PROPER LEFT TYPE-A COVERS

JOHN FOUNTAIN and GRACINDA M.S. GOMES

Introduction

Left type-A monoids form a special class of left abundant monoids. Interest in the latter arose originally from the study of monoids by means of their associated S-sets. A left abundant monoid is a monoid with the property that all principal left ideals are projective. All regular monoids are left abundant and so are many other types of monoid including right cancellative monoids. A left abundant monoid $S$ is said to be left type-A if the set $E(S)$ of idempotents of $S$ is a commutative submonoid of $S$ and $S$ also satisfies the condition that for any elements $e$ in $E(S)$ and $a$ in $S$ we have $eS \cap aS = eaS$. In fact, [see 2] left type-A monoids are precisely those monoids which are isomorphic to certain submonoids of symmetric inverse monoids, namely those submonoids $S$ of $I(X)$ which satisfy the condition that if $\alpha$ is in $S$, then $\alpha \alpha^{-1}$ is in $S$. Thus all inverse monoids are left type-A but there are many left type-A monoids which are not inverse, for example, right cancellative monoids which are not groups. We see from the characterization just given that for a topological space $X$, the submonoid of $I(X)$ consisting of continuous one-one partial maps is left type-A. In general, of course, this example is not inverse. A significant body of structure theory has been developed for left type-A monoids, much of it inspired by corresponding theory for inverse monoids. In particular, it is shown in [2] that for the study of general left type-A monoids the subclass of proper left type-A monoids plays a special role.

This paper is the last of a series of three devoted to studying proper left type-A monoids via categories. The ideas and techniques are inspired by those which Margolis and Pin introduced [5] in their study of $E$-dense and inverse monoids. The first paper [3] of the series showed that the work of Margolis and
Pin for $E$-dense monoids could be strengthened in the case of left type-$A$ $E$-dense monoids to give generalizations of results on inverse monoids. This paper and the second [4] of the series are concerned with extending the techniques to apply to left type-$A$ monoids in general. The concept of a left type-$A$ monoid is essentially a one-sided notion and this is reflected in the fact that it is possible to generalize the methods in two ways. In [4] we considered right actions on categories and were led to new results on left type-$A$ monoids.

In the present paper we study left type-$A$ monoids by means of left actions on categories. This forces us to change both the nature of the categories considered and the definition of the action.

In Section 1 we use our new techniques to obtain a new proof of a theorem of Palmer [6] which characterizes proper left type-$A$ monoids in terms of $M$-systems. Palmer’s result is a variation of a characterization obtained in [2]. The other main result of [2] is that every left type-$A$ monoid has a proper left type-$A$ cover. In [1] the categorical methods of Margolis and Pin were used to show that every $E$-dense monoid has an $E$-unitary dense cover. This result was relativized in [3] to the case of left type-$A$ $E$-dense monoids showing that the cover constructed is proper and respects the relation $R$. In Section 2 of the present paper we adapt the techniques of [1] to obtain a new proof of the covering theorem of [2]. That is, we prove that every left type-$A$ monoid has a left type-$A$ $+$-cover. It is not difficult to see that this is, in fact, the dual of Theorem 3.3 of [2].

1 – Preliminaries

We start by recalling some of the definitions and results, presented in [3], for both left type-$A$ monoids and categories.

On left type-$A$ monoids

Let $S$ be a monoid, with set of idempotents $E(S)$. On $S$, we define a binary relation $R^*$, which contains the Green’s relation $R$, as follows: for all $a, b \in S$,

$$(a, b) \in R^* \iff (\forall s, t \in S) \, sa = ta \iff sb = tb \, .$$

The monoid $S$ is said to be left abundant if each $R^*$-class, $R^*_a$, contains an idempotent. When $E(S)$ is a semilattice, such idempotent is unique and it is denoted by $a^+$. If, in addition, $S$ satisfies the type-A condition: for all $a \in S$ and $e \in E(S)$,

$$ae = (ae)^+a ,$$
we say that $S$ is a left type-$A$ monoid. It is shown in [2] that this definition is equivalent to those given in the Introduction.

We remind the reader of the following basic properties of left type-$A$ monoids which we use frequently and without further mention:

1) For every $a, b, c \in S$, $aR^+ b$ implies $caR^+ cb$;
2) For every $a \in S$, $a = a^+ a$;
3) For every $e \in E(S)$ and $a \in S$, $(e a)^+ = e a^+$.

On a left type-$A$ monoid $S$, the least right cancellative monoid congruence, $\sigma$, is defined by: for all $a, b \in S$,

$$(a, b) \in \sigma \iff (\exists e \in E(S)) e a = e b ;$$

and we say that $S$ is proper if

$$\sigma \cap R^+ = \iota ,$$

where $\iota$ is the identity relation [2].

