, then every prime factor of \( P_n \) is congruent to 1 modulo 4. This implies that \( 4^k \mid m \). Thus, \( 3^k > n/6 \) and \( n > m \geq 4^k \), a contradiction in our range for \( n \). This is done in Subsection 3.3. When \( n \) is even, we write \( n = 2^s n_1 \) with an odd integer \( n_1 \) and bound \( s \) and the smallest prime factor \( r_1 \) of \( n_1 \). We first show that \( s \leq 3 \), that if \( n_1 \) and \( m \) have a common divisor larger than 1, then \( r_1 \in \{3, 5, 7\} \) (Lemma 3.3). A lot of effort is spend into finding a small bound on \( r_1 \). As we saw, \( r_1 \leq 7 \) if \( n_1 \) and \( m \) are not coprime. When \( n_1 \) and \( m \) are coprime, we succeed in proving that \( r_1 < 10^6 \). Putting \( e_r \) for the exponent of \( r \) in the factorization of \( P_{2(r)} \), it turns out that our argument works well when \( e_r = 1 \) and we get a contradiction, but when \( e_r = 2 \), then we need some additional information about the prime factors of \( Q_r \). It is always the case that \( e_r = 1 \) for all primes \( r \leq 10^6 \), except for \( r \in \{13, 31\} \) for which \( e_r = 2 \), but, lucky for us, both \( Q_{13} \) and \( Q_{31} \) have two suitable prime factors each which allows us to obtain a contradiction. Our efforts in obtaining \( r_1 < 10^6 \) involve quite a complicated argument (roughly the entire argument after Lemma 3.3 until the end), which we believe it is justified by the existence of the mighty prime \( r_1 = 1546463 \), for which \( e_{r_1} = 2 \). Should we have only obtained say \( r_1 < 1.6 \times 10^6 \), we would have had to say something nontrivial about the prime factors of \( Q_{1546463} \), a nuisance which we succeeded in avoiding simply by proving that \( r_1 \) cannot get that large!

### 3.2. Some lower bounds on \( m \) and \( \omega(P_n) \)

We start with a computation showing that there are no other solutions than \( n = 1, 2 \) when \( n \leq 100 \). So, from now on \( n > 100 \). We write \( P_n = q_1^{\alpha_1} \cdots q_k^{\alpha_k} \), where \( q_1 < \cdots < q_k \) are primes and \( \alpha_1, \ldots, \alpha_k \) are positive integers. Clearly, \( m < n \).

McDaniel [12], proved that \( P_n \) has a prime factor \( q \equiv 1 \pmod{4} \) for all \( n > 14 \). Thus, McDaniel’s result applies for us showing that \( 4 \mid q - 1 \mid \phi(P_n) \mid P_m \), so \( 4 \mid m \) by Lemma 2.2. Further, it follows from a the result of the second author [5], that \( \phi(P_n) \geq P_{\phi(n)} \). Hence, \( m \geq \phi(n) \). Thus,

\[
(3.1) \quad m \geq \phi(n) \geq \frac{n}{1.79 \log \log n + 2.5/\log \log n}
\]

by Lemma 2.5(iii). The function

\[
x \mapsto \frac{x}{1.79 \log \log x + 2.5/\log \log x}
\]

is increasing for \( x \geq 100 \). Since \( n \geq 100 \), inequality (3.1) together with the fact that \( 4 \mid m \), show that \( m \geq 24 \).

