CONVERGENCE IN SPACES OF RAPIDLY INCREASING DISTRIBUTIONS

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Abstract: In this note we show that if \((T_j)\) is a sequence in \(K_M'\), the space of distributions of rapid growth (resp. \(O_c'\) the space of its convolution operators), and \((T_j \ast \phi)\) converges to 0 in \(K_M'\) (resp. in \(O_c'\)) for all \(\phi\) in \(K_M\), then \((T_j)\) converges to 0 in \(K_M'\) (resp. \(O_c'\)). Moreover, if \((\psi_j)\) is in \(O_c\) such that \((\psi_j \ast \phi)\) converges to 0 in \(O_c\) for every \(\phi\) in \(K_M\), then \((\psi_j)\) converges to 0 in \(O_c\). This is no more true if the sequence \((\psi_j)\) is in \(K_M\).

1 – Introduction

When one considers the convolution of elements from \(K_M'\) (the space of distributions of rapid growth) with elements from \(K_M\) (the space of \(C^\infty\) functions which are very rapidly decreasing at infinity), it follows trivially that if \((T_j)\) is any sequence which converges to 0 in \(K_M'\), then the sequence \((T_j \ast \phi)\) converges to 0 for every \(\phi\) in \(K_M\). Moreover, if \((T_j)\) is a sequence in \(O_c'\) (the space of convolution operators in \(K_M'\)), and \(T_j \rightarrow 0\) in \(O_c'\), then \(T_j \ast \phi \rightarrow 0\) in \(K_M\) for every \(\phi\) in \(K_M\). In this note we consider the following questions: given \((T_j) \subset K_M'\) such that \(T_j \ast \phi \rightarrow 0\) in \(K_M'\) for every \(\phi \in K_M\), does it follow that \(T_j \rightarrow 0\) in \(K_M'\)? Similarly, if \((T_j) \subset O_c'\) and \(T_j \ast \phi \rightarrow 0\) in \(K_M\) for every \(\phi \in K_M\), does it follow that \(T_j \rightarrow 0\) in \(O_c'\)? In both cases we show that the answer is affirmative. Similar questions have been considered by K. Keller [5] for the space \(S'\) of tempered distributions, our methods of proof are different from those of Keller, and they work if we replace \(K_M\) by any complete metric space of test functions. Finally we consider these questions of convergence for sequences of functions in \(O_c\) and \(K_M\).

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By \(D, E, D'\) and \(E'\) we denote Schwartz spaces of test functions and distributions, \(N^n\) consists of all \(n\)-tuples \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \; \alpha_i \in N\), and the differential operator \(D^\alpha, \alpha \in N^n\), denotes \((-i \frac{\partial}{\partial x_1})^{\alpha_1} \ldots (-i \frac{\partial}{\partial x_n})^{\alpha_n}\). Let \(M(x), \; x \geq 0\), be a function which is continuous, increasing and convex with \(M(0) = 0, \; M(\infty) = \infty\). For \(x < 0\) define \(M(x)\) to be \(M(-x)\), and for \(x = (x_1, x_2, \ldots, x_n)\), we define \(M(x) = M(x_1) + M(x_2) + \ldots + M(x_n)\). Examples of such function are \(M(x) = \frac{x^p}{p}, \; p > 1, \text{ and } M(x) = e^x\).

For a function \(M\) as above, we define the space \(K_M\) to be the space of all infinitely differentiable functions \(\phi\) on \(R^n\) such that

\[
\nu_k(\phi) = \sup_{x \in \mathbb{R}^n} |e^{M(x)} D^\alpha \phi(x)| < \infty, \quad \alpha \in N^n, \; k = 0, 1, 2, \ldots
\]

The space \(K_M\) is provided with the topology generated by the semi-norm \(\nu_k, \; k = 0, 1, 2,\ldots\). It follows that \(K_M\) is a Frechet Montel space. Moreover, it is a normal space of distributions. By \(K'_M\) we denote the space of all continuous linear functionals on \(K_M\) provided with the strong dual topology. By \(O'_c\) we denote the subspace of \(K'_M\) consisting of all \(S \in K'_M\) such that for every \(\phi\) in \(K_M\) the convolution \(S * \phi\) is in \(K_M\), and the map \(\phi \mapsto S * \phi\) from \(K_M\) into itself is continuous. \(O'_c\) is the space of convolution operators on \(K'_M\), and will be provided with the topology of uniform convergence on bounded subsets of \(K_M\). The space \(O_c\) consists of all \(e^\alpha\)-functions such that \(D^\alpha f(x) = O(e^{M(x)})\) for all \(\alpha \in N^n\), and some positive integer \(k\) independent of \(\alpha\). It turns out that \(O_c\) is the strong dual of \(O'_c\), we provide it with the strong dual topology. Another equivalent topology is \(\tau_\alpha\) of uniform convergence on bounded subset of \(K_M\) (see [2] and [3]).