As usual by an $E$-unitary semigroup, we mean a semigroup $S$ such that, for all $a \in S$ and $e \in E(S)$,

$$a e \in E(S) \text{ or } e a \in E(S) \Rightarrow a \in E(S) .$$

In [2], it is shown that every proper left type-$A$ monoid is $E$-unitary but, however, the converse is not true.

**On left type-$A$ categories**

Let $C$ be a (small) category. We denote the set of objects of $C$ by $\text{Obj} C$ and the set of morphisms by $\text{Mor} C$. For any object $u$ of $C$, $\text{Mor}(u, -)$ stands for the set of morphisms of $C$ with domain $u$ and $\text{Mor}(-, u)$ for the set of morphisms of $C$ with codomain $u$; we denote the identity morphism at the object $u$ by $O_u$.

As in [5], we adopt an additive notation for the composition of morphisms. A morphism $p$ is said to be an idempotent if $p = p + p$. Clearly, if $p$ is an idempotent then $p \in \text{Mor}(u, u)$, for some $u \in \text{Obj} C$.

On the partial groupoid $\text{Mor} C$, we define the $R^*$-relation as for a monoid.

A category $C$ is said to be $E$-left type-$A$ if, for all $u \in \text{Obj} C$, $E(\text{Mor}(u, u))$ is a semilattice, every $R^*$-class $R^*_p$ of $\text{Mor} C$ contains an idempotent $p^+$ (necessarily unique) and $C$ satisfies the type-$A$ condition, i.e. for all $u, v \in \text{Obj} C$, $p \in \text{Mor}(u, v)$ and $f \in E(\text{Mor}(v, v))$,

$$p + f = (p + f)^+ + p .$$

Let $C^0$ be an $E$-left type-$A$ category with a distinguished object $u_0$ such that $\text{Mor}(u_0, u_0)$ is a semilattice. We say that $C^0$ is (left) $u_0$-connected if, for all
\[ v \in \text{Obj} \mathcal{C}^0, \text{Mor}(u_0, v) \neq \emptyset. \] Also, \( \mathcal{C}^0 \) is called (left) \( u_0 \)-proper if, for all \( v \in \text{Obj} \mathcal{C}^0 \) and \( p, q \in \text{Mor}(u_0, v) \),

\[ p^+ = q^+ \Rightarrow p = q, \]

i.e. each \( \mathcal{R}^* \)-class has at most an element of \( \text{Mor}(u_0, v) \).

To simplify the terminology, we say that an \( E \)-left type-\( A \), \( u_0 \)-connected and \( u_0 \)-proper category \( \mathcal{C}^0 \), with distinguished element \( u_0 \) is a \( u_0 \)-proper left category.

**2 - \( u_0 \)-proper left categories**

In this section, we begin by considering left actions of right cancellative monoids on \( E \)-left type-\( A \) categories. In particular, we introduce the ideas of a downwards action and a \( u_0 \)-closed action. We show that given a right cancellative monoid acting in this way on a \( u_0 \)-proper left category we can form a proper left type-\( A \) monoid and that any proper left type-\( A \) monoid arises in this way. We then use this result to recover a theorem of Palmer which states that every proper left type-\( A \) monoid is isomorphic to an \( M \)-monoid.

**Definition 2.1.** Let \( \mathcal{C} \) be an \( E \)-left type-\( A \) category and \( T \) a right cancellative monoid. We say that \( T \) acts (on the left) on \( \mathcal{C} \) (by \( \mathcal{R}^* \)-endomorphisms) if, for all \( u \in \text{Obj} \mathcal{C} \) and \( t \in T \), there exists a unique \( tu \in \text{Obj} \mathcal{C} \), and, for all \( u, v \in \text{Obj} \mathcal{C}, p \in \text{Mor}(u, v) \), there is a unique \( tp \in \text{Mor}(tu, tv) \) such that, for all \( u, v, w \in \text{Obj} \mathcal{C}, p \in \text{Mor}(u, v), q \in \text{Mor}(v, w) \) and \( t, t_1, t_2 \in T \),

- \( t(p + q) = tp + tq \),
- \( (t_1 t_2) p = t_1(t_2 p) \),
- \( t O_v = O_{tv} \),
- \( 1 p = p \),
- \( (tp)^+ = tp^+ \).