Put \( \ell = n - m \). Since \( m \) is even, we have \( \beta^m > 0 \), therefore

\[
(3.2) \quad \frac{P_n}{P_m} = \frac{\alpha^n - \beta^n}{\alpha^m - \beta^m} \geq \frac{\alpha^n - \beta^n}{\alpha^m} \geq \alpha^\ell - \frac{1}{\alpha^{m+n}} > \alpha^\ell - 10^{-40},
\]

where we used the fact that

\[
\frac{1}{\alpha^{m+n}} \leq \frac{1}{\alpha^{24}} < 10^{-40}.
\]

We now are ready to provide a large lower bound on \( n \). We distinguish the following cases.
Case 1: $n$ is odd. Here, we have $\ell \geq 1$. So, $P_n/P_m > \alpha - 10^{-40} > 2.4142$. Since $n$ is odd, it follows that $P_n$ is divisible only by primes $q$ such that $z(q)$ is odd. Among the first 10000 primes, there are precisely 2907 of them with this property. They are

\[ F_1 = \{5, 13, 29, 37, 53, 61, 101, 109, \ldots, 104597, 104677, 104693, 104701, 104717\}. \]

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Since

\[ \prod_{p \in F_1} \left(1 - \frac{1}{p}\right)^{-1} < 1.963 < 2.4142 < \frac{P_n}{P_m} = \prod_{i=1}^{k} \left(1 - \frac{1}{q_i}\right)^{-1}, \]

we get that $k > 2907$. Since $2^k | \phi(P_n) | P_m$, we get, by Lemma 2.2, that

(3.3) \quad n > m > 2^{2907}.

Case 2: $n \equiv 2 \pmod{4}$. Since both $m$ and $n$ are even, we get $\ell \geq 2$. Thus,

(3.4) \quad \frac{P_n}{P_m} > \alpha^2 - 10^{-40} > 5.8284.

If $q$ is a prime factor of $P_n$, as in Case 1, we have that $z(q)$ is not divisible by 4. Among the first 10000 primes, there are precisely 5815 of them with this property. They are

\[ F_2 = \{2, 5, 7, 13, 23, 29, 31, 37, 41, 47, 53, 61, \ldots, 104693, 104701, 104711, 104717\}. \]

Writing $p_j$ as the $j^{th}$ prime number in $F_2$, we check with Mathematica that

\[ \prod_{i=1}^{415} \left(1 - \frac{1}{p_i}\right)^{-1} = 5.82753 \ldots \quad \prod_{i=1}^{416} \left(1 - \frac{1}{p_i}\right)^{-1} = 5.82861 \ldots, \]

which via inequality (3.4) shows that $k \geq 416$. Of the $k$ prime factors of $P_n$, we have that only $k - 1$ of them are odd ($q_1 = 2$ because $n$ is even), but one of those is congruent to 1 modulo 4 by McDaniel’s result. Hence, $2^k | \phi(P_n) | P_m$, which shows, via Lemma 2.2, that

(3.5) \quad n > m \geq 2^{416}.

Case 3: $4 | n$. In this case, since both $m$ and $n$ are multiples of 4, we get that $\ell \geq 4$. Therefore, $P_n/P_m > \alpha^4 - 10^{-40} > 33.97$. Letting $p_1 < p_2 < \cdots$ be the sequence of all primes, we have that

\[ \prod_{i=1}^{2000} \left(1 - \frac{1}{p_i}\right)^{-1} < 17.41 \ldots < 33.97 < \frac{P_n}{P_m} = \prod_{i=1}^{k} \left(1 - \frac{1}{q_i}\right), \]

showing that $k > 2000$. Since $2^k | \phi(P_n) = P_m$, we get

(3.6) \quad n > m \geq 2^{2000}.

To summarize, from (3.3), (3.5) and (3.6), we get the following results.

