SOME DISTRIBUTIONAL PRODUCTS WITH RELATIVISTIC INVARIANCE

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Abstract: In [1, 2] we introduced a distribution product, invariant under the action of compact Lie groups of linear transformations. This product has application to non-relativistic physics. Here we present a general version of the product, invariant under various groups, including the Lorentz group.

0 – Introduction

A Lorentz invariant product of distributions based in [1, 2] will imply the existence of a Lorentz invariant function $\alpha \in \mathcal{D}(\mathbb{R}^4)$ such that $\int_{\mathbb{R}^4} \alpha = 1$. If such a function $\alpha$ exists, it will be invariant with the linear transformation defined by the matrix.

$$
\begin{bmatrix}
\text{ch} \, \theta & 0 & 0 & \text{sh} \, \theta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\text{sh} \, \theta & 0 & 0 & \text{ch} \, \theta
\end{bmatrix}
$$
with $\theta \in \mathbb{R}$,

which is the same to say that

$$
\alpha(x \, \text{ch} \, \theta + t \, \text{sh} \, \theta, y, z, x \, \text{sh} \, \theta + t \, \text{ch} \, \theta) = \alpha(x, y, z, t) \quad \text{for all } x, y, z, t \in \mathbb{R}.
$$

If we fix a point $(x_0, y_0, z_0, t_0)$ in the “space-time” we see that $\alpha$ is a constant function on the line

$$(x, y, z, t) = (x_0 \, \text{ch} \, \theta + t_0 \, \text{sh} \, \theta, y_0, z_0, x_0 \, \text{sh} \, \theta + t_0 \, \text{ch} \, \theta) \quad \text{with } \theta \in \mathbb{R}.$$

This line is not bounded in $\mathbb{R}^4$ because his projection on the $x_0t$-plane is the hiperbole $x^2 - t^2 = x_0^2 - t_0^2$. Thus, the only function $\alpha \in \mathcal{D}(\mathbb{R}^4)$ which is Lorentz invariant is the zero-function and we cannot have $\int_{\mathbb{R}^4} \alpha = 1$.

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Hence, it is not possible to have a Lorentz invariant product of distributions applying the theory of [1, 2]. In the following we define a family of distributional products with relativistic invariance based in a generalization of the methods of [1, 2] applying the concepts of order of growth and integral of a distribution introduced by Sebastião e Silva [4, 5].

1 – Preliminaries on limits, integrals and orders of growth of distributions

Let \( N \in \mathbb{N} = \{1, 2, 3, \ldots\} \). By \( C_\infty \) we mean the set of complex valued distributions of finite order defined on \( \mathbb{R}^N \). Recall the following concepts introduced by J. Sebastião e Silva [4, 5].

1.1. Let \( f \in C_\infty \) and \( \lambda \in \mathbb{C} \). We say that \( f(t_1, \ldots, t_N) \) converge to \( \lambda \) as \( (t_1, \ldots, t_N) \to (+\infty, \cdots, +\infty) \) and we write \( f(+\infty, \ldots, +\infty) = \lambda \) iff there exists a sistem of integers \( r_1, \ldots, r_N \geq 0 \) and a continous complex valued function \( F(t_1, \ldots, t_N) \) defined on \( \mathbb{R}^N \) such that

\[
\begin{align*}
\text{a)} & \quad D_1^{r_1} \cdots D_N^{r_N} F(t_1, \ldots, t_N) = f(t_1, \ldots, t_N) \text{ on } \mathbb{R}^N; \\
\text{b)} & \quad \forall \delta > 0 \exists L > 0: t_1, \ldots, t_N > L \Rightarrow \left| \frac{F(t_1, \ldots, t_N)}{t_1^{r_1} \cdots t_N^{r_N}} - \frac{\lambda}{r_1! \cdots r_N!} \right| < \delta.
\end{align*}
\]

\( D_k \) means the usual derivation operator relative to the variable \( t_k \). Note that the concept of convergence when some variables converge to \( +\infty \) and others to \( -\infty \) is analogous.