It is not difficult to check that

**Lemma 2.2.** Let \( \mathcal{C}^0 \) be a \( u_0 \)-proper left category and \( T \) a right cancellative monoid acting on \( \mathcal{C}^0 \). Then

\[ \mathcal{C}_{u_0} = \left\{ (p, t) : t \in T, p \in \text{Mor}(u_0, tu_0) \right\}, \]

with multiplication given by

\[ (p, t)(q, s) = (p + tq, ts) \]
is a proper left type-A monoid such that $E(C_{u_0}) \simeq \text{Mor}(u_0, u_0)$.

**Definition 2.3.** Let $C^0$ be an $E$-left type-A category, with a distinguished object $u_0$, and $T$ a right cancellative monoid acting on $C^0$. We say that the action of $T$ on $C^0$ is downwards if, for all $u \in \text{Obj}C^0$ and $t \in T$,

$$\text{Mor}(tv, -) = t \text{Mor}(v, -).$$

On the other side, if the action of $T$ over $u_0$ satisfies the following properties:

- $\text{Obj}C^0 = Tu_0$,
- for all $v \in \text{Obj}C^0$, if $\text{Mor}(v, u_0) \neq \emptyset$ then $v = gu_0$, for some unit $g \in T$,

we say that the action is $u_0$-closed.

**Lemma 2.4.** Let $C^0$ be a $u_0$-proper left category and $T$ a right cancellative monoid acting on $C^0$. If, for all $v \in \text{Obj}C^0$,

$$\text{Mor}(v, u_0) \neq \emptyset \Rightarrow v = gu_0, \quad \text{for some unit } g \in T,$$

then, for all $p, q \in \text{Mor}(v, u_0)$,

$$p^+ = q^+ \Rightarrow p = q.$$

**Proof:** Let $p, q \in \text{Mor}(v, u_0)$ be such that $p^+ = q^+$. As $\text{Mor}(v, u_0) \neq \emptyset$, there exists a unit $g \in T$ such that $v = gu_0$. Now, as the action respects the operation $^+$, we have

$$(g^{-1}p)^+ = g^{-1}p^+ = g^{-1}q^+ = (g^{-1}q)^+,$$

where $g^{-1}p, g^{-1}q \in \text{Mor}(u_0, g^{-1}u_0)$. Whence, $C^0$ being $u_0$-proper, $g^{-1}p = g^{-1}q$ and, so $p = q$. $

Let $M$ be a proper left type-A monoid and $T = M/\sigma$. We define the derived category $D^0$ (of the natural morphism $M \to M/\sigma$) as in [3]: $\text{Obj}D^0 = T$ and, for all $t_1, t_2 \in T$,

$$\text{Mor}(t_1, t_2) = \{(t_1, m, t_2) : m \in M, \ t_1(m \sigma) = t_2\},$$

with composition given by

$$(t_1, m, t_2) (t_2, n, t_3) = (t_1, mn, t_3).$$

The distinguished object of $D^0$ is 1, the identity of $T$. The action of $T$ over $D^0$ is given by: for all $u \in \text{Obj}D^0$ and $t \in T$, $tu$ is the result of the multiplication of $t$ by $u$ in $T$ and for all $(u, m, v) \in \text{Mor}(u, v)$,

$$t(u, m, v) = (tu, m, tv).$$
Lemma 2.5. Let $M$ be a proper left type-$A$ monoid. Then the derived category $D^0$ is a 1-proper left category and the action of $T$ on $D^0$ is downwards and 1-closed.

Proof: First, notice that if $M$ is a proper left type-$A$ monoid then $M$ is $E$-unitary and, so $1 = E(M)$. Then, following [3, 4], we have that $D^0$ is an $E$-left type-$A$ category where, for all $(t_1, m, t_2) \in \text{Mor}(D^0)$,

$$(t_1, m, t_2)^+ = (t_1, m^+, t_1)$$

and

$$E(\text{Mor}(t, t)) = \{(t, e, t) : e \in E(M)\} \simeq E(M).$$

In particular,

$$\text{Mor}(1, 1) = E(\text{Mor}(1, 1)) \simeq E(M).$$

The category $D^0$ is 1-connected since, for all $m \sigma \in M/\sigma = T$,

$$(1, m, m \sigma) \in \text{Mor}(1, m \sigma).$$

On the other hand, $D^0$ is 1-proper, since $M$ is proper, i.e. $R^* \cap \sigma = \iota$.