We denote by \(V(K_M \ast K_M)\) the subspace of \(K_M\) generated by the elements of \(K_M \ast K_M\), and we provide it with the relative topology inherited from \(K_M\). In particular \(V(K_M \ast K_M)\) is metrizable.

2 – The results

**Lemma 1.** \(V(K_M \ast K_M)\) is dense in \(K_M\).

**Proof:** Let \(\psi\) be any element of \(K_M\), let \((\phi_\varepsilon; \varepsilon > 0)\) be a sequence in \(D\) converging to \(\delta\) in \(E'\). Since the convolution map \(\Lambda_\psi\) from \(O'_c\) into \(K_M\) which maps \(S\) to \(S * \psi\) is continuous, and \((\phi_\varepsilon; \varepsilon > 0)\) is bounded in \(E'\) which is continuously embedded in \(O'_c\) it follows that the sequence \((\phi * \psi; \varepsilon > 0)\) converges to \(\psi\) in \(K_M\). □

**Lemma 2.** The space \(V(K_M \ast K_M)\) is Montel.
Proof: We show first that every bounded subset of \( V(K_M \ast K_M) \) is relatively compact. Let \( U \) be a bounded subset of \( V(K_M \ast K_M) \), by \( \text{Cl} V(U) \) and \( \text{Cl} K_M(U) \) we denote the closures of \( U \) in \( V \) and \( K_M \) respectively. One has \( \text{Cl} V(U) = V \cap \text{Cl} K_M(U) \). Let \( \{ O_j, j = 1, 2, \ldots \} \) be an open cover of \( \text{Cl} V(U) \) in \( V(K_M \ast K_M) \), then \( O_j = V \cap G_j \), where the \( G_j \)'s, \( j = 1, 2, \ldots \), are open subsets of \( K_M \), and one has

\[
V \cap \text{Cl} K_M(U) = \text{Cl} V(U) \subset \bigcup_{j=1}^{\infty} O_j = \bigcup_{j=1}^{\infty} (V \cap G_j) = V \cap \left( \bigcup_{j=1}^{\infty} G_j \right).
\]

Since \( V(K_M \ast K_M) \) is dense in \( K_M \) it follows that \( \text{Cl} K_M(U) \) is contained in \( \bigcup_{j=1}^{\infty} G_j \). Since \( K_M \) is Montel it follows that there exists a finite set of indices \( j_1, j_2, \ldots, j_m \) such that \( \text{Cl} K_M(U) \subset \bigcup_{i=1}^{m} G_{j_i} \). Hence

\[
\text{Cl} V(U) = V \cap \text{Cl} K_M(U) \subset V \cap \left( \bigcup_{i=1}^{m} G_{j_i} \right) = \bigcup_{i=1}^{m} (V \cap G_{j_i}),
\]

i.e. \( \text{Cl} V(U) \) is compact in \( V(K_M \ast K_M) \).

Finally we show that \( V(K_M \ast K_M) \) is barreled. Let \( F \) be a barrel in \( V(K_M \ast K_M) \), \( F \) is a closed, absorbing, balanced and convex subset of \( V \). We show that \( F \) is a neighborhood of \( 0 \) in \( V \). Let \( F_M = \text{Cl} K_M(F) \). It is clear that \( F = V \cap F_M \). We claim that \( F_M \) is a barrel in \( K_M \). First we show that it is absorbing. Let \( \phi \in K_M, \phi \notin F_M \). Since \( V(K_M \ast K_M) \) is dense in \( K_M \) it follows that there exists a sequence \( (\phi_j) \subset V \), such that \( \phi_j \to \phi \) in \( K_M \). Since \( F \) is absorbing subset of \( V \) there exists a sequence \( (\lambda_j) \subset R, \lambda_j > 0 \) such that \( \lambda_j \phi_j \in F \) for all \( j = 1, 2, \ldots \). Without loss of generality we can assume that \( 0 < \lambda_j \leq 1 \). Thus the sequence \( (\lambda_j \phi_j) \) is bounded in \( F_M \), hence it has a convergent subsequence, call it also \( (\lambda_j \phi_j) \), \( \lambda_j \phi_j \to \psi \) in \( F_M \). We can assume also that \( \lambda_j \to \lambda \) in \( R \). Let \( \rho \) be the metric on \( K_M \). Given any \( \varepsilon > 0 \), it follows that for \( j \) large enough