1.2. We can define the generalized De Barrow symbol \( [T(t_1, \ldots, t_N)]_{(u_1, \ldots, u_N)}^{(v_1, \ldots, v_N)} \) by setting

\[
\left[ T(t_1, \ldots, t_N) \right]_{(u_1, \ldots, u_N)}^{(v_1, \ldots, v_N)} = T(v_1, t_2, \ldots, t_N) - T(u_1, t_2, \ldots, t_N)
\]

and

\[
\left[ T(t_1, \ldots, t_N) \right]_{(u_1, \ldots, u_N)}^{(v_1, \ldots, v_N)} = \left[ T(t_1, \ldots, t_{N-1}, v_N) - T(t_1, \ldots, t_{N-1}, u_N) \right]_{(u_1, \ldots, u_{N-1})}^{(v_1, \ldots, v_{N-1})}.
\]

Now, we will say that \( f \in C_\infty \) is Silva-integrable on \( \mathbb{R}^N \) iff there exists a complex valued distribution \( T(t_1, \ldots, t_N) \) such that

\[
\begin{align*}
\text{a)} & \quad D_1 \cdots D_N T(t_1, \ldots, t_N) = f(t_1, \ldots, t_N) \text{ on } \mathbb{R}^N; \\
\text{b)} & \quad \forall \delta > 0 \exists L > 0: t_1, \ldots, t_N > L \Rightarrow \left| \frac{T(t_1, \ldots, t_N)}{t_1^{r_1} \cdots t_N^{r_N}} - \frac{\lambda}{r_1! \cdots r_N!} \right| < \delta.
\end{align*}
\]
In this case, we will write

$$
\int_{\mathbb{R}^N} f = \lim_{(v_1, \ldots, v_N) \to (+\infty, \ldots, +\infty)} \left[ T(t_1, \ldots, t_N) \right]_{(u_1, \ldots, u_N) \to (-\infty, \ldots, -\infty)}^{(v_1, \ldots, v_N)}.
$$

For instance, with $N = 1$ in the sense of Silva, because $\cos t = D \sin t$ and $[\sin t]^{+\infty}_{-\infty} = \sin(+\infty) - \sin(-\infty)$. Meanwhile, $\sin(+\infty) = 0$ on account of $\sin t = D[-\cos t]$ and $\frac{\cos t}{t} \to 0$ as $t \to +\infty$. Also $\sin(-\infty) = 0$.

1.3. $f \in C_{\infty}$ is bounded on $\mathbb{R}^N$ in the sense of Silva iff there exists a sistem of integers $r_1, \ldots, r_N \geq 0$ and a complex valued continuous function $F$ defined on $\mathbb{R}^N$ such that

a) $D_1^{r_1} \cdots D_N^{r_N} F(t_1, \ldots, t_N) = f(t_1, \ldots, t_N)$ on $\mathbb{R}^N$;

b) For any invertible linear transformation $A : \mathbb{R}^N \to \mathbb{R}^N$ the function $F_{\circ A(t_1, \ldots, t_N)}^{F \circ A(t_1, \ldots, t_N)}$ is bounded in the usual sense on the region $|t_1| > k, \ldots, |t_N| > k$ where $k$ is a positive number.

Let $\alpha \in \mathbb{R}$ and $f \in C_{\infty}$. We write $f \in O(\|t\|^\alpha)$ as $\|t\| \to \infty$ in the sense of Silva if there are $f_0 \in C_{\infty}$ bounded in the sense of Silva and a number $\varepsilon > 0$ such that $f(t) = \|t\|^\alpha f_0(t)$ when $\|t\| > \varepsilon$. Recall that the concept of bounded distribution in the sense of Silva is more general that the same concept in the sense of Schwartz [3].

2 – The family of products

Let $G$ be a group of unimodular transformations $L : \mathbb{R}^N \to \mathbb{R}^N$ (|det $L| = 1$) such that there exists a $C^\infty$-function $\alpha : \mathbb{R}^N \to \mathfrak{C}$ obeying the conditions:

a) $\alpha$ is $G$-invariant;

b) $\alpha \in O(\|t\|^p)$ as $\|t\| \to \infty$ with $p < -N$, in the sense of Silva;

c) $\int \alpha = 1$.

(Unless otherwise specified, all integrals are over $\mathbb{R}^N$ and in the sense of Silva.)