**Lemma 3.1.** If $n > 2$, then

(i) $2^k | m$; 
(ii) $k \geq 416$; 
(iii) $n > m \geq 2^{416}$.
3.3. Bounding $\ell$ in term of $n$. We saw in the preceding section that $k \geq 416$. Since $n > m \geq 2^k$, we have

$$k < k(n) := \frac{\log n}{\log 2}. \tag{3.7}$$

Let $p_j$ be the $j^{th}$ prime number. Lemma 2.5 shows that

$$p_k \leq p_{\lfloor k(n) \rfloor} \leq k(n) (\log k(n) + \log \log k(n)) := q(n).$$

We then have, using Lemma 2.5 (ii), that

$$P_m \prod_{i=1}^{k} \left( 1 - \frac{1}{q_i} \right) \geq \prod_{2 \leq p \leq q(n)} \left( 1 - \frac{1}{p} \right) > \frac{1}{1.79 \log q(n)(1 + 1/(2(\log q(n))^2))}. \tag{3.8}$$

Inequality (ii) of Lemma 2.5 requires that $x \geq 286$, which holds for us with $x = q(n)$ because $k(n) \geq 416$. Hence, we get

$$1.79 \log q(n) \left( 1 + \frac{1}{2(\log q(n))^2} \right) > \frac{P_m}{P_m} > \alpha^\ell - 10^{-40} > \alpha^\ell \left( 1 - \frac{1}{10^{40}} \right).$$

Since $k \geq 416$, we have $q(n) \geq 3256$. Hence, we get

$$\log q(n) \left( 1.79 \left( 1 - \frac{1}{10^{40}} \right)^{-1} \left( 1 + \frac{1}{2(\log(3256))^2} \right) \right) > \alpha^\ell,$$

which yields, after taking logarithms, to

$$\ell \leq \frac{\log \log q(n)}{\log \alpha} + 0.67. \tag{3.9}$$

The inequality

$$q(n) < (\log n)^{1.45} \tag{3.10}$$

holds in our range for $n$ (in fact, it holds for all $n > 10^{43}$, which is our case since for us $n > 2^{416} > 10^{125}$). Inserting inequality (3.10) into (3.8), we get

$$\ell < \frac{\log \log (\log n)^{1.45}}{\log \alpha} + 0.67 < \frac{\log \log \log n}{\log \alpha} + 1.1.$$
To bound the primes $q_i$ for all $i = 1, \ldots, k$, we use the inductive argument from Section 3.3 in [11]. We write
\[
\prod_{i=1}^{k} \left(1 - \frac{1}{q_i}\right) = \frac{\phi(P_n)}{P_n} = \frac{P_m}{P_n}.
\]
Therefore,
\[
1 - \prod_{i=1}^{k} \left(1 - \frac{1}{q_i}\right) = 1 - \frac{P_m}{P_n} = \frac{P_n - P_m}{P_n} > \frac{P_n - P_{n-1}}{P_n} > \frac{P_{n-1}}{P_n}.
\]
Using the inequality
\[
1 - (1 - x_1) \cdots (1 - x_s) \leq x_1 + \cdots + x_s \quad \text{valid for all} \quad x_i \in [0, 1] \quad \text{for} \quad i = 1, \ldots, s,
\]
we get,
\[
1 - (1 - x_1) \cdots (1 - x_s) \leq x_1 + \cdots + x_s \leq k \prod_{i=1}^{k} \frac{1}{q_i} < k,
\]
therefore,
\[
q_1 \cdots q_k = u_k < (2^{\alpha_2} k \log \log n)^{(3^k-1)/2},
\]
which together with formula (3.8) and (3.10) gives
\[
P_n = q_1 \cdots q_k B < (2^{\alpha_2} k \log \log n)^{(3^k-1)/2} = (2^{\alpha_2} k \log \log n)^{(3^k+1)/2}.
\]
Since $P_n > \alpha^{n-2}$ by inequality (2.4), we get
\[
(n-2) \log \alpha < \frac{(3^k + 1)}{2} \log(2^{\alpha_2} k \log \log n).
\]
Since $k < \log n / \log 2$ (see (3.7)), we get
\[
3^k > (n-2) \left(\frac{2 \log \alpha}{\log(2^{\alpha_2} (\log n)(\log \log n)(\log 2)^{-1})}\right) - 1 > 0.17(n-2) - 1 > \frac{n}{6},
\]
where the last two inequalities above hold because $n > 2^{416}$.