It is a routine matter to verify that $T$ acts on $D^0$ in such a way that $\text{Obj}(D^0) = T1$. To prove that $T$ acts downwards, let $t \in T$, $u \in \text{Ob}(D^0)$ and $p \in \text{Mor}(tu, -)$. Then, there exists $m \in M$ such that

$$p = (tu, m, tu.m \sigma),$$

and, so

$$p = t(u, m, tu.m \sigma) \in t\text{Mor}(u, -).$$

It is obvious that $t\text{Mor}(u, -) \subseteq \text{Mor}(tu, -)$, hence $t\text{Mor}(u, -) = \text{Mor}(tu, -)$. Finally, let $p \in \text{Mor}(v, 1)$. Then, $p = (v, m, 1)$ for some $m \in M$ and $v.m \sigma = 1$. As $T$ is right cancellative, $v.m \sigma = 1 = m \sigma.v$ and $v = v.1$ is a unit of $T$, as required. 

Theorem 2.6. Let $M$ be a monoid. Then, $M$ is proper and left type-$A$ if and only if $M \simeq C_{u_0}$, where $u_0$ is the distinguished object of a $u_0$-proper left category $C^0$ on which a right cancellative monoid $T$ acts via an action which is downwards and $u_0$-closed.

Proof: In view of Lemma 2.2, under the above conditions, if $M \simeq C_{u_0}$, then $M$ is a proper left type-$A$ monoid.

Conversely, let $M$ be a proper left type-$A$ monoid. Then, by Lemma 2.5, the derived category $D^0$ of $M$ is a 1-proper left category and $T = M/\sigma$ is a right
cancellative monoid which acts on \( D^0 \) with an action which is downwards and 1-closed. Now, we consider the map

\[
\psi: M \to C_1 = \{(p, t): t \in T, \ p \in \text{Mor}(1, t)\}
\]

\[
m \mapsto ((1, m, m\phi), m\phi),
\]

which is easily seen to be an isomorphism and the result follows.

Let \( C \) be an \( E \)-left type-A category. On \( \text{Mor} C \), we define a relation \( \preceq \) as follows: for all \( p, q \in \text{Mor} C \),

\[
p \preceq q \iff (\exists a \in \text{Mor} C) \ p^+ = a^+, \ a + q^+ = a .
\]

In [3], we showed that \( \preceq \) is a preorder on \( \text{Mor} C \) and that the relation defined by

\[
p \sim q \iff p \preceq q \ and \ q \preceq p
\]

defines an equivalence relation on \( \text{Mor} C \) which contains \( R^* \). Also, on the quotient set \( X = \text{Mor} C/ \sim \), we consider the partial order \( \leq \) given by, for all \( A_p, A_q \in X \),

\[
A_p \leq A_q \iff p \preceq q .
\]

If \( T \) is a right cancellative monoid acting on \( C \), we define an action (on the left) of \( T \) on the partially ordered set \( X \) in the following way: for all \( A_p \in X \) and \( t \in T \),

\[
t A_p = A_{tp} .
\]

**Lemma 2.7.** Let \( C^0 \) be a \( u_0 \)-proper left category and \( T \) a right cancellative monoid acting on \( C^0 \). If the action is such that, for all \( v \in \text{Obj} C^0 \),

\[
(\ast) \quad \text{Mor}(v, u_0) \neq \emptyset \ \Rightarrow \ v = g u_0, \ \text{for some unit} \ g \in T ,
\]

then the action of \( T \) over \( X \) respects the relations \( \preceq, \sim \) and \( \leq \).

Moreover, for all \( t, t' \in T \), \( p \in \text{Mor}(u_0, tu_0) \) and \( q \in \text{Mor}(u_0, t'u_0) \),

\[
A_p \wedge A_{tq} = A_{p+tq} .
\]

**Proof:** By bearing in mind condition \((\ast)\) and Lemma 2.4, the proof is similar to the proof of Lemma 3.12 of [3]. Notice that here we need \( C^0 \) to be \( u_0 \)-proper.

**Lemma 2.8.** Under the conditions of Lemma 2.7, let

\[
\mathcal{Y} = \left\{ A \in X: A \cap \text{Mor}(u_0, u_0) \neq \emptyset \right\} .
\]
Then

a) \( \mathcal{Y} \) is a semilattice of \( \mathcal{X} \) with greatest element \( F = A_{O_{u_0}} \);

b) \( \mathcal{Y} = \left\{ A \in \mathcal{X}; (\exists v \in \text{Obj} \mathcal{C}^0) \ A \cap \text{Mor}(u_0, v) \neq \emptyset \right\} \);

c) \( (\forall t \in T) \ (\forall B \in \mathcal{Y}) \ B \leq tF \iff B \cap \text{Mor}(u_0, tu_0) \neq \emptyset \);

d) \( (\forall t \in T) \ (\exists B \in \mathcal{Y}) \ B \leq tF \).

**Proof:** Since \( \mathcal{C}^0 \) is a \( u_0 \)-proper left category, \( \text{Mor}(u_0, u_0) \) is a semilattice and condition a) follows from the previous lemma.

