MEASURES OF WEAK NONCOMPACTNESS
IN BANACH SEQUENCE SPACES

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Abstract: Based on a criterion for weak compactness in the $\ell^p$ product of the sequence of Banach spaces $E_i, i = 1, 2, \ldots$, we construct a measure of weak noncompactness in this space. It is shown that this measure is regular but not equivalent to the De Blasi measure of weak noncompactness provided the spaces $E_i$ have the Schur property. Apart from this a formula for the De Blasi measure in the sequence space $c_0(E_i)$ is also derived.

1 – Introduction

The notion of a measure of weak noncompactness was introduced by De Blasi [5] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations (cf. [1, 2, 3, 7, 8, 11], for instance).

In order to recall this notion denote by $E$ a Banach space with the norm $\| \|$ and the zero element $\theta$. Let $B(x_0, r)$ stand for the closed ball centered at $x_0$ and with radius $r$ and let $B = B(\theta, 1)$.

Next, denote by Conv $X$ the closed convex hull of the set $X, X \subset E$. Moreover, let $M_E$ denote the family of all nonempty and bounded subsets of $E$ and $W_E$ its subfamily consisting of all relatively weakly compact sets.

The measure of weak noncompactness of De Blasi [5] is defined in the following way:

$$\beta(X) = \inf \left\{ \varepsilon > 0 : \text{there exists a set } Y \in W_E \text{ such that } X \subset Y + \varepsilon B_E \right\},$$

where $X \in M_E$. This function possesses several useful properties [5] (see also below). For example, $\beta(B_E) = 1$ whenever $E$ is nonreflexive and $\beta(B_E) = 0$ otherwise.

There exists also an axiomatic approach in defining of measures of noncompactness [4]. Let us recollect this definition.

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**Definition.** A function \( \mu : M_E \to R_+ = [0, \infty) \) is said to be a measure of weak noncompactness in \( E \) if it satisfies the following conditions:

1. \( \mu(X) = 0 \iff X \in W_E \);
2. \( X \subset Y \Rightarrow \mu(X) \leq \mu(Y) \);
3. \( \mu(\text{Conv } X) = \mu(X) \);
4. \( \mu(X \cup Y) = \max\{\mu(X), \mu(Y)\} \);
5. \( \mu(X + Y) \leq \mu(X) + \mu(Y) \);
6. \( \mu(cX) = |c| \mu(X), \ c \in R \).

Let us mention that in the paper [4] a measure of weak noncompactness in the above sense is called to be regular.

Notice that De Blasi measure \( \beta \) is a measure of weak noncompactness in this sense and has also some additional properties [5]. However, for any measure \( \mu \) the following inequality holds [4]

\[
\mu(X) \leq \mu(B_E) \beta(X) .
\]

Finally, let us recall [4] that each measure of weak noncompactness satisfies also the Cantor intersection condition.

### 2 – Main results

At the beginning let us establish some notation. Assume that \( (E_i, \| \cdot \|_i) \), \( i = 1, 2, \ldots \), is a given sequence of Banach spaces. Fix a number \( p, 1 \leq p < \infty \) and consider the set of the sequences \( x = (x_i) \) such that \( x_i \in E_i \) for any \( i = 1, 2, \ldots \) and \( \sum_{i=1}^{\infty} \|x_i\|_i^p < \infty \). Denote this set by \( \ell^p(E_1, E_2, \ldots) \) or shortly by \( \ell^p(E_i) \). If we normed it by

\[
\|x\| = \|(x_i)\| = \left( \sum_{i=1}^{\infty} \|x_i\|_i^p \right)^{1/p}
\]

then it becomes a Banach space [10, 12].

Similarly, let \( c_0(E_i) \) denote the space of all sequences \( x = (x_i) \), \( x_i \in E_i \), with the property \( \|x_i\|_i \to 0 \) as \( i \to \infty \) and endowed by the norm

\[
\|x\| = \|(x_i)\| = \max \left\{ \|x_i\| : i = 1, 2, \ldots \right\} .
\]

Further, let \( e_k \) denote the canonical projection of one of the spaces \( \ell^p(E_i) \), \( c_0(E_i) \) or \( \ell^p(E_1, E_2, \ldots, E_n) \) onto the space \( E_k \), i.e. \( e_k(x_1, x_2, \ldots) = x_k \). Observe that \( e_k(B_p) = e_k(B_0) = B_{E_k} \), where \( B_p = B_{\ell^p(E_i)} \) and \( B_0 = B_{c_0(E_i)} \).
In what follows we shall need the following theorem.

