Let
\[ f(s) = \sum_{n=1}^{\infty} a_n e^{s \cdot n}, \]
where \( 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots \), \( \lambda_n \to \infty \) as \( n \to \infty \), \( s = \sigma + it \) (\( \sigma, t \) being reals) and \( \{a_n\}_{1}^{\infty} \) any sequence of complex numbers, be a Dirichlet series. Further, let
\[ \limsup_{n \to \infty} \frac{n}{\lambda_n} = D < \infty, \]
\[ \limsup_{n \to \infty} (\lambda_{n+1} - \lambda_n) = h > 0, \]
and
\[ \limsup_{n \to \infty} \log \frac{|a_n|}{\lambda_n} = -\infty. \]
Then the series in (1.1) represents an entire function \( f(s) \). We denote by \( X \) the set of all entire functions \( f(s) \) having representation (1.1) and satisfying the conditions (1.2)–(1.4). By giving different topologies on the set \( X \), Kamthan [4] and Hussain and Kamthan [2] have studied various topological properties of these spaces. Hence we define, for any nondecreasing sequence \( \{r_i\} \) of positive numbers, \( r_i \to \infty \),
\[ \|f\|_{r_i} = \sum_{n=1}^{\infty} |a_n| e^{r_i \cdot n}, \quad i = 1, 2, \ldots, \]
where \( f \in X \). Then from (1.4), \( \|f\|_{r_i} \) exists for each \( i \) and is a norm on \( X \). Further, \( \|f\|_{r_i} \leq \|f\|_{r_{i+1}} \). With these countable number of norms, a metric \( d \) is defined on \( X \) as:
\[ d(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|f - g\|_{r_i}}{1 + \|f - g\|_{r_i}}, \quad f, g \in X. \]

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Further, following functions are defined for each \( f \in X \), namely
\[
p(f) = \sup_{n \geq 1} |a_n|^{1/\lambda_n} ;
\]
\[
\|f\|_i = \sup_{n \leq i} \left( |a_n|^{1/\lambda_n} \right).
\]
Then \( p(f) \) and \( \|f\|_i \) are para-norms on \( X \). Let
\[
s(f, g) = \sum_{i=1}^\infty \frac{1}{2^i} \frac{\|f-g\|_i}{1+\|f-g\|_i}.
\]
It was shown [2, Lemma 1] that the three topologies induced by \( d \), \( s \) and \( p \) on \( X \) are equivalent. Many other properties of these spaces were also obtained (see [2], pp. 206–209).

For the space of entire functions of finite Ritt order \([6]\) and type, yet another norm \( \|f\|_q \) and hence a metric \( \lambda \) was introduced and the properties of this space \( X_\lambda \) were studied.

Let, for \( f \in X \),
\[
M(\sigma; f) \equiv M(\sigma) = \sup_{-\infty < t < \infty} |f(\sigma + it)| ,
\]
then \( M(\sigma) \) is called the maximum modulus of \( f(s) \). The Ritt order of \( f(s) \) is defined as
\[
\limsup_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \rho , \quad 0 \leq \rho \leq \infty .
\]
For \( \rho < \infty \), the entire function \( f \) is said to be of finite order. A function \( \rho(\sigma) \) is said to be proximate order [3] if
\[
\rho(\sigma) \to \rho \quad \text{as} \quad \sigma \to \infty , \quad 0 < \rho < \infty ,
\]
\[
\sigma \rho'(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty .
\]
For \( f \in X \), define
\[
\limsup_{\sigma \to \infty} \frac{\log M(\sigma)}{e^{\sigma \rho(\sigma)}} \leq A < \infty .
\]
Then it was proved [3] that (1.13) holds if and only if
\[
\limsup_{n \to \infty} \phi(\lambda_n) |a_n|^{1/\lambda_n} \leq (A e \rho)^{1/\rho} ,
\]
where \( \phi(t) \) is the unique solution of the equation \( t = \exp[\sigma \rho(\sigma)] \).
SPACES OF ENTIRE FUNCTIONS OF SLOW GROWTH

(Apparently the inequality (4.1) and the definition of $\phi(t)$ contain some misprints in [2, pp. 209–210]).