\[
\rho(\lambda_j \phi_j - \lambda \phi) \leq \rho(\lambda_j \phi_j - \lambda_j \phi) + \rho(\lambda_j \phi - \lambda \phi) \\
\leq \lambda_j \rho(\phi_j - \phi) + |\lambda_j - \lambda| \rho(\phi) \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus \( (\lambda_j \phi_j) \) converges to \( \lambda \phi \) in \( K_M \). Hence \( \lambda \phi = \psi \), and \( \lambda \phi \in F_M \), i.e. \( F_M \) is absorbing.

Next, we show that \( F_M \) is convex. Let \( \phi_1, \phi_2 \) be in \( F_M \), \( \alpha \) real number, \( 0 \leq \alpha \leq 1 \), we show that \( \alpha \phi_1 + (1 - \alpha) \phi_2 \in F_M \). We will consider the general case that \( \phi_1, \phi_2 \) are not in \( F \). Since \( V(K_M \ast K_M) \) is dense in \( K_M \) it follows that there exist sequences \( (\phi_{j_1}), (\phi_{j_2}) \) of functions in \( V \) such that \( \phi_{j_1} \to \phi_1 \) and \( \phi_{j_2} \to \phi_2 \) in \( K_M \). Since \( F \) is absorbing there exist sequences of positive real
numbers \((k_{j_1}), (k_{j_2})\) such that \(\{k_{j_1}\phi_{j_1}\}\) and \(\{k_{j_2}\phi_{j_2}\}\) are contained in \(F\). Since \(F\) is convex it follows that \(0 < k_{j_1} \leq 1\) and \(0 < k_{j_2} \leq 1\). Let

\[
\lambda_{j_1} = \sup\{k_{j_1} : k_{j_1}\phi_{j_1} \in F\} - \frac{1}{j},
\]

\[
\lambda_{j_2} = \sup\{k_{j_2} : k_{j_2}\phi_{j_2} \in F\} - \frac{1}{j}.
\]

For each \(j = 1, 2, 3, \ldots, i = 1, 2\), one has \(0 < k_{j_i} - \frac{1}{j} < k_{j_i} \leq 1\), and

\[
(k_{j_i} - \frac{1}{j})\phi_{j_i} = \left(\frac{k_{j_i} - \frac{1}{j}}{k_{j_i}}\right) \cdot k_{j_i}\phi_{j_i} + \left(1 - \frac{k_{j_i} - \frac{1}{j}}{k_{j_i}}\right) \cdot 0
\]
is in \(F\). Hence \((\lambda_{j_i}\phi_{j_i}) \subset F, i = 1, 2\). Since \(C\ell_K(F) = F_M\), it follows that

\[
\lim_{j \to \infty} \sup\{k_{j_i} : k_{j_i}\phi_{j_i} \in F\} = 1,
\]

and \(\lambda = \lim_{j \to \infty} \lambda_{j_i} = 1, i = 1, 2\). Thus \(\lambda_{j_i}\phi_{j_i} \to \phi_1\) and \(\lambda_{j_2}\phi_{j_2} \to \phi_2\) in \(K_M\) as \(j \to \infty\). Hence

\[
\alpha\phi_1 + (1 - \alpha)\phi_2 = \lim_{j_1 \to \infty} \lim_{j_2 \to \infty} \left[\alpha \lambda_{j_1}\phi_{j_1} + (1 - \alpha)\lambda_{j_2}\phi_{j_2}\right].
\]

Since for each \(j_1, j_2\) the term in the bracket is in \(F\) (by convexity), it is in \(F_M\). Since \(F_M\) is closed it follows that \(\alpha\phi_1 + (1 - \alpha)\phi_2\) is in \(F_M\), i.e. \(F_M\) is convex.

Finally we show that \(F_M\) is balanced. Let \(\phi \in F_M, \alpha \in \mathbb{R}, |\alpha| \leq 1\). If \(\phi \in F\) there is nothing to prove. Otherwise, as in the proof of convexity, there exist sequences \((\phi_j), (\lambda_j)\) in \(V\) and \(R\) respectively, such that for all \(j = 1, 2, \ldots, \lambda_j\phi_j \in F, \phi_j\lambda_j \to \phi\) as \(j \to \infty\). Since \(F\) is balanced one has \(\alpha\lambda_j\phi_j \in F\), and since \(F_M\) is closed it follows that \(\alpha\phi = \lim_{j \to \infty} \alpha\lambda_j\phi_j\) is in \(F_M\), i.e. \(F_M\) is balanced.

