CONVOLUTION OPERATORS IN INFINITE DIMENSION

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1 – Introduction

Let $E$ be a complete convex bornological vector space (denoted by the letters b.v.s.). This means that $E$ is an injective algebraic inductive limit of a family $\{E_i\}_{i \in I}$ of Banach spaces $E_i$, $i \in I$, such that for $i < j$ the canonical linear map from $E_i$ to $E_j$ is continuous. A subset of $E$ is called bounded if it is contained and bounded in a Banach space $E_i$. We say that $E$ is a Schwartz (resp. weak Schwartz) b.v.s. if the canonical map from $E_i$ to $E_j$ is compact (resp. weakly compact) for every $i < j$.

Given $D$ a subset of $E$ such that $D_i := D \cap E_i$ is open in $E_i$ for every $i \in I$. A function $f$ on $D$ is said to be holomorphic if $f|D_i$ is holomorphic for every $i \in I$. By $H(D)$ we denote the space of holomorphic functions on $D$ equipped with the compact-open topology, where as above a subset $K$ of $D$ is called compact if $K_i := K \cap E_i$ is compact. Consider the Fourier–Borel transformation

$$\mathcal{F}_D : H'(D) \to H(E^+)$$

given by

$$\mathcal{F}_D(\mu)(x^*) = \mu(\exp x^*) \quad \text{for} \quad \mu \in H'(D) \quad \text{and} \quad x^* \in E^+,$$

where $H'(D)$ denotes the dual space of $H(D)$ equipped with the compact-open topology and

$$E^+ = \{ f \in H(E) : f \text{ is linear} \}.$$

Equip $\text{Im} \mathcal{F}_D$ the quotient topology via $\mathcal{F}_D$. For each $\alpha \in E^+$ define the translation operator $\tau_\alpha$ on $H(E^+)$ by the form

$$\tau_\alpha(\phi)(x^*) = \phi(x^* + \alpha)$$

for $x^* \in E^+$ and $\phi \in H(E^+)$. 

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Since 

\[ \mathcal{F}_D \tilde{\tau}_\alpha = \tau_\alpha \mathcal{F}_D , \]

where \( \tilde{\tau}_\alpha : H'(D) \rightarrow H'(D) \) given by 

\[ (\tilde{\tau}_\alpha \mu)(\varphi) = \mu(\varphi \exp \alpha) \]

for \( \varphi \in H(D) \) and \( \mu \in H'(D) \), it follows that 

\[ \tau_\alpha : \text{Im} \mathcal{F}_D \rightarrow \text{Im} \mathcal{F}_D \]

is continuous.

Now a continuous linear map \( \theta : \text{Im} \mathcal{F}_D \rightarrow \text{Im} \mathcal{F}_D \) is called a convolution operator if it commutes with every translation.

\section*{2 - Statement of the results}

In this note we always assume that \( E \) is a b.v.s. which is separated by \( E^+ \) and \( D \) is a subset of \( E \) such that \( D \cap E_i \) is connected and open in \( E_i \) for every \( i \in I \).

**Existence Theorem.** Every non-zero convolution operator on \( \text{Im} \mathcal{F}_D \) is surjective.

**Approximation Theorem.** Let hold one of the following two conditions

i) \( D \) is balanced;

ii) \( D \) is polynomially convex and \( E \) is a weak Schwartz b.v.s. such that every \( E_i \) has the approximation property.

Then every solution \( u \) of the homogeneous equation \( \theta u = 0 \) is a limit for the topology of \( \text{Im} \mathcal{F}_D \) of solutions in \( \mathcal{P}(E) \mathcal{E}\text{xp}(D) \), where \( \mathcal{P}(E) \) denotes the set of all continuous polynomials on \( E \) and \( \mathcal{E}\text{xp}(D) = \text{span}\{\exp(x) : x \in D\} \).

In the case \( E \) is a Schwartz b.v.s. such that every space \( E_i \) has the approximation property and \( D \) is a balanced convex open subset of \( E \) the above results have established by Colombeau and Perrot [3]. Some particular cases were proved by Boland [1], Dwyer [4], [5], [6] and Gupta [8].