For each $f \in X$, define

$$
\|f\|_q = \sum_{n=1}^{\infty} |a_n| \left\{ \frac{\phi(\lambda_n)}{[(A + \frac{1}{q}) e^{1/\rho}]^{1/\rho}} \right\}^{\lambda n},
$$

where $q = 1, 2, \ldots$. For $q_1 \leq q_2$, $\|f\|_{q_1} \leq \|f\|_{q_2}$. It was proved that $\|f\|_q$, $q = 1, 2, \ldots$, induces on $X$ a unique topology such that $X$ becomes a convex topological vector space, where this topology is given by the metric $\lambda$,

$$
\lambda(f, g) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|f - g\|_q}{1 + \|f - g\|_q}.
$$

This space was denoted by $X_\lambda$. Various properties of this space were studied [2, pp. 209–216].

It is evident that if $\rho = 0$, then the definition of the norm $\|f\|_q$ and proximate order $\rho(\sigma)$ is not possible. It is the aim of this paper to give a metric on the space of entire functions of zero order thereby studying some properties of this space.

2 – For an entire function $f(s)$ represented by (1.1), for which $\rho$ defined by (1.10) is equal to zero, we define following Rahman [5]

\begin{equation}
\limsup_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\log \sigma} = \rho^*, \quad 1 \leq \rho^* \leq \infty.
\end{equation}

Then $\rho^*$ is said to be the logarithmic order of $f(s)$. For $1 < \rho^* < \infty$, we define the logarithmic proximate order [1] $\rho^*(\sigma)$ as a continuous piecewise differentiable function for $\sigma \geq \sigma_0$ such that

\begin{equation}
\rho^*(\sigma) \to \rho^* \quad \text{as} \quad \sigma \to \infty,
\end{equation}

\begin{equation}
\sigma \log \sigma, \rho^*(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty.
\end{equation}

Then the logarithmic type $T^*$ of $f$ with respect to proximate order $\rho^*(\sigma)$ is defined as [7]:

\begin{equation}
\limsup_{\sigma \to \infty} \frac{\log M(\sigma)}{\sigma^{T^*}} = T^*^*, \quad 0 < T^* < \infty.
\end{equation}

It was proved by one of the authors [7] that $f(s)$ is of logarithmic order $\rho^*$, $1 < \rho^* < \infty$, and logarithmic type $T^*$, $0 < T^* < \infty$, if and only if

\begin{equation}
\limsup_{n \to \infty} \frac{\lambda_n \phi(\lambda_n)}{\log |a_n|^{-1}} = \frac{\rho^*}{(\rho^* - 1)} \left(\rho^* T^* \right)^{1/(\rho^* - 1)},
\end{equation}

where
where \( \phi(t) \) is the unique solution of the equation \( t = \sigma^{\rho^*(\sigma)^{-1}} \).

We now denote by \( X \) the set of all entire functions \( f(s) \) given by (1.1), satisfying (1.2) to (1.4), for which

\[
\lim_{\sigma \to \infty} \frac{\log M(\sigma)}{\sigma^{\rho^*(\sigma)}} \leq T^* < \infty, \quad 1 < \rho^* < \infty.
\]

Then from (2.5), we have

\[
\lim_{n \to \infty} \frac{\lambda_n \phi(\lambda_n)}{\log |a_n|^{-1}} \leq \left( \frac{\rho^*}{\rho^* - 1} \right) \left( \rho^* T^* \right)^{1/(\rho^* - 1)}.
\]

In all our further discussion, we shall denote \( (\rho^*/(\rho^* - 1))(\rho^* - 1) \) by the constant \( K \). Then from (2.7) we have

\[
|a_n| < \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{\{K, \rho^*(T^* + \varepsilon)\}^{1/(\rho^* - 1)}} \right],
\]

where \( \varepsilon > 0 \) is arbitrary and \( n > n_0 \).

Now, for each \( f \in X \), let us define

\[
\|f\|_q = \sum_{n=1}^{\infty} |a_n| \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{\{K, \rho^*(T^* + \varepsilon)\}^{1/(\rho^* - 1)}} \right],
\]

where \( q = 1, 2, 3, \ldots \). In view of (2.8), \( \|f\|_q \) exists and for \( q_1 \leq q_2, \|f\|_{q_1} \leq \|f\|_{q_2} \). This norm induces a metric topology on \( X \).

We define

\[
\lambda(f, g) = \sum_{q=1}^{\infty} \frac{1}{2^q} \cdot \frac{\|f - g\|_q}{1 + \|f - g\|_q}.
\]

The space \( X \) with the above metric \( \lambda \) will be denoted by \( X_\lambda \).