Thus \(F_M\) is a neighborhood of 0 in \(K_M\) because \(K_M\) is Montel. Hence \(\bar{F} = V \cap F_M\) is a neighborhood of 0 in \(V(K_M \ast K_M)\). ■

From the definition of \(V(K_M \ast K_M)\) and its topology it follows that \(K_M'\) is contained in \((V(K_M \ast K_M))'\). Now we give the main result of this paper.

**Theorem 1.** Let \(T_j\) be a sequence in \(K_M'\) such that for every \(\phi\) in \(K_M\) the sequence \((T_j \ast \phi)\) converges to 0 in \(K_M'\), then \((T_j)\) converges to 0 in \(K_M'\).

**Proof:** Since \(K_M'\) is the strong dual of the Montel space \(K_M\) it suffices to show that \((T_j)\) converges to 0 weakly in \(K_M'\). Let \(\phi \in K_M\), we show that \(\langle T_j, \phi \rangle \to 0\). Let \(\phi_{\varepsilon; \varepsilon > 0}\) be the sequence as in the proof of Lemma 1,
\( \phi_\varepsilon \ast \phi \to \phi \) in \( K_M \) as \( \varepsilon \to 0 \). Moreover, the set \( \{ \phi_\varepsilon \ast \phi : \varepsilon > 0 \} \) is bounded in \( V(K_M \ast K_M) \). Thus

\[
\lim_{j \to \infty} \langle T_j, \phi \rangle = \lim_{j \to \infty} \lim_{\varepsilon \to 0} \langle T_j, \phi_\varepsilon \ast \phi \rangle .
\]

Since \( T_j \ast \phi \to 0 \) in \( K'_M \), and the bilinear map \( (T, \psi) \to T \ast \psi \) from \( K'_M \times K_M \to O_c \) is continuous in each variable (see [2]), it follows that \( (T_j \ast \phi) \ast \psi \to 0 \) in \( O_c \) as \( j \to \infty \). Hence

\[
\lim_{j \to \infty} \langle T_j, \phi \ast \psi \rangle = \lim_{j \to \infty} \langle T_j \ast \phi, \psi \rangle = \lim_{j \to \infty} \left( (T_j \ast \phi) \ast \psi \right)(0) = 0 .
\]

Thus \( (T_j) \) converges weakly to \( O \) in \( (V(K_M \ast K_M))' \). Since \( V(K_M \ast K_M) \) is Montel by Lemma 2, it follows that \( (T_j) \) converges strongly to \( O \) in \( (V(K_M \ast K_M))' \), i.e. it converges uniformly on bounded subsets of \( V(K_M \ast K_M) \). Since \( \{ \phi_\varepsilon \ast \phi : \varepsilon > 0 \} \) is bounded in \( V(K_M \ast K_M) \) it follows that \( \lim_{j \to \infty} \langle T_j, \phi \ast \phi_\varepsilon \rangle = 0 \) uniformly in \( \varepsilon \). Thus we can interchange the limits on the right hand side of (I), and one gets,

\[
\lim_{j \to \infty} \langle T_j, \phi \rangle = \lim_{j \to \infty} \lim_{\varepsilon \to 0} \langle T_j, \phi \ast \phi_\varepsilon \rangle
\]

\[
= \lim_{\varepsilon \to 0} \lim_{j \to \infty} \langle T_j, \phi \ast \phi_\varepsilon \rangle = 0 .
\]

This completes the proof of the theorem.

Next, we consider the same question of convergence in \( O'_c \). In this direction we have:

**Theorem 2.** Let \( (T_j) \) be a sequence in \( O'_c \) such that, for every \( \phi \) in \( K_M \) the sequence \( (T_j \ast \phi) \) converges to \( 0 \) in \( O'_c \), then \( (T_j) \) converges to \( 0 \) in \( O'_c \).