So, we proved the following result.

**Lemma 3.3.** If $n > 2$, then $3^k > n/6$.

**3.5. The case when $n$ is odd.** Assume that $n > 2$ is odd and let $q$ be any prime factor of $P_n$. Reducing relation $Q_n^2 - 8P_n^2 = 4(-1)^n$ of Lemma 2.1(ii) modulo $q$, we get $Q_n^2 \equiv -4 \pmod{q}$. Since $q$ is odd, (because $n$ is odd), we get that $q \equiv 1 \pmod{4}$. This is true for all prime factors $q$ of $P_n$. Hence,
\[
4^k \mid \prod_{i=1}^{k} (q_i - 1) \mid \phi(P_n) \mid P_m,
\]
which, by Lemma 2.2 (ii), gives $4^k \mid m$. Thus, $n > m \geq 4^k$, inequality which together with Lemma 3.3 gives $n > (3^k)\log 4 / \log 3 > (\frac{n}{6})^{\log 4 / \log 3}$, so

$$n < 6^{\log 4 / \log(4/3)} < 5621,$$

in contradiction with Lemma 3.1.

### 3.6. Bounding $n$

From now on, $n > 2$ is even. We write it as

$$n = 2^s r_1^\lambda_1 \cdots r_t^\lambda_t =: 2^s n_1,$$

where $s \geq 1$, $t \geq 0$ and $3 \leq r_1 < \cdots < r_t$ are odd primes. Thus, by inequality (3.2), we have

$$\alpha^\ell \left(1 - \frac{1}{10^{40}}\right) < \frac{\alpha^\ell}{\phi(P_n)} = \prod_{p|P_n} \left(1 + \frac{1}{p - 1}\right) = 2 \prod_{d \geq 3, p \in \mathcal{P}_d} \prod_{d|n} \left(1 + \frac{1}{p - 1}\right),$$

and taking logarithms we get

$$\ell \log \alpha - \frac{1}{10^{39}} < \log \left(\alpha^\ell \left(1 - \frac{1}{10^{40}}\right)\right) < \log 2 + \sum_{d \geq 3} \sum_{p \in \mathcal{P}_d} \log \left(1 + \frac{1}{p - 1}\right) < \log 2 + \sum_{d \geq 3} S_d.$$  

In the above, we used the inequality $\log(1 - x) > -10x$ valid for all $x \in (0, 1/2]$ with $x = 1/10^{40}$ and the inequality $\log(1 + x) \leq x$ valid for all real numbers $x$ with $x = p$ for all $p \in \mathcal{P}_d$ and all divisors $d \mid n$ with $d \geq 3$.

Let us deduce that the case $t = 0$ is impossible. Indeed, if this were so, then $n$ is a power of 2 and so, by Lemma 3.1 both $m$ and $n$ are divisible by $2^{416}$. Thus, $\ell \geq 2^{416}$. Inserting this into (3.11), and using Lemma 2.6 we get

$$2^{416} \log \alpha - \frac{1}{10^{39}} < \sum_{a \geq 1} \frac{2 \log(2^a)}{2^a} = 4 \log 2,$$

a contradiction.

Thus, $t \geq 1$ so $n_1 > 1$. We now put $\mathcal{I} := \{i : r_i \mid m\}$ and $\mathcal{J} = \{1, \ldots, t\} \setminus \mathcal{I}$. We put $M = \prod_{i \in \mathcal{I}} r_i$. We also let $j$ be minimal in $\mathcal{J}$. We split the sum appearing in (3.11) in two parts:

$$\sum_{d|n} S_d = L_1 + L_2,$$

where

$$L_1 := \sum_{d|n} S_d \quad \text{and} \quad L_2 := \sum_{d|n} S_d \quad \text{for some } u \in \mathcal{J}$$