Now we prove

**Theorem 1.** The space \( X_\lambda \) is a Fréchet space.

**Proof:** It is sufficient to show that \( X_\lambda \) is complete. Hence, let \( \{f_\alpha\} \) be a \( \lambda \)-Cauchy sequence in \( X \). Therefore, for any given \( \varepsilon > 0 \) there exists \( n_0 = n_0(\varepsilon) \) such that

\[
\|f_\alpha - f_\beta\|_q < \varepsilon \quad \forall \alpha, \beta > n_0, \quad q \geq 1.
\]

Denoting \( f_\alpha(s) = \sum_{n=1}^{\infty} a_n^{(\alpha)} e^{s \lambda_n}, f_\beta(s) = \sum_{n=1}^{\infty} a_n^{(\beta)} e^{s \lambda_n} \), we have therefore

\[
\sum_{n=1}^{\infty} |a_n^{(\alpha)} - a_n^{(\beta)}| \cdot \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{\{K, \rho^*(T^* + \varepsilon)\}^{1/(\rho^* - 1)}} \right] < \varepsilon
\]

for all \( \alpha, \beta > n_0 \).
for \(\alpha, \beta > n_0, q \geq 1\). Hence we obviously have

\[ |a^{(\alpha)}_n - a^{(\beta)}_n| < \varepsilon\quad \forall \alpha, \beta > n_0, \]

i.e., \(\{a^{(\alpha)}_n\}\) is a Cauchy sequence of complex numbers for each fixed \(n = 1, 2, \ldots\). Hence

\[ \lim_{\alpha \to \infty} a^{(\alpha)}_n = a_n, \quad n = 1, 2, \ldots. \]

Now letting \(\beta \to \infty\) in (2.9), we have for \(\alpha > n_0, \)

\[ (2.10) \quad \sum_{n=1}^{\infty} |a^{(\alpha)}_n| \cdot \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{\{K \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^* - 1)}} \right] < \varepsilon. \]

Taking \(\alpha = n_0\), we get for a fixed \(q\),

\[ |a^{(n_0)}_n| \cdot \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{\{K \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^* - 1)}} \right] \leq |a^{(n_0)}_n| \cdot \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{\{K \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^* - 1)}} \right] + \varepsilon. \]

Now, \(f^{(n_0)} = \sum_{n=1}^{\infty} a^{(n_0)}_n e^{s \lambda_n} \in X_\lambda\), hence the condition (2.8) is satisfied. For arbitrary \(p > q\), we have

\[ |a^{(n_0)}_n| < \exp \left[ \frac{-\lambda_n \phi(\lambda_n)}{\{K \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^* - 1)}} \right] \] for arbitrarily large \(n\).

Hence we have

\[ |a_n| \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{\{K \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^* - 1)}} \right] < \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{\{K \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^* - 1)}} \right] + \varepsilon. \]

Since \(\varepsilon > 0\) is arbitrary and the first term on the R.H.S. \(\to 0\) as \(n \to \infty\), we find that the sequence \(\{a_n\}\) satisfies (2.8). Then \(f(s) = \sum_{n=1}^{\infty} a_n e^{s \lambda_n}\) belongs to \(X_\lambda\). \(\blacksquare\)

Now, from (2.10), we have for \(q = 1, 2, \ldots\), \(\|f_\alpha - f\|_q < \varepsilon\). Hence

\[ \lambda(f_\alpha, f) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|f_\alpha - f\|_q}{1 + \|f_\alpha - f\|_q} \leq \frac{\varepsilon}{(1 + \varepsilon)} \sum_{q=1}^{\infty} \frac{1}{2^q} = \frac{\varepsilon}{(1 + \varepsilon)} \quad \varepsilon. \]

Since the above inequality holds for all \(\alpha > n_0\), we finally get \(f_\alpha \to f\) where \(f \in X_\lambda\). Hence \(X_\lambda\) is complete. This proves Theorem 1. \(\blacksquare\)
Now, we characterize the linear continuous functionals on $X_\lambda$. We prove

**Theorem 2.** A continuous linear functional $\psi$ on $X_\lambda$ is of the form

$$\psi(f) = \sum_{n=1}^{\infty} a_n C_n$$

if and only if

$$|C_n| \leq A \exp\left[\frac{\lambda_n \phi(\lambda_n)}{\{K \rho^*(T^s + \frac{1}{q})\}^{1/(\rho^*-1)}}\right]$$

(2.11)

for all $n \geq 1$, $q \geq 1$, where $A$ is a finite, positive number, $f = f(s) = \sum_{n=1}^{\infty} a_n e^{s,\lambda_n}$ and $\lambda_1$ is sufficiently large.