2.1 Definition. Let $T \in \mathcal{D}'$ and $U \in \mathcal{D}'_n$ ($\mathcal{D}'_n$ denotes the space of distributions with nowhere dense support). We say that there exists the product of $T$ by $U$ relative to the pair $(G, \alpha)$ iff
d) For each \( x \in D \) there exists an integer \( p < -N \) such that

\[
T[\alpha * (Ux)] \in O(\|t\|^p) \quad \text{as} \quad \|t\| \to \infty \quad \text{in the sense of Silva},
\]

e) The functional \( x \to \int T[\alpha * (Ux)] \) is continuous on \( D \) (endowed with the usual topology),

and we call product of \( T \) by \( U \) relative to the pair \((G, \alpha)\), the distribution \( T_\alpha U \) defined by

\[
\langle T_\alpha U, x \rangle = \int T[\alpha * (Ux)] \quad \text{for all} \quad x \in D.
\]

Note that if \( f \in C_\infty \cap O(\|t\|^p) \) with \( p < -N \) then \( f \) is Silva integrable (see [4]).

This product verifies the distributive properties and is consistent with the product of \( T \in D' \) by \( U \in D'_n \) defined in [1, 2]. Indeed, if \( \alpha \in D \) then \( \langle T_\alpha U, x \rangle = \langle T, \alpha * (Ux) \rangle = \langle T\zeta^{-1}(U), x \rangle \) which is the \((G, \alpha)\)-result we have obtained in [1, 2].

Observe that the family of products defined here is really more general than that considered in [1, 2] in the sense that it can be applied to a wider class of groups. For instance:

**2.2 Proposition.** Let \( G \) be the Lorentz group in the 4-dimensional space. Then, there exists a \( C^\infty \)-function \( \alpha : \mathbb{R}^4 \to \mathbb{C} \) such that conditions a), b), c) are verified.

**Proof:** Let \( \alpha(x, y, z, t) = \frac{1}{\sqrt{1 - t^2}} e^{i(x^2 + y^2 + z^2 - t^2)} \). It is obvious that \( \alpha \) is Lorentz invariant. To see that \( \alpha \) verifies conditions b), let us note that \( x\alpha = \frac{1}{2} D_x \alpha \) and for \( p = 2, 3, 4, \ldots \), \( x^p \alpha = \frac{1}{2^p} [D_x (x^{p-1} \alpha) - (p - 1)x^{p-2} \alpha] \). Thus, for \( p = 0, 1, 2, \ldots \), \( x^p \alpha = \sum_{j \in \mathbb{N}_0} a_j D^j \alpha \) where only a finite number of \( a_j \in \mathbb{C} \) are different from zero (\( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \)). By a similar process we conclude that for \( q = 0, 1, 2, \ldots \), \( y^q \alpha = \sum_{l \in \mathbb{N}_0} b_l D^l \alpha \) and so

\[
x^p y^q z^r t^s \alpha = \sum_{(j, l, m, n) \in \mathbb{N}_0^4} d_{jlmn} D^j_x D^l_y D^m_z D^n_t \alpha,
\]

where only a finite number of \( d_{jlmn} \in \mathbb{C} \) are different from zero.

But, for every invertible linear transformation \( A : \mathbb{R}^4 \to \mathbb{R}^4 \) the function \( A(x, y, z, t) \) is bounded in the usual sense in some region \( |x| > x_0, |y| > y_0, |z| > z_0, |t| > t_0 \) with \( x_0, y_0, z_0, t_0 > 0 \) and we conclude that \( x^p y^q z^r t^s \alpha \) is bounded on \( \mathbb{R}^4 \) in the sense of Silva. Then, \( (x^2 + y^2 + z^2 + t^2)^3 \alpha \) is bounded on \( \mathbb{R}^4 \) in the same sense, which proves that

\[
\alpha \in O\left(\|(x, y, z, t)||^{-6}\right) \quad \text{as} \quad \|(x, y, z, t)|| \to \infty \quad \text{in the sense of Silva}.
\]
Applying Theorem 14.2 of [4] we have

$$
\int_{\mathbb{R}^4} \alpha = \frac{1}{\pi^2 i} \left( \int_{\mathbb{R}} e^{ix^2} \, dx \right) \left( \int_{\mathbb{R}} e^{iy^2} \, dy \right) \left( \int_{\mathbb{R}} e^{iz^2} \, dz \right) \left( \int_{\mathbb{R}} e^{-it^2} \, dt \right) = 1
$$

because

$$
\int_{\mathbb{R}} \sin(x^2) \, dx = \int_{\mathbb{R}} \cos(x^2) \, dx = 2 \sqrt{\frac{\pi}{8}}
$$

are simply convergent integrals (Fresnel’s integrals) and so the corresponding Silva-integrals exist and have the same value.