**Theorem 1.** A subset $X$ of the space $\ell^p(E_i)$, $1 < p < \infty$, is relatively weakly compact if and only if

(a) $X$ is bounded;

(b) the set $e_k(X)$ is relatively weakly compact in $E_k$ for any $k = 1, 2, \ldots$.

This theorem comes from [12], where the case $E_k = E$, $k = 1, 2, \ldots$ was investigated. Repeating step by step the reasoning from [12] we can easily obtain the proof of Theorem 1.

In order to define measures of weak noncompactness in the space $\ell^p(E_i)$ let us assume that $\beta_i$ is De Blasi measure in the space $E_i$, $i = 1, 2, \ldots$ and let $\beta_p$ denote De Blasi measure in $\ell^p(E_i)$. Further, for $X \in M_{\ell^p(E_i)}$ let us put

\[
(2) \quad \mu(X) = \sup \left\{ \beta_n(e_n(X)) : n = 1, 2, \ldots \right\}.
\]

Then we have the following theorem.

**Theorem 2.** The function $\mu$ is a measure of weak noncompactness in the space $\ell^p(E_i)$, $1 < p < \infty$, such that $\mu(X) \leq \beta_p(X)$ for any $X \in M_{\ell^p(E_i)}$.

**Proof:** Notice first that when all the spaces $E_i$ are reflexive then $\ell^p(E_i)$ is also reflexive [10], so in view of Theorem 1, we have that $\mu(X) = 0$ for any $X \in M_{\ell^p(E_i)}$.

Let us suppose that at least one of the space $E_i$ is nonreflexive. Then taking into account the properties of the function $\beta$ we can easily infer that the function $\mu$ satisfies all the conditions of our Definition (in fact, the condition (1) is a consequence of Theorem 1).

Finally, let us notice that $\beta_k(e_k(B_k)) = \beta_k(B_{E_k}) = 1$ at least for one natural number $k$. Thus we deduce that $\mu(B_p) = 1$ and by (1) we obtain that $\mu(X) \leq \beta_p(X)$. This complete the proof. $lacksquare$

In the sequel we are going to show that the measure of weak noncompactness defined by (2) has not to be equivalent to De Blasi measure $\beta_p$.

First, let us recall that a Banach space $E$ is said to have *Schur property* if weakly convergent sequences in $E$ are norm convergent. For example, the classical space $\ell^1$ has this property [6].

In what follows we shall need the following two lemmas.

**Lemma 1.** Let $E$ be a Banach space having Schur property. Then a set $X \subset E$ is weakly compact if and only if $X$ is compact.
Lemma 2. Let $E_1, E_2, \ldots, E_n$ be Banach spaces with Schur property. Then the space $\ell^p(E_1, E_2, \ldots, E_n)$ has also Schur property for $1 \leq p < \infty$.

We omit trivial proofs of the lemmas.

Starting from now on let us assume that $(E_i, \| \cdot \|_i)$ is a sequence of Banach spaces being nonreflexive and such that every space $E_i$ has Schur property. Then we have the following theorem.

Theorem 3. Under the above assumptions the measure of weak noncompactness $\mu$ in the space $\ell^p(E_i)$ defined by (2) is not equivalent to De Blasi measure $\beta_p$ ($1 < p < \infty$).

Proof: Suppose the contrary. Then there exists a constant $c > 0$ such that
\begin{equation}
(3) \quad c \beta_p(X) \leq \mu(X)
\end{equation}
for any $X \in M_{\ell^p(E_i)}$.