**Proof:** Let $\psi \in X_\lambda'$. Then for any sequence $\{f_m\} \in X_\lambda$ such that $f_m \to f$, we have $\psi(f_m) \to \psi(f)$ as $m \to \infty$. Now let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s,\lambda_n},$$

where $a_n$'s satisfy (2.8). Then $f \in X_\lambda$. Also, let

$$f_m(s) = \sum_{n=1}^{m} a_n e^{s,\lambda_n}.$$

Then $f_m \in X_\lambda$ for $m = 1, 2, \ldots$. Let $q$ be any fixed positive integer and let $0 < \varepsilon < \frac{1}{q}$. From (2.8), we can find an integer $m$ such that

$$|a_n| < \exp\left[\frac{-\lambda_n \phi(\lambda_n)}{\{K \rho^*(T^s + \varepsilon)\}^{1/(\rho^*-1)}}\right], \quad n > m.$$

Then

$$\left\| f - \sum_{n=1}^{m} a_n e^{s,\lambda_n} \right\|_q = \left\| \sum_{n=m+1}^{\infty} a_n e^{s,\lambda_n} \right\|_q =$$

$$= \sum_{n=m+1}^{\infty} |a_n| \exp\left[\frac{-\lambda_n \phi(\lambda_n)}{\{K \rho^*(T^s + \varepsilon)\}^{1/(\rho^*-1)}}\right]$$

$$< \sum_{n=m+1}^{\infty} \exp\left[\frac{-\lambda_n \phi(\lambda_n)}{(K \rho^*)^{1/(\rho^*-1)}} \left(\frac{T^s + \frac{1}{q}}{1 - (T^s + \varepsilon)^{-1/(\rho^*-1)}}\right)\right]$$

$$< \varepsilon \quad \text{for sufficiently large values of } m.$$
Hence
\[ \lambda(f, f_m) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|f - f_m\|_q}{1 + \|f - f_m\|_q} \leq \frac{\varepsilon}{(1 + \varepsilon)} < \varepsilon , \]
i.e., \( f_m \to f \) as \( m \to \infty \) in \( X_\lambda \). Hence by assumption that \( \psi \in X'_\lambda \), we have
\[ \lim_{m \to \infty} \psi(f_m) = \psi(f) . \]
Let us denote by \( C_n = \psi(e^{s \lambda_n}) \). Then
\[ \psi(f_m) = \sum_{n=1}^{m} a_n \psi(e^{s \lambda_n}) = \sum_{n=1}^{m} a_n C_n . \]

Also \( |C_n| = |\psi(e^{s \lambda_n})| \). Since \( \psi \) is continuous on \( X_\lambda \), it is continuous on \( X_{\| \|_{q}} \) for each \( q = 1, 2, 3, \ldots \). Hence there exists a positive constant \( A \) independent of \( q \) such that
\[ |\psi(e^{s \lambda_n})| = |C_n| \leq A \| \alpha \|_q , \quad q \geq 1 , \]
where \( \alpha(s) = e^{s \lambda_n} \). Now using the definition of the form for \( \alpha(s) \), we get
\[ |C_n| \leq A \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{(K \rho^*(T^* + \frac{1}{q}))^{1/(\rho^* - 1)}} \right] , \quad n \geq 1 , \quad q \geq 1 . \]
Hence we get \( \psi(f) = \sum_{n=1}^{\infty} a_n C_n \), where \( C_n \)'s satisfy (2.11).

Conversely, suppose that \( \psi(f) = \sum_{n=1}^{\infty} a_n C_n \) and \( C_n \) satisfies (2.11). Then for \( q \geq 1 \),
\[ |\psi(f)| \leq A \sum_{n=1}^{\infty} |a_n| \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{(K \rho^*(T^* + \frac{1}{q}))^{1/(\rho^* - 1)}} \right] \]
i.e.
\[ |\psi(f)| \leq A \| f \|_q , \quad q \geq 1 , \]
i.e.
\[ \psi \in X'_{\| \|_{q}} , \quad q \geq 1 . \]

Now, since
\[ \lambda(f, g) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|f - g\|_q}{1 + \|f - g\|_q} , \]
therefore \( X'_\lambda = \bigcup_{q=1}^{\infty} X'_{\| \|_{q}} \). Hence \( \psi \in X'_\lambda \).