To bound $L_1$, we note that all divisors involved divide $n'$, where

$$n' = 2^s \prod_{i \in \mathcal{I}} r_i^\lambda_i.$$
Using Lemmas 2.6 and 2.7, we get

$$L_1 \leq 2 \sum_{d \mid n'} \frac{\log d}{d} < 2 \left( \sum_{r \mid n'} \frac{\log r}{\phi(n')} \right) = 2 \left( \sum_{r \mid 2M} \frac{\log r}{r - 1} \right) \left( \frac{2M}{\phi(2M)} \right).$$  

We now bound $L_2$. If $\mathcal{J} = \emptyset$, then $L_2 = 0$ and there is nothing to bound. So, assume that $\mathcal{J} \neq \emptyset$. We argue as follows. Note that since $s \geq 1$, by Lemma 2.1 (i), we have $P_n = P_{n_1}Q_n_1Q_{2n_1} \cdots Q_{2n_1-n_1}$. Let $q$ be any odd prime factor of $Q_{n_1}$. By reducing relation (ii) of Lemma 2.1 modulo $q$ and using the fact that $n_1$ and $q$ are both odd, we get $2P_{n_1}^2 \equiv 1 \pmod{q}$, therefore $\left( \frac{2}{q} \right) = 1$. Hence, $z(q) \mid q - 1$ for such primes $q$. Let $d$ be any divisor of $n_1$ which is a multiple of $r_j$. The number of them is $\tau(n_1/r_j)$, where $\tau(u)$ is the number of divisors of the positive integer $u$. For each such $d$, there is a primitive prime factor $q_d$ of $Q_d \mid Q_{n_1}$. Thus, $r_j \mid d \mid q_d - 1$. This shows that

$$\nu_{r_j}(\phi(P_n)) \geq \nu_{r_j}(\phi(Q_{n_1})) \geq \tau(n_1/r_j) \geq \tau(n_1)/2,$$

where the last inequality follows from the fact that

$$\frac{\tau(n_1/r_j)}{\tau(n_1)} = \frac{\lambda_j}{\lambda_j + 1} \geq \frac{1}{2},$$

Since $r_j$ does not divide $n$, it follows from (2.27) that

$$\nu_{r_j}(P_m) \leq e_{r_j}.$$  

Hence, (3.13), (3.14) and (1.1) imply that

$$\tau(n_1) \leq 2e_{r_j}.$$  

Invoking Lemma 2.4, we get

$$\tau(n_1) \leq \frac{(r_j + 1) \log \alpha}{\log r_j}.$$  

Now every divisor $d$ participating in $L_2$ is of the form $d = 2^a d_1$, where $0 \leq a \leq s$ and $d_1$ is a divisor of $n_1$ divisible by $r_a$ for some $u \in \mathcal{J}$. Thus,

$$L_2 \leq \tau(n_1) \min \left\{ \sum_{0 \leq a \leq s, d_1 \mid n_1 \atop r_u \mid d_1 \text{ for some } u \in \mathcal{J}} S_{2^a d_1} \right\} := g(n_1, s, r_1).$$

In particular, $d_1 \geq 3$ and since the function $x \mapsto \log x/x$ is decreasing for $x \geq 3$, we have that

$$g(n_1, s, r_1) \leq 2\tau(n_1) \sum_{0 \leq a \leq s} \frac{\log(2^a r_j)}{2^a r_j}.$$  

Putting also $s_1 := \min\{s, 416\}$, we get, by Lemma 3.1, that $2^{s_1} \mid \ell$. Thus, inserting this as well as (3.12) and (3.16) all into (3.11), we get

$$\ell \log \alpha - \frac{1}{10^{69}} < 2 \left( \sum_{r \mid 2M} \frac{\log r}{\phi(2M)} \right) \left( \frac{2M}{\phi(2M)} \right) + g(n_1, s, r_1).$$
Mathematica confirmed that the above inequality implies 
\[ \ell \]

\[ I = M = r \] according to whether 
\[ g \) The function 
\[ \ell \]

\[ \frac{\log(2^9 r_i)}{2^9 r_i} < 4 \log 2 + 2 \log r_j \]

inequalities (3.18), (3.15) and (3.16) give us that 
\[ g(n_1, s, r_1) \leq 2 \left( 1 + \frac{1}{r_j} \right) \left( 2 + \frac{4 \log 2}{\log r_j} \right) \log \alpha := g(r_j). \]