\section*{3 - Proof}

Let \( T \in (\text{Im} \mathcal{F}_D)' \), the dual space of \( \text{Im} \mathcal{F}_D \), equipped with the strong topology, and let \( T^* : \text{Im} \mathcal{F}_D \rightarrow \text{Im} \mathcal{F}_D \) given by the form 

\[ (T^* \phi)(\alpha) = T(\tau_\alpha \phi) \quad \text{for} \quad \phi \in \text{Im} \mathcal{F}_D \quad \text{and} \quad \alpha \in E^+ . \]
Lemma 1. $T^*$ is a convolution operator on $\text{Im} \mathcal{F}_D$ and conversely each convolution operator on $\text{Im} \mathcal{F}_D$ is a $T^*$ for some $T$.

Proof: First observe that $\mathcal{F}_D^\prime: (\text{Im} \mathcal{F}_D)^\prime \to H''(D) = H(D)$ (algebraically). Define the continuous linear map $U_T$ from $H'(D)$ to $H'(D)$ by

$$U_T(\mu)(\psi) = \mu(\mathcal{F}_D^\prime(T) \psi)$$

for $\mu \in H'(D)$ and $\psi \in H(D)$.

We have

$$(T^* \mathcal{F}_D)(\mu)(\alpha) = (T^* \mathcal{F}_D(\mu))(\alpha) = T(\tau_\alpha \mathcal{F}_D(\mu))$$

$$= T(\mathcal{F}_D \tilde{\tau}_\alpha(\mu)) = \mathcal{F}_D^\prime(T)(\tilde{\tau}_\alpha(\mu))$$

$$= \mu(\mathcal{F}_D(\exp \alpha)) = (\mathcal{F}_D U_T(\mu))(\alpha)$$

for all $\mu \in H'(D)$ and $\alpha \in E^+$. Thus

$$T^* \mathcal{F}_D = \mathcal{F}_D U_T.$$ 

This yields the continuity of $T^*$.

Let $\mathcal{M}$ denote the algebra of all convolution operators on $\text{Im} \mathcal{F}_D$ and let $\gamma$ be the map from $\mathcal{M}$ to $(\text{Im} \mathcal{F}_D)^\prime$ given by

$$\gamma: \theta \mapsto (\phi \mapsto \theta^\phi(0)).$$

It is easy to see that

$$\gamma(T^*) = T \quad \text{and} \quad (\gamma \theta)^* = \theta.$$ 

Hence the map $T \mapsto T^*$ is a bijection between $(\text{Im} \mathcal{F}_D)^\prime$ and $\mathcal{M}$. 

Lemma 2. Let $F$ be a Fréchet space and let $C(F)$ denote the set consisting of all compact balanced convex subsets of $F$. Then for every $K \in C(F)$ there exists $L \in C(F)$ such that the canonical map from the canonical Banach space $F(K)$ spanned by $K$ to $F(L)$ is compact.

Proof: Let $H$ be a closed separated subspace of $F$ containing $K$. From a result of Geijler [7] we can find a continuous linear map $\eta$ from a Fréchet–Montel space $Q$ onto $H$. Since $K$ is compact in $H$ there exists $B \in C(Q)$ such that $\eta(B) = K$. Observe that the map $\tilde{\eta}: Q(B) \to F(K)$ induced by $\eta$ is open. Thus it suffices to show that there exists $\tilde{B} \in C(Q)$ such that $B \leq \tilde{B}$ and the canonical map $\tilde{e}(B, \tilde{B})$ from $Q(B)$ to $Q(\tilde{B})$ is compact.

Let $\{\| \cdot \|_n\}$ be an increasing sequence of continuous semi-norms defining the topology of $Q$ and let $Q_n$ be the canonical Banach space associated to $\| \cdot \|_n$. Since
$Q$ is reflexive, $Q'$ is bornological [10]. Hence $Q' = \operatorname{lim ind} Q'_n$. Put $P = \bigoplus_{n \geq 1} Q'_n$.

Let $\alpha$ be the canonical map from $P$ onto $Q'$.

First we find a continuous semi-norm $\rho$ on $P$ such that the map $\bar{\alpha}: P_\rho \to Q'_{p(K)}$ induced by $\alpha$ is compact, where $p(K)$ denotes the sup-norm on $B$. Take a sequence $\lambda_j \downarrow 0$ such that $\sum_j \lambda_j \leq 1$ and such that for the unit open ball $U_j$ in $Q_j$ we have $\lambda_j B \subseteq U_j$. Consider the semi-norm $\rho$ on $P$ given by

$$\rho(\{u_j\}) = \sum_j \|u_j\|_j / \lambda_j^2,$$

where $u_j \in Q'_j$ and $\| \cdot \|_j$ is the sup-norm on $U_j$.