On any \( E \)-left type-\( A \) category \( \mathcal{C} \), for all \( u, v \in \text{Obj} \mathcal{C} \) and \( p \in \text{Mor}(u_0, v) \), we must have \( p^+ \in \text{Mor}(u_0, u_0) \). Since the equivalence \( \sim \) contains \( \mathcal{R}^* \), condition b) must hold.

c) Let \( t \in T \) then \( tF = A_{O_{tu_0}} \). Let \( B = A_q \in \mathcal{Y} \), with \( q \in \text{Mor}(u_0, u_0) \). Suppose that \( B \leq tF \). Then, \( q \leq O_{tu_0} \). Thus, there exists \( r \in \text{Mor}(u_0, tu_0) \) such that \( q^+ = r^+ \) and, so
\[
A_q = r^+ \cap \text{Mor}(u_0, tu_0).
\]
Conversely, suppose that there exists \( r \in A_q \cap \text{Mor}(u_0, tu_0) \). Then, \( r + O_{tu_0} = r \).

Hence, \( r \leq O_{tu_0} \) and \( A_r = B \leq tF \).

d) Let \( t \in T \). Since \( \mathcal{C}^0 \) is \( u_0 \)-connected, there exists \( a \in \text{Mor}(u_0, tu_0) \). Thus, \( A_a \in \mathcal{Y} \), by condition b), and \( a \leq O_{tu_0} \).

Next, we make the connection between the characterization of a proper left type-\( A \) monoid \( M \) as an \( M \)-monoid [6] and the characterization of \( M \), via categories, as a \( \mathcal{C}_{u_0} \) monoid. We start by describing an \( M \)-monoid.

**Definition 2.9** [6]. Let \( X \) be a partially ordered set and \( Y \) a subsemilattice of \( X \) with greatest element \( f \). Let \( T \) be a right cancellative monoid acting (on the left) on \( X \), in such a way that

- \( (\forall a \in X) \ 1a = a \);
- \( (\forall a, b \in X) \ (\forall t \in T), a \leq b \Rightarrow ta \leq tb \);
- \( X = TY \);
- \( (\forall t \in T) \ (\exists b \in Y) \ b \leq tf \);
- \( (\forall a, b \in Y) \ (\forall t \in T) a \leq tf \Rightarrow a \wedge tb \in Y \);
- \( (\forall a, b, c \in Y) \ (\forall t, t' \in T), a \leq tf, b \leq tf' \Rightarrow (a \wedge tb) \wedge tt'c = a \wedge t(b \wedge t'c) \).

Then, we define
\[
M(T, X, Y) = \left\{ (a, t) \in Y \times T : a \leq tf \right\}.
\]
with multiplication given by
\[(a, t) (b, t') = (a \land tb, tt'),\]
and obtain a monoid which we call an \(M\)-monoid.

**Theorem 2.10** [6]. Every proper left type-\(A\) monoid \(M\) is isomorphic to an \(M\)-monoid \(M(T, X, Y)\). Also, in \(M(T, X, Y)\), for all \((a, t), (b, t')\):

- \((a, t) R^* (b, t') \iff a = b;\)
- \((a, t) \sigma (b, t') \iff t = t';\)
and so \(T \simeq M(T, X, Y)/\sigma\).

**Lemma 2.11.** Let \(C^0\) be a \(u_0\)-proper left category and \(T\) be a right cancellative monoid acting downwards on \(C^0\). If this action is \(u_0\)-closed, then \(M(T, X, Y)\) is an \(M\)-monoid.

**Proof:** By Lemma 2.8, \(Y\) is a subsemilattice, with greatest element \(F = A_{O_{u_0}}\) of the partially ordered set \(X\). Now, we verify that \((T, X, Y)\) satisfies the properties of Definition 2.9. Let \(A_p, A_q \in X\) and \(t \in T\). Clearly, \(1A_p = A_{1p} = A_p\) and, by Lemma 2.7,

\[A_p \leq A_q \Rightarrow p \leq q \Rightarrow tp \leq tq \Rightarrow tA_p \leq tA_q .\]

Now, let \(A_p \in X\) with \(p \in \text{Mor}(v, v)\). As the action of \(T\) on \(C^0\) is \(u_0\)-closed, \(v = tu_0\), for some \(t \in T\). Thus, \(p^+ \in \text{Mor}(tu_0, tu_0)\) and, as \(T\) acts downwards on \(C^0\), there exists \(r \in \text{Mor}(u_0, u_0)\) such that \(p^+ = tr\). Whence, \(A_r \in Y\) and

\[A_p = A_{p^+} = A_tr = tA_r \in Y.\]

