A STUDY OF \(K_W\)-SPACES AND \(K_0^W\)-SPACES

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Abstract: Further study of \(K_W\)-spaces leads to the introduction of \(K_0^W\)-spaces. We obtain a characterization of \(K_0^W\)-spaces in terms of continuous real-valued functions which is dual to a characterization of \(K_0\)-spaces. We also get two characterizations of \(K_W\)-spaces, one of which exhibits their remarkable similarities with \(K_1\)-spaces; a consequence of the latter characterization is that \(K_W\)-spaces are collectionwise normal.

Throughout, we will use the terminology of [1].
We introduced the concept of \(K_W\)-spaces in [1; Definition 10], as follows:
A space \((X, \tau)\) is a \(K_W\)-space provided that, for each closed \(F \subseteq X\), there exists a function \(k: \tau|F \to \tau\) (\(k\) is called a \(K_W\)-function) which satisfies the following:

\(1\) \(F \cap k(U) = U\), for each \(U \in \tau|F\), \(k(F) = X\) and \(k(\emptyset) = \emptyset\);
\(2\) \(k(U) \subseteq k(V)\) whenever \(U \subseteq V\);
\(3\) \(k(U) \cup k(V) = X\) whenever \(U \cup V = F\);
\(4\) \(\overline{k(U)} \cap F = \overline{U}\).

Condition (3) naturally leads to one question if it can be replaced by the stronger condition below, without affecting the concept of a \(K_W\)-space:

\(3^*\) \(k(U) \cup k(V) = k(U \cup V)\).

We do not yet know the answer to this question. However, replacing (3) by \(3^*\) in the definition of \(K_W\)-spaces leads to a (possibly new) class of spaces which we will call \(K_0^W\)-spaces, with remarkable properties which are dual to those of \(K_0\)-spaces (see Theorem 2 of [1] and compare it with Theorem 2 ahead). It is noteworthy that a \(K_0\)-function is a \(K_W\)-function if and only if it is a \(K_0^W\)-function (this follows from Theorem 12 of [1], and Theorems 2 c) and 3 b) v) ahead).

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Remark. Note that, for each closed subspace $F$ of any space $(X, \tau)$ there exists $k: \tau|F \to \tau$ which satisfies (1), (2) and (3) above. Simply, let $k(U) = U \cup (X - F)$, for $U \neq \emptyset$, and $k(\emptyset) = \emptyset$.

**Proposition 1.** Every $K_W$-space is completely normal.

**Proof:** Let $A$, $B$ be subsets of a $K_W$-space $(X, \tau)$ such that $\overline{A} \cap B = \emptyset = A \cap \overline{B}$. Let $F = \overline{A} \cup \overline{B}$ and let $k: \tau|F \to \tau$ be a $K_W$-function. Then $\overline{B} - A = F - \overline{A} = U \in \tau|F$, $B \subset U$ and $\overline{U} \cap A = \emptyset$ (note that $a \in A$ implies $a \notin \overline{B}$ which implies that $a \in X - \overline{B} \in \tau$, since $(X - \overline{B}) \cap (\overline{B} - A) = \emptyset$, $a \notin U$). Since $k(U) \cap A = \emptyset$, by (4), we get that $k(U) \cap A = \emptyset$; therefore, $k(U)$ and $X - k(U)$ are disjoint $\tau$-open subsets of $X$ such that $B \subset k(U)$ and $A \subset X - k(U)$. This completes the proof. \(\blacksquare\)

**Theorem 2.** For any space $X$, the following are equivalent:

a) $X$ is a $K_w^*$-space;

b) $X$ is completely normal and, for each nonempty closed subspace $F$ of $X$, there exist extenders $\phi: C^*_w(F) \to C^*_w(X)$ and $\psi: C^*_lsc(F) \to C^*_lsc(X)$ such that