Obviously $\alpha$ induces a continuous linear map $\bar{\alpha}$ from $P_\rho$ to $Q'_{p(B)}$.

We show that $\bar{\alpha}$ is compact.

Indeed let $\{u^{(n)}\}$ be a sequence in $P$ such that

$$M = \sup\{\rho(u^{(n)}): n \geq 1\} < \infty.$$

Then for every $m \geq 1$ and for every $x \in B$ we have

$$\sum_{j \geq m} |u^{(n)}_j(x)| = \sum_{j \geq m} \lambda_j |u^{(n)}_j(\lambda_j x)| / \lambda_j^2 \leq M \sum_{j \geq m} \lambda_j$$

and

$$\sup\{\|u^{(n)}_j\|_j: n \geq 1\} \leq M \lambda_j^2 \quad \text{for every} \quad j \geq 1.$$

These inequalities show that $\{\bar{\alpha}(u^{(n)}_j)\}$ is equicontinuous on $B$. Since $B$ is compact it follows that $\{u^{(n)}\}$ is relatively compact in $Q'_{p(B)}$.

Now by the openness of $\alpha: P \to Q'$ there exists $\bar{B} \in C(Q)$ containing $B$ such that the canonical map induced by $\alpha$ from $P_\rho$ onto $Q'_{p(B)}$ is open. Hence the canonical map from $Q'_{p(B)}$ to $Q'_{p(B)}$ is compact. This yields from the commutativity of the diagram

$$\begin{array}{ccc}
Q(B) & \longrightarrow & Q(B) \\
\downarrow & & \downarrow \\
[Q'_{p(B)}]' & \longrightarrow & [Q'_{p(B)}]' \\
\end{array}$$

in which the maps $Q(B) \hookrightarrow [Q'_{p(B)}]'$ and $Q(\bar{B}) \hookrightarrow [Q'_{p(B)}]'$ are canonical embeddings, the compactness of $e(B, \bar{B})$. 

**Lemma 3.** Let $\theta$ be a non-zero convolution operator on $\operatorname{Im} F_D$. Then $U_T$ with $T = \gamma \theta$, is surjective.

**Proof:** i) Let $\psi \in H(D)$ with $U'_T(\psi) = 0$. Then

$$\mu(F'_D(T) \psi) = (U'_T \mu)(\psi) = 0.$$
for every $\mu \in H'(D)$.

By the Hahn–Banach Theorem we have $\mathcal{F}_D'(T) \psi = 0$. Since $\mathcal{F}_D'(T) \neq 0$ it follows that $\psi = 0$. Thus $U_T'$ is injective.

Assume now that $\{\psi_{i\alpha}\} \subset \text{Im} \ U_T'$ which is weakly convergent to $\varphi$ in $H(D)$. Then for every finite dimensional subspace $F$ of $E$, the sequence $\{\mathcal{F}_D'(T) \psi_{i\alpha} \mid F \cap D\}$ is weakly convergent to $\varphi \mid F \cap D$. Since the ideal in $H(F \cap D)$ generated by $\mathcal{F}_D'(T) | F \cap D$ is weakly closed in $H(F \cap D)$ it follows that $\varphi \mid F \cap D = \mathcal{F}_D'(T) | F \cap D \psi_F$ for some $\psi_F \in H(F \cap D)$. By the unique principle the family $\{\psi_F\}$ defines a Gateaux holomorphic function $\psi$ on $D$ such that $\mathcal{F}_D'(T) \psi = \varphi$. This relation yields by the Zorn Theorem the holomorphicity of $\psi$ on $D$. Hence $U_T'$ has the weakly closed image in $H(D)$.