**Proof:** It is clear that \( T_j \ast \phi \in O'_c \) for every \( T_j \) in \( O'_c \) and \( \phi \) in \( K_M \). Let \( T \) be any element in \( K'_M \), we claim that \( T_j \ast T \to 0 \) in \( K'_M \). For given \( \phi \) in \( K_M \), one has

\[
(T_j \ast T) \ast \phi = (T_j \ast \phi) \ast T \to 0 \quad \text{in} \quad K'_M .
\]

From Theorem 1 it follows that \( T_j \ast T \to 0 \) in \( K'_M \). Let \( B \) be a bounded subset of \( K_M \), then for any \( T \in K'_M \) one has

\[
(II) \quad \langle T_j \ast \phi, T \rangle = \langle T_j \ast T, \phi \rangle \to 0 \quad \text{uniformly in} \quad \phi \in B .
\]

Since \( K_M \) is reflexive and \( K'_M \) is Montel (being the strong dual of a Montel space), (II) implies that \( T_j \ast \phi \to 0 \) in \( K_M \), uniformly in \( \phi \in B \). This complete the proof of the theorem.
Corollary. Let \((T_j)\) be a sequence in \(O_c'\) such that for any \(\phi \in K_M\), \((T_j \ast \phi)\) converges to 0 in \(K_M\), then \((T_j)\) converges to 0 in \(O_c'\).

As in the case of the space \(K'_1\) of distribution of exponential growth, it is possible to extend the definition of Fourier transform of distributions of compact support to the elements of \(O_c'\). It turns out that for \(S \in O_c'\), its Fourier transform \(\hat{S}\) could be extended to \(C^n\) as an entire function, which satisfies a Paley–Wiener type theorem, see Pahk [6] (the theorem was quoted and used in [1]). [8], Zielezny proved that the space \(O_c'(K_0 : K_1)\) is bornologic. A simple modification of the proof of Theorem 9 of [8] shows that \(O_c'\) is bornologic. Since \(O_c'\) is the projective limit of the Montel spaces \(w^{-k}S'\), and the topology of \(w^{-k}S'\) is finer than the topology of \(w^{-j}S'\) for \(k \geq j\), it follows from the Corollary to Proposition 3.9.6 of Horvath [4] that \(O_c'\) is semi-Montel. Thus \(O_c'\) is Montel. Hence its strong dual \(O_c\) is Montel. As in Lemma 2, one can show that \(K_M\) as a subspace of \(O_c'\) with the relative topology of \(O_c'\) is Montel. Following the idea of the proof of Theorem 1, we can prove the following.

Theorem 3. Let \((\psi_j)\) be a sequence in \(O_c\) such that \((\psi_j \ast \phi)\) converges to 0 in \(O_c\) for every \(\phi \in K_M\), then \((\psi_j)\) converges to 0 in \(O_c\).

The last result of this note is of negative nature, it simply says that in Theorem 1, one can not replace \(K_M'\) by \(K_M\). More precisely we have

Theorem 4. There exist a sequence \((\psi_j)\) in \(K_M\) and \(\phi \in K_M\), where the map \(\phi \ast \psi \rightarrow \psi\) from \(\phi \ast K_M\) to \(K_M\) is well-defined, such that \((\phi \ast \psi_j)\) converges to 0 in \(K_M\) but \((\psi_j)\) does not converge to 0 in \(K_M\).

Proof: Assume the contrary, since for given \(\phi \in K_M\) the space \(\phi \ast K_M\) with the relative topology of \(K_M\) is metric, it follows that the linear map \(\Lambda\) from \(\phi \ast K_M\) into \(K_M\) which takes \(\phi \ast \psi\) is continuous. We claim that \(\hat{\phi} \ast K_M = K_M'\). Indeed, given \(T\) in \(K_M'\), let \(S\) be the Hahn-Banach extension to \(K_M\) of \(T \circ \Lambda\) from \(\phi \ast K_M\) into \(C\). \(S\) is in \(K_M'\) and \(\hat{\phi} \ast S = T\). But on the other hand the equality of \(\hat{\phi} \ast K_M\) and \(K_M'\) is impossible, because \(\phi \ast S\) is infinitely differentiable for all \(S\) in \(K_M'\) and can never be equal to \(\delta\). The contradiction completes the proof of the theorem.

Remarks.
(1) It will be nice to have a concrete example of a sequence \((\psi_j)\) and a function \(\phi \in K_M\) which satisfy the conditions of the above result.
(2) Theorem 4 and its proof remain valid if the sequence \((\psi_j)\) is in \(\varepsilon\) and \(\phi\) is in \(D\).
Added in proof. In a recent article Stevan Pilipovic (Proceedings of the AMS, Vol. 111, No. 4, April 1991) has shown that, if \((T_j)\) is a sequence in \(S^0\) such that \((T_j * \phi)\) converges to 0 in \(S'\) for any \(\phi\) in \(D\), then \((T_j)\) converges to 0 in \(S'\).

In his proof he followed the method of Keller [5].

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