i) $\phi(f) \leq \phi(g)$, whenever $f \leq g$,

ii) $\phi(f + g) \geq \phi(f) + \phi(g)$,

iii) $\psi(f) \leq \psi(g)$, whenever $f \leq g$,

iv) $\psi(f + g) \leq \psi(f) + \psi(g)$,

v) $\phi(f) \leq \psi(f)$, whenever $f \in C^*(F)$,

vi) $\phi(aF) = a_X = \psi(aF)$, for $a \in \mathbb{R}$,

vii) $\psi(\sup(f,g)) = \sup(\psi(f),\psi(g))$,

viii) $\phi(\inf(f,g)) = \inf(\phi(f),\phi(g))$,

ix) $\phi(f) = -\psi(-f)$, for each $f \in C^*(F)$,

x) for any $\{f_\alpha | \alpha \in \Lambda \} \subset C^*(F)$, $\bigcup_\alpha \phi(f_\alpha)^{-1}(-\infty,0] \cap F = \bigcup_\alpha f_\alpha^{-1}(-\infty,0]$;

c) $X$ is completely normal and, for any nonempty closed $F \subset X$ there exists an extender $\phi: C^*(F) \to C^*_w(X)$ which satisfies i), vi), viii) and x) of b) for functions in $C^*(F)$.

**Proof:** a) implies b). By Proposition 1, $X$ is completely normal. Let $k: \tau|F \to \tau$ be a $K_W$-function. For each $x \in X$, let

$$
\phi(f)(x) = \inf \left\{ t \in \mathbb{R} \mid x \in k(f^{-1}([-\infty,t]) \right\},
$$

$$
\psi(g)(x) = \sup \left\{ t \in \mathbb{R} \mid x \in k(g^{-1}([t,\infty])) \right\},
$$
where \( f \in C^*_{usc}(F) \) and \( g \in C^*_{bc}(F) \). Since \( k \) is monotone, we immediately get that \( \phi \) and \( \psi \) satisfy i) and iii), respectively. Since we also get that

\[
\phi(f)^{-1}([-\infty, t]) = \bigcup \{ k(f^{-1}([-\infty, s])) \mid s < t \},
\]

\[
\psi(g)^{-1}(t, \infty] = \bigcup \{ k(f^{-1}(s, \infty]) \mid s > t \},
\]

we immediately get that \( \phi \) is a u.s.c.-extender and \( \psi \) is an lsc-extender. (It is clear that, for \( x \in F \), \( \phi(f)(x) = f(x) \) and \( \psi(g)(x) = g(x) \).

Next, we show that \( \phi \) satisfies ii): Pick \( x \in \mathbb{X} \) and say \( \phi(f)(x) = t_1 \), \( \phi(g)(x) = t_2 \), with \( t_1 \leq t_2 \). Let \( t = t_1 + t_2 \) and note that, for any \( \varepsilon > 0 \),

\[
(f + g)^{-1}([-\infty, t - \varepsilon]) \subset f^{-1}([-\infty, t_1 - \frac{\varepsilon}{2}]) \cup g^{-1}([-\infty, t_2 - \frac{\varepsilon}{2}) .
\]

(Pick any \( z \in F \) such that \( f(z) + g(z) < t - \varepsilon \). Note that if \( f(z) < t_1 - \frac{\varepsilon}{2} \) then \( z \in f^{-1}([-\infty, t_1 - \frac{\varepsilon}{2})] \); if \( f(z) \geq t_1 - \frac{\varepsilon}{2} \) then \( g(z) < t_2 - \frac{\varepsilon}{2} \) which implies that \( z \in g^{-1}([-\infty, t_2 - \frac{\varepsilon}{2})] \). Since \( \phi(f)(x) = t_1 \) and \( \phi(g)(x) = t_2 \), we get that \( x \notin k(f^{-1}([-\infty, t_1 - \frac{\varepsilon}{2})) \) and \( x \notin k(g^{-1}([-\infty, t_2 - \frac{\varepsilon}{2})) \); hence \( x \notin (f + g)^{-1}([-\infty, t - \varepsilon]) \), (2) and (3'), which implies that \( \phi(f + g)(x) \geq t = \phi(f)(x) + \phi(g)(x) \), as required.