The function \( g(x) \) is decreasing for \( x \geq 3 \). Thus, \( g(r_j) \leq g(3) < 10.64 \). For a positive integer \( N \) put 
\[ f(N) := N \log \alpha - \frac{1}{10^{199}} - 2 \left( \sum_{r_i \mid N} \log r_i - 1 \right) \left( N - \phi(N) \right). \]

Then inequality (3.19) implies that both inequalities 
\[ f(\ell) < g(r_j), \quad (\ell - M) \log \alpha + f(M) < g(r_j) \]

hold. Assuming that \( \ell \geq 26 \), we get, by Lemma 2.5, that 
\[ \ell \log \alpha - \frac{1}{10^{199}} - 2(\log 2)(1.79 \log \log \ell + 2.5/\log \ell) \log \ell \leq 10.64. \]

Mathematica confirmed that the above inequality implies \( \ell \leq 500 \). Another calculation with Mathematica showed that the inequality \( f(\ell) < 10.64 \) for even values of \( \ell \in [1, 500] \cap \mathbb{Z} \) implies that \( \ell \in [2, 18] \). The minimum of the function \( f(2N) \) for \( N \in [1, 250] \cap \mathbb{Z} \) is at \( N = 3 \) and \( f(6) > -2.12 \). For the remaining positive integers \( N \), we have \( f(2N) > 0 \). Hence, inequality (3.19) implies 
\[ (2^{s_1} - 2) \log \alpha < 10.64 \quad \text{and} \quad (2^{s_1} - 2)3 \log \alpha < 10.64 + 2.12 = 12.76, \]

according to whether \( M \neq 3 \) or \( M = 3 \), and either one of the above inequalities implies that \( s_1 \leq 3 \). Thus, \( s = s_1 \in \{1, 2, 3\} \). Since \( 2M \mid \ell, 2M \) is square-free and \( \ell \leq 18 \), we have that \( M \in \{1, 3, 5, 7\} \). Assume \( M > 1 \) and let \( i \) be such that \( M = r_i \). Let us show that \( \lambda_i = 1 \). Indeed, if \( \lambda_i > 2 \), then 
\[ 199 \mid Q_9 \mid P_n, \quad 29201 \mid P_{25} \mid P_n, \quad 1471 \mid Q_{49} \mid P_n, \]

according to whether \( r_i = 3, 5, 7 \), respectively, and \( 3^2 \mid 199 - 1, 5^2 \mid 29201 - 1, 7^2 \mid 1471 - 1 \). Thus, we get that \( 3^2, 5^2, 7^2 \) divide \( \phi(P_n) = P_n \), showing that \( 3^2, 5^2, 7^2 \) divide \( \ell \). Since \( \ell \leq 18 \), only the case \( \ell = 18 \) is possible. In this case, \( r_j \geq 5 \), and inequality (3.19) gives \( 8.4 < f(18) \leq g(5) < 7.9 \), a contradiction. Let us record what we have deduced so far.

**Lemma 3.4.** If \( n > 2 \) is even, then \( s \in \{1, 2, 3\} \). Further, if \( \mathcal{I} \neq \emptyset \), then \( \mathcal{I} = \{i\}, r_i \in \{3, 5, 7\} \) and \( \lambda_i = 1 \).