Meanwhile, the product of this approach is certainly more restricted in what concerns the left-hand side factor.

Applying 1.2 and 1.1 we can prove a proposition we need, which is an example of the power of the concepts introduced by S. Silva in [4, 5].

**2.3 Proposition.** If $f \in C_\infty$ is Silva integrable on $\mathbb{R}^N$, then all partial derivatives of $f$ are Silva integrable and the integral of anyone of them is zero.

Now we can prove the usual derivation rule.

**2.4 Proposition.** If $T \in \mathcal{D}'$, $U \in \mathcal{D}_n'$, there exists $T \cdot U$ and one of the products $(D_k T)_{\alpha} U$ or $T_{\alpha} (D_k U)$ then there exists the other product and we have

$$
D_k(T_{\alpha} U) = (D_k T)_{\alpha} U + T_{\alpha} (D_k U), \quad k = 1, \ldots, N.
$$

**Proof:** Let $x \in \mathcal{D}$. It is easy to prove that

$$
D_k[T(\alpha \ast U x)] - (D_k T)(\alpha \ast U x) - T[\alpha \ast (D_k U)x] = T[\alpha \ast U(D_k x)].
$$

By assumption we have

$$
\int D_k[T(\alpha \ast U x)] - \int (D_k T)(\alpha \ast U x) - \int T[\alpha \ast (D_k U)x] = \int T[\alpha \ast U(D_k x)],
$$

the first term is zero by Proposition 2.3 and we can write

$$
\langle (D_k T)_{\alpha} U, x \rangle - \langle T_{\alpha} (D_k U), x \rangle = \langle T_{\alpha} U, D_k x \rangle = -\langle D_k(T_{\alpha} U), x \rangle
$$

which proves the proposition.

This product is also invariant with translations and all transformations of $G$.

**2.5 Proposition.** If $T \in \mathcal{D}'$, $U \in \mathcal{D}_n'$, $L \in G$, $a \in \mathbb{R}^N$ and the product $T_{\alpha} U$ exists, then:
a) \( \tau_a(T \cdot U) = (\tau_a T) \cdot (\tau_a U) \);
b) \( (T \cdot U) \circ L = (T \circ L) \cdot (U \circ L) \).

**Proof:** Let \( x \in \mathcal{D} \).

a) \( \langle \tau_a(T \cdot U), x \rangle = \langle T \cdot U, \tau_{-a} x \rangle = \int T[\alpha * U(\tau_{-a} x)] = \int T \tau_{-a}[\alpha * (\tau_a U)x] \)
and applying theorem 13.8 of [4] we can write
\[
\langle \tau_a(T \cdot U), x \rangle = \int (\tau_a T)[\alpha * (\tau_a U)x] = \langle (\tau_a T) \cdot (\tau_a U), x \rangle
\]
and the invariance of the product with translations is proved.

b) \( \langle T \cdot U \circ L, x \rangle = \langle T \cdot U, x \circ L^{-1} \rangle = \int T[\alpha * U(x \circ L^{-1})] \)
\[
= \int T\left[\alpha * (((U \circ L)x) \circ L^{-1})\right]
\]
and again by theorem 13.8 of [4] we can write
\[
\langle T \cdot U \circ L, x \rangle = \int (T \circ L)\left[\alpha * (((U \circ L)x) \circ L^{-1})\right] \circ L \right].
\]

Noting that, if \( V \) is a distribution with compact support, \( \alpha \in C^\infty \) and \( L : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is unimodular, then \( (\alpha * V) \circ L = (\alpha \circ L) * (V \circ L) \), we have
\[
\langle T \cdot U \circ L, x \rangle = \int (T \circ L)\left[(\alpha \circ L) * ((U \circ L)x)\right] = \int (T \circ L)\left[\alpha * ((U \circ L)x)\right]
\]
\[
= \langle (T \circ L) \cdot (U \circ L), x \rangle,
\]
which proves that the product is \( G \)-invariant.