Now, consider the sequence $(X_n)$ of subsets of $\ell^p(E_i)$ having the form
\[ X_n = \left\{ x = (x_1, x_2, \ldots, x_n, \theta, \theta, \ldots) : x_1 \in B_{E_1}, \ldots, x_n \in B_{E_n} \right\}, \]
for $n = 1, 2, \ldots$. Obviously we can write
\[ X_n = B_{E_1} \times B_{E_2} \times \cdots \times B_{E_n} \times \{ \theta \} \times \{ \theta \} \times \cdots \]
which implies that we can treat $X_n \subset \ell^p(E_1, E_2, \ldots, E_n)$. Particularly we have that $e_i(X_n) = B_{E_i}$ ($i = 1, 2, \ldots, n$) and consequently
\[ \mu(X_n) = 1 \]
for $n = 1, 2, \ldots$. Thus, in virtue of (3) we get
\begin{equation}
(4) \quad \beta_p(X_n) \leq 1/c
\end{equation}
for $n = 1, 2, \ldots$.

Further, let us choose an integer $n$ such that $n^{1/p} - (2/c) > 0$ and take $\varepsilon > 0$ such that $n^{1/p} - 2 \left( \frac{1}{c} + \varepsilon \right) > 0$. By (4) we can find a relatively weakly compact set $W_n$ in the space $\ell^p(E_i)$ such that
\[ X_n \subset W_n + \left( \frac{1}{c} + \varepsilon \right) B_{\ell^p(E_i)} . \]
In view of the remark made before, instead of the above inclusion we may write
\begin{equation}
(5) \quad X_n \subset W_n + \left( \frac{1}{c} + \varepsilon \right) B_{\ell^p(E_1, E_2, \ldots, E_n)} , \]

where \( W_n \) is treated as a relatively weakly compact set in the space \( \ell^p(E_1, E_2, \ldots, E_n) \).

Now, fix arbitrarily \( i, \ 1 \leq i \leq n \). In view of generalized version of Riesz lemma [9] we can select a sequence \( (x^i_k) \subset B_{E_i} \) such that

\[
\|x^i_k - x^i_m\| > 1
\]

for \( k \neq m, k, m = 1, 2, \ldots \) and for every \( i = 1, 2, \ldots, n \).

Next, consider the sequence \( (y_k)_{k \in \mathbb{N}} \) of points from \( X_n \) of the form

\[
y_n = (x^1_k, x^2_k, \ldots, x^n_k, \theta, \theta, \ldots),
\]

\( k = 1, 2, \ldots \). Taking \( k \neq m \) and keeping in mind (6) we derive

\[
\|y_k - y_m\| = \|y_k - y_m\|_{\ell^p(E_1, E_2, \ldots, E_n)} = \left( \sum_{i=1}^{n} \|x^i_k - x^i_m\|_i^p \right)^{1/p} > n^{1/p}.
\]

On the other hand in view of (5) we can find \( w_k \in W_k \) and \( z_k \in B_{\ell^p(E_1, E_2, \ldots, E_n)} \) (for any \( k = 1, 2, \ldots \)) such that

\[
y_k = w_k + \left( \frac{1}{c} + \varepsilon \right) z_k.
\]

Hence, taking \( k \neq m \) we obtain

\[
\|w_k - w_m\|_{\ell^p(E_1, \ldots, E_n)} = \|\left( y_k - y_m \right) - \left( \frac{1}{c} + \varepsilon \right) (z_k - z_m)\|_{\ell^p(E_1, \ldots, E_n)}
\]

\[
\quad \geq \|y_k - y_m\|_{\ell^p(E_1, \ldots, E_n)} - \left( \frac{1}{c} + \varepsilon \right) \|z_k - z_m\|_{\ell^p(E_1, \ldots, E_n)}
\]

\[
\quad > n^{1/p} - \left( \frac{1}{c} + \varepsilon \right) \|z_k - z_m\|_{\ell^p(E_1, \ldots, E_n)}.
\]

Consequently

\[
\|w_k - w_m\|_{\ell^p(E_1, \ldots, E_n)} > n^{1/p} - 2 \left( \frac{1}{c} + \varepsilon \right) > 0
\]

for \( k, m = 1, 2, \ldots, k \neq m \).