ii) Let $\phi \in H'(D)$ with $U_T(\phi) \neq 0$ and let $\mu$ be an arbitrary element of $H'(D)$. Take $i_0 \in I$ such that $\phi, \mu \in H'(D_{i_0})$. Let us note that the canonical map from $H'(D_{i_0})$ to $H'(D)$ induces a continuous linear map from $\text{Im} \mathcal{F}_{D_{i_0}}$ to $\text{Im} \mathcal{F}_D$ for which the following diagram is commutative

\begin{equation}
\begin{array}{ccc}
H'(D) & \xrightarrow{U_T} & H'(D) \\
\downarrow{\mathcal{F}_D} & & \downarrow{\mathcal{F}_D} \\
H'(D_{i_0}) & \xrightarrow{\mathcal{F}_{D_{i_0}}} & H'(D_{i_0}) \\
\downarrow{\mathcal{F}_D} & & \downarrow{\mathcal{F}_D} \\
\text{Im} \mathcal{F}_{D_{i_0}} & \xrightarrow{T_0^*} & \text{Im} \mathcal{F}_{D_{i_0}}
\end{array}
\end{equation}

where $T_0 \in (\text{Im} \mathcal{F}_{D_{i_0}})'$ is induced by $T$.

Hence by Lemma 2 without loss of generality we may assume that $E$ is a Schwartz b.v.s. Take a strictly increasing sequence $\{i_j\}_{j \geq 0}$ in $I$. Put

$$F = \lim \text{ind} \ E_{i_j}, \quad D_0 = D \cap F$$

and

$$T_0 = T \mid \text{Im} \mathcal{F}_{D_0}.$$ 

Consider the commutative diagram (1) in which $D_{i_0}$ is replaced by $D_0$ with $\phi, \mu \in H'(D_{i_0})$. By i) $U_{T_0}'$ is injective and has the weakly closed image. This implies that $U_{T_0} = U_{T_0}'$ is surjective. Thus $\mu = U_T(\beta)$ for some $\beta \in H'(D)$. ■

**Proof of Existence Theorem:** By the relation $T^* \mathcal{F}_D = \mathcal{F}_D U_T$ we infer that Existence Theorem is an immediate consequence of Lemma 3. ■

To prove Approximation Theorem we need the following five lemmas.

**Lemma 4.** $\text{Exp}(D)$ is dense in $\text{Im} \mathcal{F}_D$. 

Proof: Let \( S \in (\text{Im } \mathcal{F}_D)' \) such that \( S(\exp x) = 0 \) for every \( x \in D \).
Then
\[
\mathcal{F}_D'(S)(x) = 0 \quad \text{for every } x \in D.
\]
Since \( \mathcal{F}_D' \) is injective, it follows that \( S = 0 \). □

Lemma 5. Let \( f, g \in H(D) \) such that for every finite dimensional subspace \( F \) of \( E \) on which \( g \neq 0 \), the function \( f \mid F \cap D \) is divisible by \( g \mid F \cap D \). Then \( f \) is divisible by \( g \).

Proof: By the unique principle there exists a Gateaux holomorphic function \( h \) on \( D \) such that \( f = h g \). Since \( h \) is holomorphic at every \( x \in D \) with \( g(x) \neq 0 \), by the Zorn Theorem \( h \) is holomorphic on \( D \). □

Lemma 6. Let \( X, T \in (\text{Im } \mathcal{F}_D)' \), \( T \neq 0 \), such that
\[
\forall x \in D, \forall P \in \mathcal{P}(E): \quad T^* P \exp(x) = 0 \Rightarrow X(P \exp(x)) = 0.
\]
Then \( \mathcal{F}_D'(X) \) is divisible by \( \mathcal{F}_D'(T) \).

Proof: By hypothesis we have
\[
\forall x \in D: \quad \mathcal{F}_D'(T)(x) = 0 \Rightarrow \mathcal{F}_D'(X)(x) = 0.
\]
This implies that \( \mathcal{F}_D'(X) \mid F \cap D \) is divisible by \( \mathcal{F}_D'(T) \mid F \cap D \) for every finite dimensional subspace \( F \) of \( E \) on which \( \mathcal{F}_D'(T) \neq 0 \). Lemma 5 yields that \( \mathcal{F}_D'(X) \) is divisible by \( \mathcal{F}_D'(T) \). □

Lemma 7. Let \( E \) be a weak Schwartz b.v.s. and let \( i < j < k \). Then \( E^+ \) is dense in \( E_i' \mid E_i \).