We now deal with \( \mathcal{J} \). For this, we return to (3.11) and use the better inequality namely 
\[ 2^\nu M \log \alpha - \frac{1}{10^{199}} \leq \ell \log \alpha - \frac{1}{10^{199}} \leq \log \left( \frac{P_n}{\phi(P_n)} \right) \leq \sum_d 2^\nu M \sum_{p \in P_d} \log \left( 1 + \frac{1}{p-1} \right) + L_2, \]
so
\[ L_2 \geq 2^s M \log \alpha - \frac{1}{10^{39}} - \sum_{d \mid 2^s M} \sum_{p \in P_d} \log \left(1 + \frac{1}{p} \right). \]

In the right-hand side above, \( M \in \{1, 3, 5, 7\} \) and \( s \in \{1, 2, 3\} \). The values of the right-hand side above are in fact
\[ h(u) := u \log \alpha - \frac{1}{10^{39}} - \log(P_u/\phi(P_u)) \]
for \( u = 2^s M \in \{2, 4, 6, 8, 10, 12, 14, 20, 24, 28, 40, 56\} \). Computing we get:
\[ h(u) \geq H_{s,M}\left(\frac{M}{\phi(M)}\right) \]
for \( M \in \{1, 3, 5, 7\}, \quad s \in \{1, 2, 3\}, \)
where
\[ H_{1,1} > 1.069, \quad H_{1,M} > 2.81 \quad \text{for} \quad M > 1, \quad H_{2,M} > 2.426, \quad H_{3,M} > 5.8917. \]

We now exploit the relation
\[ (3.20) \]
Our goal is to prove that \( r_j < 10^6 \). Assume this is not so. We use the bound
\[ L_2 < \sum_{d \mid n \in \mathcal{J}} \frac{4 + 4 \log \log d}{\phi(d)} \]
of Lemma 2.6. Each divisor \( d \) participating in \( L_2 \) is of the form \( 2^a d_1 \), where \( a \in [0, s] \cap \mathbb{Z} \) and \( d_1 \) is a multiple of a prime at least as large as \( r_j \). Thus,
\[ 4 + 4 \log \log d \leq 4 + 4 \log \log 8d_1 \]
for \( a \in \{0, 1, \ldots, s\} \),
and
\[ d_1 \phi(d_1) \leq n_1 \phi(n_1) \leq M \phi(M) \left(1 + \frac{1}{r_j - 1}\right)^{\omega(n_1)} \]
Using (3.20), we get
\[ 2^{\omega(n_1)} \leq \tau(n_1) \leq \frac{(r_j + 1) \log \alpha}{\log r_j} < r_j, \]
where the last inequality holds because \( r_j \) is large. Thus,
\[ (3.21) \quad \omega(n_1) < \frac{\log r_j}{\log 2} < 2 \log r_j. \]
Hence,
\[ (3.22) \quad \frac{n_1}{\phi(n_1)} \leq \frac{M}{\phi(M)} \left(1 + \frac{1}{r_j - 1}\right)^{\omega(n_1)} < \frac{M}{\phi(M)} \left(1 + \frac{1}{r_j - 1}\right)^{2\log r_j} \]
\[ < \frac{M}{\phi(M)} \exp \left(\frac{2 \log r_j}{r_j - 1}\right) < \frac{M}{\phi(M)} \left(1 + \frac{4 \log r_j}{r_j - 1}\right), \]
where we used the inequalities $1 + x < e^x$, valid for all real numbers $x$, as well as $e^x < 1 + 2x$ which is valid for $x \in (0, 1/2)$ with $x = 2 \log r_j / (r_j - 1)$ which belongs to $(0, 1/2)$ because $r_j$ is large. Thus, the inequality

$$\frac{4 + 4 \log \log d}{\phi(d)} \leq \left( \frac{4 + 4 \log \log 8d_1}{d_1} \right) \left( 1 + \frac{4 \log r_j}{r_j - 1} \right) \left( \frac{1}{\phi(2^n)} \right) \left( \frac{M}{\phi(M)} \right)$$

holds for $d = 2^a d_1$ participating in $L_2$. The function $x \mapsto (4 + 4 \log \log(8x))/x$ is decreasing for $x \geq 3$. Hence,

$$(3.23) \quad L_2 \leq \frac{4 + 4 \log \log(8r_j)}{r_j} \tau(n_1) \left( 1 + \frac{4 \log r_j}{r_j - 1} \right) \left( \sum_{0 \leq a \leq s} \frac{1}{\phi(2^j)} \right) \left( \frac{M}{\phi(M)} \right).$$