Thus we lead to a contradiction because in view of Lemmas 1 and 2 the set \( W_k \) is relatively compact in the space \( \ell^p(E_1, E_2, \ldots, E_n) \). This complete the proof. \( \blacksquare \)

In the sequel we shall deal with a measure of weak noncompactness in the space \( c_0(E_i) \). Similarly as before let \( \beta_k \) denote De Blasi measure in \( E_k \) \( (k = 1, 2, \ldots) \).
and $\beta_0$ stand for this measure in the space $c_0(E_i)$. For further purposes denote by $d_k$ the operator acting from $c_0(E_i)$ into itself defined by

$$d_k(x) = d_k(x_1, x_2, \ldots) = (\theta, \theta, \ldots, \theta, x_k, x_{k+1}, \ldots).$$

Finally, define for $X \in M_{c_0(E_i)}$:

$$a(X) = \sup \left\{ \beta_n(e_n(X)) : n = 1, 2, \ldots \right\},$$

$$b(X) = \inf \left\{ \beta_0(d_n(X)) : n = 1, 2, \ldots \right\},$$

$$\gamma(X) = \max \left\{ a(X), b(X) \right\}.$$

Then we have the following theorem.

**Theorem 4.** $\beta_0(X) = \gamma(X)$.

**Proof:** Let us take an arbitrary number $r > \gamma(X)$. Then there exists a positive integer $n$ such that

$$\beta_0(d_n(X)) < r$$

which implies that we can choose a subset $W \in W_{c_0(E_i)}$ with the property

(7) \hspace{1cm} d_n(X) \subset W + rB_0.

Without loss of generality we can assume that $W = d_n(W)$.

On the other hand $\beta_k(e_k(X)) < r$ for any $k = 1, 2, \ldots, n - 1$ which allows us to deduce that there is $W_k = W_{E_k}$ such that

(8) \hspace{1cm} e_k(X) \subset W_k + rB_{E_k}

for $k = 1, 2, \ldots, n - 1$.

Now, keeping in mind (7) and (8) we infer that

$$X \subset \left( (W_1 + rB_{E_1}) \times \cdots \times (W_{n-1} + rB_{E_{n-1}}) \times \{\theta\} \times \cdots \right) + W + rB_0$$

and consequently

$$x \subset \left( W_1 \times W_2 \times \cdots \times W_{n-1} \times \{\theta\} \times \cdots \right) + W + rB_0.$$ 

Hence, by the properties of De Blasi measure we have

$$\beta_0(X) \leq r$$

which means that

$$\beta_0(X) \leq \gamma(X).$$
In order to show the converse inequality take \( r > \bar{\beta}_0(X) \). Then we can find a set \( W \in \mathcal{W}_c(\mathcal{E}_i) \) such that \( X \subset W + rB_0 \). Hence we have
\[
\beta_n(e_n(X)) \leq \beta_n(e_n(W)) + r \beta_n(e_n(B_0)) \leq r
\]
for \( n = 1, 2, \ldots \). Consequently
\[
a(X) \leq r, \quad b(X) \leq r,
\]
which gives the desired inequality and ends the proof. 

Let us notice that \( d_n(X) \geq d_k(X) \) for \( n \leq k \) which implies that
\[
b(X) = \lim_{n \to \infty} \beta_0(d_n(X)).
\]

Finally observe that from Theorem 4 we obtain the following criterion for weak compactness in the space \( c_0(\mathcal{E}_i) \).

**Corollary 1.** A subset \( X \) of the space \( c_0(\mathcal{E}_i) \) is relatively weakly compact if and only if

(i) \( X \) is bounded,

(ii) the set \( e_k(X) \) is relatively weakly compact in \( \mathcal{E}_k \) for any \( k = 1, 2, \ldots \), and

(iii) for any \( \varepsilon > 0 \) there exists a positive integer \( n_0 \) such that \( \beta_0(d_n(X)) \leq \varepsilon \) for \( n \geq n_0 \).

**Corollary 2.** Let \( X \) be a subset of the space \( c_0(\mathcal{E}_i) \) satisfying the conditions (i), (ii) of Corollary 1 and instead of (iii) the following one

(iv) \( \lim_{n \to \infty} \sup_{x \in X} \left[ \max \{ \| x_k \| : k \geq n \} \right] = 0. \)

Then \( X \) is relatively weakly compact.

Indeed, notice that
\[
\sup_{x \in X} \left[ \max \{ \| x_k \| : k \geq n \} \right] = \| d_n(X) \|.
\]

Thus in view of the inequality
\[
\beta_0(d_n(X)) \leq \| d_n(X) \|
\]
we infer that \( X \) satisfies the condition (iii) of Corollary 1.
REFERENCES


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