Next, let \(t \in T\). By Lemma 2.8 d), there exists \(A_a \in Y\) such that

\[A_a \leq tF.\]

To prove the fifth condition suppose that \(A_a, A_b \in Y\), with \(a, b \in \text{Mor}(u_0, u_0)\), and let \(t \in T\) be such that \(A_a \leq tA_{O_{u_0}}\). By Lemma 2.8 c), \(A_a = A_r\), for some \(r \in \text{Mor}(u_0, tu_0)\). Hence, by Lemma 2.7, there exists

\[A_a \land tA_b = A_r \land A_{tb} = A_{r+tb} = A_{(r+tb)^+} \in Y.\]

Finally, let \(A_a, A_b, A_c \in Y\) with \(a, b, c \in \text{Mor}(u_0, u_0)\) and \(t, t' \in T\). Suppose that \(A_a \leq tF\) and \(A_b \leq t'F\). Then, as before, there exist \(r \in \text{Mor}(u_0, tu_0) \cap A_a\) and \(r' \in \text{Mor}(u_0, tu'_0) \cap A_b\). Now, by Lemma 2.7,

\[A_a \land tA_b = A_r \land tA_{r'} = A_{r+tr'},\]
and
Again, by Lemma 2.7,
\[(A_a \land tA_b) \land t' t' A_c = A_{r+tr'} \land t t' A_c = A_{r+tr'+tt' c}\]
and
\[A_a \land t(A_b \land t' A_c) = A_r \land tA_{r'+t' c} = A_{r+t(r'+t' c)} = A_{r+tr'+t' c}.
\]
Therefore $M(T,X,Y)$ is an $M$-monoid, as required. ■

By Theorem 2.10, we know that every proper left type-$A$ monoid $M$ is iso-
morphic to an $M$-monoid $M$. The above results allow us to obtain a clearer
construction of such an $M$ and a new proof of the theorem.

**Theorem 2.12.** Let $M$ be a proper left type-$A$ monoid, $T = M/\sigma$ and
$D^0$ its derived category. Then, $M \simeq M(T,X,Y)$, where $X = \text{Mor}D^0/\sim$ and
$Y = \{A \in X: A \cap \text{Mor}(1,1) \neq \emptyset\}$.

**Proof:** In view of Theorem 2.6 and Lemma 2.11, it only remains to prove
that $C_1 \simeq M(T,X,Y)$. Consider the map
\[
\theta: C_1 \to M(T,X,Y)
\]
\[(p,t) \mapsto (A_p,t).\]

It follows from Lemma 2.8 c) that $\theta$ is well defined. By Lemma 2.7, $\theta$ is a
morphism. Again, by Lemma 2.8 c), $\theta$ is onto. To see that $\theta$ is injective, let
$q,p \in \text{Mor}(1,t)$, for some $t \in T$, be such that $A_p = A_q$, i.e. $p \sim q$. Thus, there
exists $a \in \text{Mor}D^0$ such that $p^+ = a^+$, $a + q^+ = a$. Hence $a \in \text{Mor}(1,1)$ and
$a = a^+$. Thus $p^+ = a^+ = a^+ + q^+ = p^+ + q^+$. Similarly, $q^+ = q^+ + p^+$. As
$\text{Mor}(1,1)$ is a semilattice, $p^+ = q^+$. Finally, $D^0$ being 1-proper, it follows that
$p = q$, as required. ■

3 – Proper left type-$A$ covers of left type-$A$ monoids

In this section we are concerned to show that for each left type-$A$ monoid $M$
there is a proper left type-$A$ monoid $P$ and an idempotent separating homomor-
phism $\theta: P \to M$ from $P$ onto $M$ such that $a^+ \theta = (a \theta)^+$. We express this result
by saying that $M$ has a proper left type-$A^+$-cover. It (or rather its dual) was
originally proved in [2] although it is stated somewhat differently there. For the
alternative proof which we present here we use the theory developed in Section 2 and a modification of the method of [1].