3 – Examples and comments

**3.1.** Notice that if \( T \) is a distribution with compact support and \( U \in \mathcal{D}'_a \) then there exists always the product \( T \cdot U \) and we dont need the Silva integral to compute it because, in this case we have
\[
\langle T \cdot U, x \rangle = \int T(\alpha * Ux) = \langle T(\alpha * Ux), 1 \rangle = \langle T, \alpha * Ux \rangle.
\]

For instance, let \( G \) be the Lorentz group in \( \mathbb{R}^4 \) and let \( \alpha \) a \( C^\infty \)-function obeying the conditions 2a), b), c). The product of two Dirac delta distributions is easily obtained
\[
\langle \delta \cdot \delta, x \rangle = \langle \delta, \alpha * \delta x \rangle = \langle \delta, x(0) \cdot \alpha \rangle = x(0) \alpha(0) = \langle (\alpha(0) \delta, x \rangle.
\]
Thus, we can write \( \delta \cdot \delta = \alpha(0) \delta \). Formally this is the same result we have obtained in [1] 1.5.6 relative to a group of a more restricted class which does not include the Lorentz group.

3.2. Let \( G \) be the Lorentz group in \( \mathbb{R}^4 \), \( \alpha \) a \( C^\infty \)-function obeying the conditions 2a), b), c) and \( H \) the Heaviside function defined on \( \mathbb{R}^4 \). We can compute \( H \cdot \delta \) where \( \delta \) is the Dirac distribution defined on \( \mathbb{R}^4 \) noting that

\[
\alpha \cdot \delta (0) = \frac{1}{16}
\]

Thus, we will prove that \( \int_{\mathbb{R}^4} \phi(0) H \alpha \) is continuous on \( \mathcal{D}(\mathbb{R}^4) \) because \( H \alpha \) is Silva-integrable on \( \mathbb{R}^4 \) and so this functional is equal to the distribution \( (\int_{\mathbb{R}^4} H \alpha) \delta \).

Now, we will prove that \( \int_{\mathbb{R}^4} H \alpha = \frac{1}{16} \) in the sense of Silva. Let \( R : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) be any one of the 16 transformations \((x, y, z, t) \rightarrow (\pm x, \pm y \pm z, \pm t)\) and \( P(x, y, z, t) = x^2 + y^2 + z^2 - t^2 \). We have \( P \circ L = P \) for all Lorentz transformation \( L \) and so \( P \circ L \circ R = P \circ R = P \) which proves that \( L \circ R \) is also a Lorentz transformation and condition 2a) yields \( \alpha \circ L \circ R = \alpha \). Then \( \alpha \circ R = \alpha \) which proves that \( \alpha \) is \( R \)-invariant. Putting \( A = \{-1, 1\} \) we can write \( \alpha \) as sum of 16 terms

\[
\alpha(x, y, z, t) = \sum_{(i,j,k,\ell) \in A^4} \alpha(x, y, z, t) H(ix, jy, kz, \ell t),
\]

thus

\[
1 = \int_{\mathbb{R}^4} \alpha = \sum_{(i,j,k,\ell) \in A^4} \int_{\mathbb{R}^4} \alpha(x, y, z, t) H(ix, jy, kz, \ell t) \, dx \, dy \, dz \, dt
\]

and it is easy to see that all terms of this sum are identical: for instance, applying the change of variable \( x = -s, y = u, z = v, t = w \) in the Silva integral we have

\[
\int_{\mathbb{R}^4} \alpha(x, y, z, t) H(-x, y, z, t) \, dx \, dy \, dz \, dt =
\]

\[
= \int_{\mathbb{R}^4} \alpha(s, u, v, w) H(s, u, v, w) \, ds \, du \, dv \, dw.
\]

Then \( \int_{\mathbb{R}^4} H \alpha = \frac{1}{16} \) and it is proved that \( H \cdot \delta = \frac{1}{16} \delta \) is independent of the \( \alpha \) function.

3.3. It is possible to extend this product (and also the product defined in [1, 2]) to a larger class of situations in a simple way. Let \( p \) be an integer \( \geq 0 \) or \( p = \infty \) and \( \mathcal{D}^p \) the space of distributions of order \( \leq p \) in the sense of Schwartz.
[3]. If $T \in \mathcal{D}'$, $S = \beta + U \in C_0 \oplus \mathcal{D}'_n$ and there exists $T \cdot U$ in the sense of 2.1 we can always define $T \cdot S$ putting $T \cdot S = T\beta + T\bar{\alpha} U$ where $T\beta$ is the product in the sense of Schwartz [3].

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REFERENCES


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