Proof: Denote by \( E_i^+ \) the completion of \( E^+ / \text{Ker } || \cdot ||_i \), where \( || \cdot ||_i \) is the semi-norm on \( E^+ \) defined by the unit open ball \( U_i \) in \( E_i \). Since \( E^+ \) separates the points of \( E \), it follows that \( E^+ \subseteq \lim \text{ind } E_i^+ \) (algebraically). On the other hand, by the weak compactness of the canonical map \( \omega_{ij} \) from \( E_i \) to \( E_j \) for every \( i < j \) and since \( U_i \) is \( \sigma(E_i^+, E_j^+) \)-dense in the unit open ball \( U_i^+ \) in \( E_j^+ \), we have
\[
\text{Cl} \omega_{ij}^+(U_i^+) = \text{Cl} \omega_{ij}^+(\text{Cl}_{\sigma(E_i^+, E_j^+)} U_i) \leq \text{Cl} \omega_{ij}^+(U_i) = \text{Cl} \omega_{ij}(U_i) \leq E_j,
\]
where \( \omega_{ij}^+ \) is the restriction map from \( E_j^+ \) to \( E_i^+ \) and \( U_i^+ \) is the unit open in \( E_i^+ \).

Thus for \( i < j < k \) we have by the weak compactness of \( \omega_{ij}^+ \) the following two commutative diagrams
\[
\begin{array}{ccc}
E_j & \longrightarrow & E_j' \\
\omega_{jk}^- & \quad \longrightarrow \quad & \omega_{kj}^+
\end{array}
\quad
\begin{array}{ccc}
E_j & \longrightarrow & E_j^+ \\
\omega_{jk}^- & \quad \longrightarrow \quad & \omega_{kj}^+
\end{array}
\]

\[
\begin{array}{ccc}
E_k & \longrightarrow & E_k^+ \\
E_k & \longrightarrow & E_k^+
\end{array}
\quad
\begin{array}{ccc}
E_i & \longrightarrow & E_i^+ \\
E_i & \longrightarrow & E_i^+
\end{array}
\]
This implies that \( E^+ \) is dense in \( E'_k | E_i \).

Put

\[
H_0(D) = \left\{ f \in H(D) : D^n f(x) \text{ can be approximated} \right\}. 
\]

by elements of \( E^\otimes n \) for every \( x \in D \).

**Lemma 8.** \( \ker \mathcal{F}_D = [H_0(D)]^\perp \) and hence \( \text{Im} \mathcal{F}_D \subseteq H_0(D) \).

**Proof:** Let \( \mu \in [H_0(D)]^\perp \). Then

\[
(\mathcal{F}_D \mu)(x^*) = \mu(\exp x^*) = \sum_{k \geq 0} (1/k!) \mu(x^k) = 0
\]

for every \( x^* \in E^+ \).

Hence \( \mu \in \ker \mathcal{F}_D \). Conversely, assume that \( \mathcal{F}_D \mu = 0 \). By hypothesis on \( D \)
and on \( E \) it follows that \( H_0(E) \) is dense in \( H_0(D) \).

**Proof of Approximation Theorem:** If \( \theta = 0 \), the result is true since \( \mathcal{P}(E) \exp(D) \) is dense in \( \text{Im} \mathcal{F}_D \). Let \( \theta \neq 0 \) and let \( X \in (\text{Im} \mathcal{F}_D)' \), \( X \mid \mathcal{P}(E) \exp(D) \cap \ker \theta = 0 \). This means that

\[
\forall P \in \mathcal{P}(E), \forall x \in D : \ T^* P \exp(x) = 0 \Rightarrow X(P \exp(x)) = 0
\]

where \( T^* = \theta \).

Lemma 6 implies that \( \mathcal{F}'_D(X) = h \mathcal{F}'_D(T) \) for some \( h \in H(D) \). From the relations \( \mathcal{F}'_D(X) \in H_0(D) \) and \( \mathcal{F}'_D(T) \in H_0(D) \), it is easy to see that \( h \in H_0(D) = (\ker \mathcal{F}_D)^\perp \). Thus \( h = \mathcal{F}'_D(Q) \) for some \( Q \in (\text{Im} \mathcal{F}_D)' \). Hence \( \mathcal{F}'_D(X) = \mathcal{F}'_D(Q) \mathcal{F}'_D(T) = \mathcal{F}'_D(Q^* T) \). From the injectivity of \( \mathcal{F}'_D \) we have

\[
X = Q^* T = T^* Q = (T^*)' Q = \theta'(Q).
\]

These equalities imply \( X = 0 \) on \( \ker \theta \).

**REFERENCES**


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