Before embarking on the proof we illustrate the notion of proper left type-A \( + \)-cover by the following example. Let \( X \) be a topological space. We denote by \( G(X) \) the monoid of all continuous bijections from \( X \) to itself under composition. Certainly \( G(X) \) is cancellative but it is not a group in general. We let \( \mathcal{I}_c(X) \) denote the monoid of all continuous one-one partial maps from \( X \) to itself under composition of partial functions. Finally, \( \mathcal{P}(X) \) denotes the power set of \( X \) regarded as a semilattice under the operation of intersection. We define a left action of \( G(X) \) on \( \mathcal{P}(X) \) by the rule that \( \sigma Y = Y \sigma^{-1} \) for all \( \sigma \) in \( G(X) \) and all subsets \( Y \) of \( X \). It is then easy to verify that the multiplication

\[
(Y, \sigma)(Z, \tau) = (Y \cap \sigma Z, \sigma \tau)
\]

makes the set \( \mathcal{P}(X) \times G(X) \) into a monoid \( \mathcal{P}(X) * G(X) \) (a semidirect product of \( \mathcal{P}(X) \) and \( G(X) \)). It is also readily checked that this monoid is proper left type-A with semilattice of idempotents \( \{(Y, 1) : Y \in \mathcal{P}(X)\} \) and \( (Y, \sigma)^{+} = (Y, 1) \). Indeed, \( \mathcal{P}(X) * G(X) \) is nothing other than \( M(G(X), \mathcal{P}(X), \mathcal{P}(X)) \). We claim that it is a left type-A \( + \)-cover of \( \mathcal{I}_c(X) \). To see this consider the surjective function \( \mu : \mathcal{P}(X) \# G(X) ! \mathcal{I}_c(X) \) defined by

\[
(Y, \sigma) \mu = \sigma Y,
\]

where \( \sigma Y \) denotes the partial map with domain \( Y \) obtained by restricting \( \sigma \). It is routine to show that \( \mu \) is an idempotent separating homomorphism and that \( ((Y, \sigma)^{+}) \mu = ((Y, \sigma) \mu)^{+} \). Of course, this example is very familiar when \( X \) has the discrete topology and we have an \( E \)-unitary cover of the symmetric inverse monoid on \( X \).

We now start our proof with a technical lemma on left type-A monoids.

**Lemma 3.1.** Let \( M \) be a left type-A monoid and let \( s \in S \). If \( s = e_0x_1e_1 \cdots e_{n-1}x_ne_n \), for some \( n \in \mathbb{N}, \ x_i \in M \ (i = 1, \ldots, n) \) and \( e_j \in E(M) \ (j = 0, \ldots, n) \), then

\[
s = s^{+}(x_1 \cdots x_n).
\]

**Proof:** Suppose that \( n = 0 \), then \( s = e_0 \) and \( s = s^+ \). Now, let us assume that the result is true for \( n \). Suppose that

\[
s = e_0x_1 \cdots x_ne_ne_{n+1}e_{n+1}.
\]

Then,

\[
s = r \cdot x_{n+1}e_{n+1}.
\]
where \( r = e_0 x_1 e_1 \cdots x_n e_n \). Hence, by the induction hypothesis, \( r = r^+ (x_1 \cdots x_n) \) and so
\[
s = r^+ (x_1 \cdots x_n) \cdot x_{n+1} e_{n+1}.
\]
Thus
\[
s = r^+ (x_1 \cdots x_{n+1} e_{n+1})^+ x_1 \cdots x_{n+1}
= (r^+ x_1 \cdots x_{n+1} e_{n+1})^+ x_1 \cdots x_{n+1}
= s^+ x_1 \cdots x_{n+1},
\]
as required.

Let \( M \) be a left type-\( A \) monoid with set of idempotents \( E \). Put \( X = M \setminus \{1\} \). We start by considering \( X^* \), the free monoid on \( X \) with identity 1. We write the non-identity elements as sequences \((x_1, ..., x_n)\), where \( n \geq 1 \) and \( x_i \in X \) \((i = 1, ..., n)\). To each word \( w \in X^* \) we associate a subset \( M_w \) of \( M \), in the following way:
\[
M_w = \begin{cases} 
E & \text{if } w = 1, \\
Ex_1Ex_2E \cdots x_{n-1}Ex_nE & \text{if } w = (x_1, ..., x_n).
\end{cases}
\]
It is clear that, for all \( v, w \in X^* \), we have
\[
M_{vw} = M_v M_w.
\]
Now, define a category \( \mathcal{C}^0 \) as follows:
\[
\text{Obj} \mathcal{C}^0 = X^*
\]
and, for all \( v, w \in X^* \),
\[
\text{Mor}(v, w) = \begin{cases} 
\{(v, s, w) : s \in M_w\} & \text{if } w = vw_1, \text{ for some } w_1 \in X^*, \\
\emptyset & \text{otherwise}.
\end{cases}
\]
The composition law is given by
\[
(v, s, w) + (w, t, u) = (v, st, u).
\]
Clearly, the composition is well defined and associative. Also, for any object \( v \),
\[
\text{Mor}(v, v) = \{(v, e, v) : e \in E\}
\]
and \((v, 1_M, v)\) is the identity on \( \text{Mor}(v, v) \), where \( 1_M \) denotes the identity of \( M \). Thus, \( \mathcal{C}^0 \) is indeed a category.
Next, we consider a (left) action of the (right) cancellative monoid $X^*$ on the category $C^0$: the action of $X^*$ on $\text{Obj}C^0$ is given by the multiplication on $X^*$ and, for all $u \in X^*$ and $(v, s, w) \in \text{Mor}C^0$,

$$u(v, s, w) = (uv, s, uw).$$

It is easy to verify that this action is well defined.