Next, we show that \( \psi \) satisfies iv): Pick \( x \in \mathbb{X} \) and say \( \psi(f)(x) = t_1 \), \( \psi(g)(x) = t_2 \), with \( t_1 \leq t_2 \). Let \( t = t_1 + t_2 \) and note that, for any \( \varepsilon > 0 \),

\[
(f + g)^{-1}([t + \varepsilon, \infty]) \subset f^{-1}([t_1 + \frac{\varepsilon}{2}, \infty]) \cup g^{-1}([t_2 + \frac{\varepsilon}{2}, \infty]) .
\]

(Pick any \( z \in F \) such that \( f(z) + g(z) > t + \varepsilon \). Note that if \( f(z) > t_1 + \frac{\varepsilon}{2} \) then \( z \in f^{-1}([t_1 + \frac{\varepsilon}{2}, \infty]) \); if \( f(z) \leq t_1 + \frac{\varepsilon}{2} \) then \( g(z) > t_2 + \frac{\varepsilon}{2} \) which implies that \( z \in g^{-1}([t_2 + \frac{\varepsilon}{2}, \infty]) \). Since \( \psi(f)(x) = t_1 \) and \( \psi(g)(x) = t_2 \), we get that \( x \notin k(f^{-1}([t_1 + \frac{\varepsilon}{2}, \infty))) \) and \( x \notin k(g^{-1}([t_2 + \frac{\varepsilon}{2}, \infty])) \); hence \( x \notin k((f + g)^{-1}([t + \varepsilon, \infty])) \), (2) and (3'), which implies that \( \psi(f + g)(x) \leq t = \psi(f)(x) + \psi(g)(x) \), as required.

In order to show that v) is satisfied, let \( f \in C^*(F) \) and say \( \phi(f)(x) = t_0 \). Then \( x \notin k(f^{-1}([-\infty, t_0])) \) for \( t < t_0 \). Therefore, by conditions (3) for a \( \mathbb{K}_W \)-function, \( x \in k(f^{-1}([s, \infty])) \) for \( s < t < t_0 \) (because \( F = f^{-1}([s, \infty]) \cup f^{-1}([-\infty, t]) \)); therefore, \( \psi(f)(x) \geq t_0 = \phi(f)(x) \).

It is easily seen from the definitions of \( \phi \) and \( \psi \) that they satisfy vi).

Next, we show that \( \psi \) satisfies vii): Note that, for \( f, g \in C^*_{bc}(F) \) and \( t \in \mathbb{R} \),

\[
\sup(f, g)^{-1}([t, \infty]) = f^{-1}([t, \infty]) \cup g^{-1}([t, \infty]) .
\]

Pick \( x \in \mathbb{X} \) and let \( \psi(f)(x) = t_1 \), \( \psi(g)(x) = t_2 \); say \( t_1 \leq t_2 \). Then \( x \notin k(f^{-1}([t_1, \infty])) \) for \( t > t_1 \), and \( x \notin k(g^{-1}([t, \infty])) \) for \( t > t_2 \); therefore, by (3'),

\[
x \notin k(f^{-1}([t_1, \infty])) \cup k(g^{-1}([t_2, \infty])) \quad \text{for} \quad t > t_2 .
\]
Therefore, \( x \notin k(\sup(f,g)^{-1}([t, \infty])) \) for \( t > t_2 \), which implies that \( \psi(\sup(f,g))(x) \leq t_2 = \sup(\psi(f)(x), \psi(g)(x)) \). Since \( \psi(\sup(f,g)) \geq \sup(\psi(f), \psi(g)) \), because of iii), we get that \( \psi \) satisfies vii).

Similarly, one can prove that \( \phi \) satisfies viii); also, ix) follows immediately from the definitions of \( \phi \) and \( \psi \).

Finally, we show that x) is satisfied: Note that

\[
\bigcup_{\alpha} \phi(f_{\alpha})^{-1}([-\infty, 0]) = \bigcup_{\alpha} \left( \bigcup_{r < 0} k(f_{\alpha}^{-1}([-\infty, r])) \right) \subset k\left( \bigcup_{\alpha} f_{\alpha}^{-1}([-\infty, 0]) \right).
\]

Hence,

\[
\bigcup_{\alpha} \phi(f_{\alpha})^{-1}([-\infty, 0]) \cap F \subset \bigcup_{\alpha} f_{\alpha}^{-1}([-\infty, 0]) \cap F = \bigcup_{\alpha} f_{\alpha}^{-1}([-\infty, 0]),
\]

by (4). Since, for \( A \subset X \), \( \overline{A} \cap F \supset \overline{A \cap F} \), letting \( A = \bigcup_{\alpha} \phi(f_{\alpha})^{-1}([-\infty, 0]) \), we then get that

\[
\bigcup_{\alpha} \phi(f_{\alpha})^{-1}([-\infty, 0]) \cap F = \bigcup_{\alpha} f_{\alpha}^{-1}([-\infty, 0]).
\]

This completes the proof that a) implies b).