Inserting inequality (3.15) into (3.23) and using (3.20), we get

$$(3.24) \quad \log r_j < 4 \left( 1 + \frac{1}{r_j} \right) \left( 1 + \frac{4 \log r_j}{r_j - 1} \right) \left( 1 + \log \log(8r_j) \right) \left( \log \alpha \right) \left( \frac{G_s}{H_s M} \right),$$

where

$$G_s = \sum_{0 \leq a \leq s} \frac{1}{\phi(2^j)}.$$ 

For $s = 2$, inequality (3.23) implies $r_j < 900,000$ and $r_j < 300$, respectively. For $s = 1$ and $M > 1$, inequality (3.23) implies $r_j < 5000$. When $M = 1$ and $s = 1$, we get $n = 2n_1$ and $j = 1$. Here, inequality (3.24) implies that $r_j < 8 \times 10^{12}$. This is too big, so we use the bound

$$S_d < \frac{2 \log d}{d}$$

of Lemma [2.0] instead for the divisors $d$ of participating in $L_2$, which in this case are all the divisors of $n$ larger than 2. We deduce that

$$1.06 < L_2 < 4 \sum_{d \geq 2} \frac{\log d}{d} < 4 \sum_{d \mid n_1} \frac{\log d_1}{d_1}.$$ 

The last inequality above follows from the fact that all divisors $d > 2$ of $n$ are either of the form $d_1$ or $2d_1$ for some divisor $d_1 \geq 3$ of $n_1$, and the function $x \mapsto \log x/x$ is decreasing for $x \geq 3$. Using Lemma [2.4] and inequalities (3.21) and (3.22), we get

$$1.06 < 4 \left( \sum_{d \mid n_1} \frac{\log r}{r - 1} \right) \omega(n_1) \left( 1 + \frac{4 \log r_1}{r_1 - 1} \right)$$

$$< \left( \frac{4 \log r_1}{r_1 - 1} \right) \left( 2 \log r_1 \right) \left( 1 + \frac{4 \log r_1}{r_1 - 1} \right),$$

which gives $r_1 < 159$. So, in all cases, $r_j < 10^6$. Here, we checked that $e_r = 1$ for all such $r$ except $r \in \{13, 31\}$ for which $e_r = 2$. If $e_{rj} = 1$, we then get $\tau(n_1/r_j) \leq 1$, so $n_1 = r_j$. Thus, $n \leq 8 \cdot 10^6$, in contradiction with Lemma [4.1]. Assume now that $r_j \in \{13, 31\}$. Say $r_j = 13$. In this case, 79 and 599 divide $Q_{13}$ which divides $P_n$, therefore $13^2 \mid (79 - 1)(599 - 1) \mid \phi(P_n) = P_m$. Thus, if there is some other prime factor $r'$ of $n_1/13$, then $13r' \mid n_1$, and $Q_{13r'}$ has a primitive prime factor $q \equiv 1 \pmod{13r'}$. In particular, $13 \mid q - 1$. Thus, $\nu_{13}(\phi(P_n)) \geq 3$, showing that
133 | P_m. Hence, 13 | m, therefore 13 | M, a contradiction. A similar contradiction is obtained if r_j = 31 since Q_{31} has two primitive prime factors namely 424577 and 865087 so 31 | M.

This finishes the proof.

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**References**