This completes the proof of Theorem 2. ■

Lastly, we give the construction of total sets in \( X_\lambda \). Following [2], we give

**Definition.** Let \( X \) be a locally convex topological vector space. A set \( E \subset X \) is said to be total if and only if for any \( \psi \in X' \) with \( \psi(E) = 0 \), we have \( \psi = 0 \).
Now, we prove

**Theorem 3.** Consider the space $X_{\lambda}$ defined before and let $f(s) = \sum_{n=1}^{\infty} a_n e^{s \lambda_n}$, $a_n \neq 0$, for $n = 1, 2, \ldots$, $f \in X_{\lambda}$. Suppose $G$ is a subset of the complex plane having at least one limit point in the complex plane. Define, for $\mu \in G$,

$$f_{\mu}(s) = \sum_{n=1}^{\infty} (a_n e^{\mu \lambda_n}) e^{s \lambda_n}.$$ 

Then $E = \{ f_{\mu} : \mu \in G \}$ is total in $X_{\lambda}$.

**Proof:** Since $f \in X_{\lambda}$, from (2.7) we have

$$\limsup_{n \to \infty} \frac{\lambda_n \phi(\lambda_n)}{\log |a_n e^{\mu \lambda_n}|^{1-1}} = \limsup_{n \to \infty} \frac{\phi(\lambda_n)}{\log |a_n|^{1-1} - R(\mu)} \leq \left( \frac{\rho^*}{\rho^* - 1} \right) (\rho^* T^*)^{1/(\rho^* - 1)}, \text{ since } R(\mu) < \infty .$$

Hence, if we denote by $M_\mu(\sigma) = \sup_{-\infty < t < \infty} | f_{\mu}(\sigma + it) |$, then from (2.6),

$$\limsup_{\sigma \to \infty} \frac{\log M_\mu(\sigma)}{\sigma \rho^*(\sigma)} \leq T^* < \infty .$$

Therefore, $f_{\mu} \in X_{\lambda}$ for each $\mu \in G$. Thus $E \subset X_{\lambda}$.

Now, let $\psi$ be a linear continuous functional on $X_{\lambda}$ and suppose that $\psi(f_{\mu}) = 0$. From Theorem 2, there exists a sequence \{C_n\} of complex numbers such that

$$\psi(g) = \sum_{n=1}^{\infty} b_n C_n, \quad g(s) = \sum_{n=1}^{\infty} b_n e^{s \lambda_n} \in X_{\lambda},$$

where

$$|C_n| < A \exp \left[ \frac{\lambda_n \phi(\lambda_n)}{\{K \rho^*(T^* + \frac{1}{q})\}^{1/(\rho^* - 1)}} \right], \quad n \geq 1, \quad q \geq 1,$$

$A$ being a constant and $\lambda_1$ is sufficiently large.

Hence

$$\psi(f_{\mu}) = \sum_{n=1}^{\infty} a_n C_n e^{\mu \lambda_n} = 0, \quad \mu \in G.$$ 

Let us consider the function $F(s) = \sum_{n=1}^{\infty} a_n C_n e^{s \lambda_n}$. Then from (2.8) and (2.12), for any $\varepsilon$, $0 < \varepsilon < \frac{1}{q}$,

$$|a_n C_n|^{1/\lambda_n} < A^{1/\lambda_n} \exp \left[ \phi(\lambda_n) \left\{ \left( K \rho^* (T^* + \frac{1}{q}) \right)^{-1/(\rho^* - 1)} \right\} \right] - \left( K \rho^*(T^* + \varepsilon)^{-1/(\rho^* - 1)} \right)$$

$$\leq \left( \frac{\rho^*}{\rho^* - 1} \right) (\rho^* T^*)^{1/(\rho^* - 1)}$$

$$\leq T^* < \infty .$$
for all $n > n_0$. By definition of $\phi(t)$, $\phi(\lambda_n) \to \infty$ as $n \to \infty$ and $\lambda_n \to \infty$. Hence we get

$$\limsup_{n \to \infty} \frac{\log |a_n C_n|}{\lambda_n} = -\infty,$$

i.e., $F(s)$ satisfies (1.4). Hence $F \in X$.

Also, $F(\mu) = 0 \ \forall \ \mu \in G$. Thus the entire function $F(s) \equiv 0$ in the entire complex plane. But this implies that $a_n C_n = 0, \ \forall n = 1, 2, \ldots$. Since we have started with $a_n \neq 0$, thus we get $C_n = 0, n = 1, 2, \ldots$. Hence $\psi \equiv 0$. This proves Theorem 3.

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Arvind Kumar,
Department of Mathematics, U.O.R. Roorkee,
Roorkee 247667, U.P. – INDIA

and

G.S. Srivastava,
Department of Mathematics, U.O.R. Roorkee,
Roorkee 247667, U.P. – INDIA