Since it is obvious that b) implies c), let us prove that c) implies a). Define \( k: \tau|F \to \tau \) by

\[
k(U) = \bigcup \left\{ \phi(f)^{-1}([-\infty, 0]) \mid f \in C^*(F, [-\infty, 1]) \right\}.
\]

Since \( \phi \) is a usc-extender and \( F \) is a Tychonoff space, one easily gets that \( k(U) \in \tau \) and \( k(U) \cap F = U \), for each \( U \in \tau|F \); also, \( k(\emptyset) = \emptyset \) and \( k(F) = X \), because of vi).

Next, note that \( k \) is monotone: Let \( U, V \in \tau|F \) such that \( U \subset V \). Note that \( f(F - U) \subset \{1\} \) implies that \( f(F - V) \subset \{1\} \), by i), which shows that \( k(U) \subset k(V) \).

Next, we prove that, for each \( U, V \in \tau|F \), \( k(U \cup V) = k(U) \cup k(V) \); i.e., \( k \) satisfies (3*): Since \( k \) is monotone, we need only prove that \( k(U \cup V) \subset k(U) \cup k(V) \). Let \( x \in k(U \cup V) \). Then there exists a function \( f \in C^*(F, [-\infty, 1]) \) such that \( f(F - U \cup V) \subset \{1\} \) and \( \phi(f)(x) < 0 \). By Lemma 1 in the Appendix, there exist functions \( f_1, f_2, f_3 \in C^*(F, [-\infty, 1]) \) such that \( f_1(F - U) \cup f_2(F - V) \cup f_3(F - U \cap V) \subset \{1\} \) and \( \inf(f_1, f_2, f_3) \leq f \). Then

\[
0 > \phi(f)(x) \geq \phi(\inf(f_1, f_2, f_3))(x) = \inf(\phi(f_1)(x), \phi(f_2)(x), \phi(f_3)(x)).
\]
Note that if $\phi(f_1)(x) < 0$ then $x \in k(U)$; if $\phi(f_2)(x) < 0$ then $x \in k(V)$; if $\phi(f_3)(x) < 0$ then $x \in k(U \cap V) \subset k(U) \cup k(V)$. Hence, $x \in k(U) \cup k(V)$.

Finally, we prove that $k(U) \cap F = U$: Let us say that $k(U) = \bigcup \{\phi(f_\alpha)^{-1}([-\infty, 0]) \mid \alpha \in \Lambda\}$. Then, since $\phi$ satisfies property x) of b), we get that

$$\overline{\mu(U)} \cap F = \bigcup_{\alpha} \phi(f_\alpha)^{-1}([-\infty, 0]) \cap F = \bigcup_{\alpha} f_\alpha^{-1}([-\infty, 0]) = \overline{U}.$$ 

Hence, $k(U) \cap F = U$, which completes the proof that c) implies a).

**Theorem 3.** For any space $X$, the following are equivalent:

a) $X$ is a $K_W$-space;

b) $X$ is a normal space and, for each nonempty closed subspace $F$ of $X$, there exist extenders $\phi: C^*_usc(F) \to C^*_usc(X)$ and $\psi: C^*_lsc(F) \to C^*_lsc(X)$ such that

i) $\phi(f) \leq \phi(g)$ whenever $f \leq g$,

ii) $\psi(f) \leq \psi(g)$ whenever $f \leq g$,

iii) $\phi(aF) = aX = \psi(aF)$, for each $a \in \mathbb{R}$,

iv) $\phi(f) \leq \psi(f)$ whenever $f \in C^*(F)$,

v) For any subset $\{f_\alpha \mid \alpha \in \Lambda\}$ of $C^*(F)$ and $a \in \mathbb{R}$,