We choose 1 to be the distinguished object of $C^0$.

**Lemma 3.2.** Let $M$ be a left type-$A$ monoid. Then $C^0$ is a left proper category with distinguished object 1. Also, the right cancellative monoid $X^*$ acts (on the left) downwards on $C^0$. The action is 1-closed.

**Proof:** Most of the required properties of $C^0$ and of the action of $X^*$ over $C^0$ are easy to prove, once we notice that:

- For all $u \in \text{Obj}C^0$, $\text{Mor}(u, u) = \{(u, e, u) : e \in E\} \simeq E$;
- For all $(u, s, v) \in \text{Mor}(u, v)$, $(u, s, v)^+ = (u, s^+, u)$;
- The unique unit of $X^*$ is the empty word 1.

Here, we only prove that $C^0$ is 1-proper. Let $v \in X^*$ and $(1, s, v), (1, t, v) \in \text{Mor}(1, v)$ be such that $(1, s, v)^+ = (1, t, v)^+$. Then, $s^+ = t^+$ and $s, t \in M_v$. If $v = 1$, then $M_v = E$ and we have $s = s^+ = t^+ = t$. Whence $(1, s, v) = (1, t, v)$. If $v \neq 1$, let $v = (x_1, ..., x_n)$, where $n > 0$ and $x_i \in X$ ($i = 1, ..., n$). Thus, there exist $e_1, ..., e_n, f_1, ..., f_n \in E$ such that

$$s = e_1x_1e_2 \cdots e_nx_ne_{n+1}$$

and

$$t = f_1x_1f_2 \cdots f_nx_nf_{n+1}.$$ 

By Lemma 3.1,

$$s = s^+(x_1 \cdots x_n) \quad \text{and} \quad t = t^+(x_1 \cdots x_n).$$

Hence, as $s^+ = t^+$, we have $s = t$. Therefore

$$(1, s, v) = (1, t, v)$$

and $C^0$ is 1-proper, as required. ♦

**Definition 3.3.** Let $M$ and $N$ be left type-$A$ monoids we say that $N$ is a $^+$-cover of $M$ if there exists an idempotent separating monoid morphism $\theta$ from $N$ onto $M$ that respects the operation $^+$, that is, for all $a \in N$, $a^+ \theta = (a \theta)^+$. 

**Theorem 3.4.** Every left type-$A$ monoid has a proper left type-$A$ $^+$-cover.

**Proof:** Suppose that $M$ is a left type-$A$ monoid. Let $C^0$ be the category defined before. We have

$$C_1 = \left\{ ((1, s, u), u) : u \in X^*, s \in M_u \right\}$$

and the multiplication on $C_1$ is given by

$$((1, s, u), u)(1, t, v), v) = ((1, st, uv), uv).$$

The identity of $C_1$ is $((1, 1^M, 1), 1)$. By Lemmas 3.2 and 2.2, $C_1$ is a proper left type-$A$ monoid. Now, let us consider the map

$$\theta : C_1 \to M$$

$$((1, s, u), u) \mapsto s.$$ 

Clearly, $\theta$ is monoid morphism and is, in fact, a $^+$-morphism. Because

$$((1, s, u), u)^+ \theta = ((1, s^+, 1), 1) \theta = s^+ = (((1, s, u), u) \theta)^+.$$ 

That $\theta$ is onto follows from the fact that, for all $a \in M \setminus \{1\} = X$,

$$a = ((1, a, (a)), (a)) \theta.$$ 

Finally, as

$$E(C_1) = \left\{ ((1, e, 1), 1) : e \in E \right\},$$

we have that $\theta|_{E(C_1)}$ is an isomorphism from $E(C_1)$ into $E$. Therefore, $C_1$ is a proper left type-$A$ $^+$-cover of $M$, as required. 

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John Fountain,
Dept. Mathematics, University of York,
Heslington, York, YO15DD – ENGLAND

and

Gracinda M.S. Gomes,
Dep. Matemática, Universidade de Lisboa,
Rua Ernesto de Vasconcelos, C1, 1700 Lisboa – PORTUGAL