$$\bigcup_{\alpha} \phi(f_\alpha)^{-1}([-\infty, a]) \cap F = \bigcup_{\alpha} f_\alpha^{-1}([-\infty, a]),$$

$$\bigcup_{\alpha} \psi(f_\alpha)^{-1}(a, \infty] \cap F = \bigcup_{\alpha} f_\alpha^{-1}(a, \infty].$$

c) $X$ is normal and, for any nonempty closed $F \subset X$, there exist extenders $\phi: C^*(F) \to C^*_usc(X)$ and $\psi: C^*(F) \to C^*_lsc(X)$ which satisfy iii), iv) and v) of b) for functions in $C^*(F)$.

**Proof:** a) implies b). This is essentially Proposition 11 of [1]. (The proof of condition v) in Proposition 11 of [1] can obviously be adapted to the more general condition v) of this result.)

Clearly, b) implies c).

c) implies a). (The proof of Theorem 4.1 in [2] surely helped us in devising this argument.) Let $F$ be a nonempty closed subspace of $(X, \tau)$. For each $U \in \tau|F$,
Let

\[
\mu(U) = \bigcup \{ \phi(f)^{-1}([-\infty, 1]) \mid f \in C(F, [-2, 2]), f(F - U) \subset \{2\} \},
\]

\[
\nu(U) = \bigcup \{ \psi(f)^{-1}([-1, \infty]) \mid f \in C(F, [-2, 2]), f(F - U) \subset \{-2\} \},
\]

\[
k(U) = \mu(U) \cup \nu(U).
\]

If \( U \in \tau|F \) and \( z \in U \), then there exists \( f \in C(F, [-2, 2]) \) such that \( f(z) = -2 \) and \( f(F - U) \subset \{2\} \) (because \( X \) is Tychonoff). Since \( \phi \) is an extender, we get that \( U \cap \mu(U) = U \); similarly, \( U \cap \nu(U) = U \). Hence, \( F \cap k(U) = U \), for each \( U \in \tau|F \). Clearly, \( k(F) = X \) and \( k(\emptyset) = \emptyset \), because of iii).

It is easily seen that \( k(U) \subset k(V) \) whenever \( U \subset V \) (indeed, \( \mu(U) \subset \mu(V) \) and \( \nu(U) \subset \nu(V) \)).

Next, we prove that if \( U \cup V = F \) then \( k(U) \cup k(V) = X \) (Wlog, let us assume that \( U \neq F \neq V \)). Let \( x \in X \) and suppose that \( x \notin \mu(U) \). Then, for each \( f \in C(F, [-2, 2]) \) such that \( f(F - U) = 2 \), we get that \( \phi(f)(x) \geq 1 \). Pick \( h \in C(F, [-2, 2]) \) such that \( h(F - V) = -2 \) and \( h(F - U) = 2 \) (this can be done because \( F \) is normal). It follows that \( \psi(h)(x) \geq \phi(h)(x) \geq 1 \), which implies that \( x \in \nu(V) \). Similarly, if \( x \notin \mu(V) \) then \( x \in \nu(U) \). Consequently, we get that \( x \in k(U) \cup k(V) \), as required.

Finally, we prove that, for each \( U \in \tau|F \), \( \overline{k(U)} \cap F = \overline{U} \), by proving that \( \overline{\mu(U)} \cap F = \overline{U} = \overline{\nu(U)} \cap F \) (we will prove the first equality and note that the second equality can be similarly proved): Let us assume that \( \mu(U) = \bigcup \{ \phi(f_{\alpha})^{-1}([-\infty, 1]) \mid \alpha \in \Lambda \} \). Since \( \phi \) satisfies condition vi) of b), we get that

\[
\overline{\mu(U)} \cap F = \bigcup_{\alpha} \phi(f_{\alpha})^{-1}([-\infty, 1]) \cap F = \bigcup_{\alpha} f_{\alpha}^{-1}([-\infty, 1]) = \overline{U}.
\]

This completes the proof. ■

**Theorem 4.** For a space \((X, \tau)\), the following are equivalent:

i) \( X \) is a \( K_W \)-space;

ii) For each closed subspace \( F \) of \( X \) there exists a function \( k : \tau|F \rightarrow \tau \) such that

- \((1') F \cap k(U) = U \), for each \( U \in \tau|F \), \( k(F) = X \), \( k(\emptyset) = \emptyset \),
- \((2') k(U) \subset k(V) \) whenever \( U \subset V \),
- \((3') U, V \in \tau|F \), \( \overline{U} \cap \overline{V} = \emptyset \) implies \( k(U) \cap k(V) = \emptyset \),
- \((4') k(U) \cap F = \overline{U} \).
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Proof: i) implies ii). Let $\sigma: \tau[F \to \tau$ be a $K_W$-function and define $k: \tau[F \to \tau$ by $k(U) = U \cup (X - [F \cup \sigma(F - U)])$. (Note that

$$k(U) = U \cup (X - F) \cap (X - \sigma(F - U))$$

and $X - \sigma(F - U) \supset U$ because, by (4),

$$(X - \sigma(F - U)) \cap F = F - (\sigma(F - U) \cap F) = F - F - U \supset U.$$  

Hence, we do get that $k(U) \in \tau$.

From the definition of $k$ we immediately get that $k$ satisfies (1').

$k$ satisfies (2'): $U \subseteq V$ implies $U \subseteq V$ implies $F - V \subseteq F - U$ implies

$$\sigma(F - V) \subseteq \sigma(F - U)$$

implies $k(U) \subseteq k(V)$.

$k$ satisfies (3'): $U \cap V = \emptyset$ implies $(F - U) \cup (F - V) = F$ implies $\sigma(F - U) \cup \sigma(F - V) = X$ implies

$$X - [F \cup \sigma(F - U)] \cap X - [F \cup \sigma(F - V)] =$$

$$= X - [F \cup \sigma(F - U)]^0 \cap X - [F \cup \sigma(F - V)]^0$$

$$= X - ([F \cup \sigma(F - U)]^0 \cup [F \cup \sigma(F - V)]^0) \supset$$

$$\subset X - (\sigma(F - U)^0 \cup \sigma(F - V)^0) \subset X - (\sigma(F - U) \cup \sigma(F - V)) = \emptyset.$$  

Also, $U \cap V = \emptyset$ implies $U \subseteq F - V$ implies $U \subseteq \sigma(F - V)$ implies $U \subseteq \sigma(F - V)^0$ implies

$$U \cap (X - [F \cup \sigma(F - U)]^0) = \emptyset;$$  

similarly, $V \cap (X - [F \cup \sigma(F - U)]^0) = \emptyset$. Consequently, $k(U) \cap k(V) = \emptyset$.

$k$ satisfies (4'): $k(U) \cap F = \overline{U} \cup ((X - [F \cup \sigma(F - U)]) \cap F) \supset U$; since

$$X - [F \cup \sigma(F - U)] \cap F \subset X - \sigma(F - U) \cap F = (X - [\sigma(F - U)]^0) \cap F =$$

$$= F - (F \cap [\sigma(F - U)]^0) \subset F - (F \cap \sigma(F - U)) = F - (F - U) = \overline{U},$$

we then get that $k(U) \cap F = \overline{U}$.

ii) implies i). One need only check that the preceding arguments are essentially reversible; that is, starting with $k$, which satisfies (1')–(4'), define $\sigma$ by

$$\sigma(U) = U \cup (X - [F \cup k(F - U)])$$

then $\sigma$ is a $K_W$-function: It is easily seen that $F \cap \sigma(U) = U$, for each $U \in \tau[F$, $\sigma(F) = X$, $\sigma(\emptyset) = \emptyset$, and $\sigma(U) \subseteq \sigma(V)$.
Whenever $U \subset V$. Also, $U, V \in \tau F$ and $U \cup V = F$ implies $U^0 \cup V^0 = F$ (here, interiors refer to $\tau F$) implies $(F - U^0) \cap (F - V^0) = \emptyset$ if and only if $(F - U) \cap (F - V) = \emptyset$ implies $k(F - U) \cap k(F - V) = \emptyset$ implies

$$U \cup (X - [F \cup k(F - U)]) \cup V \cup (X - [F \cup k(F - V)]) =$$

$$= (U \cup V) \cup (X - [F \cup k(F - U)] \cap [F \cup k(F - V)])$$

$$= F \cup (X - [F \cup (k(F - U) \cap k(F - V))]) = F \cup (X - F) = X.$$  

Therefore, $\sigma(U) \cup \sigma(V) = X$ whenever $U \cup V = F$. Finally, $\sigma(U) \cap F = U \cup (X - [F \cup k(F - U)]) \cap F \supset U$; since $X - [F \cup k(F - U)] \cap F \subset U$, we then get that $\sigma(U) \cap F = U$. We have thus shown that $\sigma$ is a $K_W$-function, which completes the proof.

**Corollary 5.** $K_W$-spaces are collectionwise normal.

**Proof:** Let $(X, \tau)$ be a $K_W$-space and $A = \{A_\alpha | \alpha \in \Lambda\}$ be a discrete collection of closed subsets of $X$. Letting $F = \bigcup A$, we get that each $A_\alpha \in \tau F$. Letting $k : \tau F \to \tau$ be a function which satisfies conditions $(1')$ and $(3')$ of Theorem 4, we then get that $\{k(A_\alpha) | \alpha \in \Lambda\}$ is a pairwise-disjoint collection of closed subsets of $X$ with each $A_\alpha \subset k(A_\alpha)$. This shows that $X$ is collectionwise normal.

**Appendix**

The following result is crucial to our work. It probably is folklore.

**Lemma 1.** Let $F$ be a completely normal space, $U$ and $V$ open subsets of $F$ and $f : F \to [-\infty, 1]$ be a continuous function such that $f(F - U \cup V) \subset \{1\}$. Then there exist continuous functions $f_1, f_2, f_3 : F \to [-\infty, 1]$ such that

i) $f_1(F - U) \cup f_2(F - V) \cup f_3(F - U \cap V) \subset \{1\}$;

ii) $\inf(f_1, f_2, f_3) \leq f$.

**Proof:** Let us first consider the case $U \cup V \neq F$. Since $F$ is completely normal and $\overline{U - V} \cap (V - U) = \emptyset = (U - V) \cap \overline{V - U}$, pick disjoint open $U', V'$ such that $U - V \subset U'$ and $V - U \subset V'$. Let $f_1 = f$ on $U - V'$ and $f_1 = 1$ on $F - U$ and extend $f_1$ to $f_1 : F \to [-\infty, 1]$. Let $f_2 = f$ on $V - U'$ and $f_2 = 1$ on $F - V$ and extend $f_2$ to $f_2 : F \to [-\infty, 1]$. Let $f_3 = f$ on $U \cap V - (U' \cap V')$ and $f_3 = 1$ on $F - U \cap V$ and extend $f_3$ to $f_3 : F \to [-\infty, 1]$. Since $U \cup V = (U - V') \cup (V - U') \cup ([U \cap V] - (U' \cup V'))$, we immediately get that $\inf(f_1, f_2, f_3) \leq f$. 


It is now clear that the result is also valid if \( U \cap V = \emptyset \). Finally, let us show that it also remains valid if \( U \cup V = F \): (Wlog, assume \( U \neq F \neq V \)). Simply pick open \( U', V' \) such that \( U' \subseteq U, \ V' \subseteq V \) and \( U' \cup V' = F \). Let \( f_1 = f \) on \( U' \) and \( f_1 = 1 \) on \( F - U \) and extend \( f_1 \) to \( f_1 : F \to ]-\infty, 1[ \). Let \( f_2 = f \) on \( V' \) and \( f_2 = 1 \) on \( F - V \) and extend \( f_2 \) to \( f_2 : F \to ]-\infty, 1[ \). Let \( f_3 = 1_F \). One immediately gets that \( \inf(f_1, f_2, f_3) \leq f \). □

**Remark.** Clearly, the preceding result remains valid for \( f : F \to [-1, \infty[, f(F - U \cup V) \subseteq \{-1\} \) and \( \sup(f_1, f_2, f_3) \geq f \).

**REFERENCES**


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